Fast linear algebra for computing LDEs and recurrences

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joint work with Alin Bostan, Bruno Salvy, Gilles Villard

Functional Equations in Limoges, March 30, 2026

D-finite functions

$f(x) \in \mathsf{K}[[x]]$ solution of

$$a_r(x)f^{(r)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) = 0, \quad a_i(x) \in \mathsf{K}[x]$$

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$L(f) = 0$ with $L = a_r\partial^r + \cdots + a_1\partial + a_0$

$r =$ order of L, $\qquad d = \max(\deg a_i) =$ degree of L

Equation as data structure

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Examples:

$$\cos(x) \quad \{f'' + f = 0, f(0) = 1, f'(1) = 0\}$$

$$e^{\arctan(x)} \quad \{(1+x^2)f' - f = 0, f(0) = 1\}$$

$$A(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\right)x^n$$

$$\{x^2(x^2-34x+1)f^{(3)}+x(6x^2-153x+3)f'' + (7x^2-112x+1)f' + (x-5)f = 0, f(0) = 1\}$$

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Similar notion for recurrences: P-recursive sequences

Typical algorithmic problems

- **Effective closure properties**
  Given L_1, L_2 s.t. $L_1(f) = L_2(g) = 0$
  Compute L s.t. $L(f \diamond g) = 0$
  where $\diamond \in \{+, \times\}$

$\cos(x) + e^{\arctan(x)}$ satisfies

$$a(x)y^{(3)} + b(x)y'' + a(x)y' + b(x)y = 0,$$
$$a(x) = (x^2+1)(x^4+2x^2-2x+2),$$
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- **Polynomial Equation $\rightarrow$ LDE**
  Given $f(x)$ algebraic s.t. $P(x, f(x)) = 0$
  Compute L s.t $L(f) = 0$

Catalan generating series $C(x) = 1 + xC(x)^2$
$2C(x) + 2(-1 + 5x)C'(x) + x(4x - 1)C''(x) = 0$

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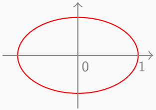
- **Reduction-based Creative telescoping**
  Find an LDE satisfied by $I(t) = \int_\gamma f(x,t)dx$
  for f D-finite

Perimeter of an ellipse



$$p(e) = 2\int_{-1}^{1} \sqrt{\frac{1-e^2x^2}{1-x^2}}dx$$
$$(e-e^3)p''(e) + (e^2-1)p'(e) + ep(e) = 0$$

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All previous problems can be seen as special instances of...

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Input: $T \in \mathsf{K}(x)^{n \times n}$, $a \in \mathsf{K}[x]^n$, let $\theta = \partial + T$

Output: $\eta = (\eta_0, \ldots, \eta_m) \in \mathsf{K}[x]^{m+1} \setminus \{0\}$ s.t.:

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Key fact: [G. 25,26] In all specific problems,

a structure inherited from $\boxed{T = XM^{-1}Y}$

with X, M, Y polynomial matrices of *small* degrees

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Exploit the structure for

- Degree bounds [G. 25,26]
- Solving the system **Now**

Example: Closure by sum

$L_1 = (x^2+1)\ \partial^2\ -(x+2)\ \partial\ -3$ and $L_2 = x^2\ \partial^2\ -(x+3)\ \partial\ -2$

Compute $L = L_1 \oplus L_2$

an operator s.t. $L(\alpha) = L(\alpha_1 + \alpha_2) = 0$ for all α_1, α_2 s.t. $L_1(\alpha_1) = L_2(\alpha_2) = 0$

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$\alpha = \alpha_1 + \alpha_2$

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$$\alpha'' = \frac{3}{x^2+1}\alpha_1 + \frac{x+2}{x^2+1}\alpha_1' + \frac{2}{x^2}\alpha_2 + \frac{x+3}{x^2}\alpha_2'$$

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$$\alpha^{(3)} = \cdots$$

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Solve the linear system and get $L = \eta_4 \cdot \partial^4 + \eta_3 \cdot \partial^3 + \eta_2 \cdot \partial^2 + \eta_1 \cdot \partial + \eta_0$

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$\alpha^{(\ell)} = c_{1,0}(x)\alpha_1 + c_{1,1}(x)\alpha_1' + c_{2,0}(x)\alpha_2 + c_{2,1}(x)\alpha_2'$ represented by vector $V_\ell \in \mathrm{K}(x)^4$

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Leibniz rule: $\boxed{V_{\ell+1} = \partial_x V_\ell + T \cdot V_\ell = \theta V_\ell}$ with

$$T = \begin{bmatrix} 0 & \frac{3}{x^2+1} & & \\ 1 & \frac{x+2}{x^2+1} & & \\ & & 0 & \frac{2}{x^2} \\ & & 1 & \frac{x+3}{x^2} \end{bmatrix} = \begin{bmatrix} 0 & 3 & & \\ 1 & x+2 & & \\ & & 0 & 2 \\ & & 1 & x+3 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & x^2+1 & & \\ & & 1 & \\ & & & x^2 \end{bmatrix}^{-1}$$

Solving pseudo-Krylov systems

Assume $T = XM^{-1}Y \in \mathsf{K}(x)^{n\times n}$ with $\Delta = \det M$ and degree $\delta = \deg \Delta$

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Degree of the column denominator:

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$$\begin{array}{c c}
\begin{array}{l}\text{Degree of the}\\ \text{column denominator:}\end{array} & \begin{array}{cccc} 0 & \delta & \cdots & m\delta \end{array}\\
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$$
\begin{array}{c}
\text{Degree of the column denominator:} \\
\\
\\
\\
\end{array}
\begin{bmatrix}
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\vdots & \vdots & & \vdots \\
a & \theta a & \cdots & \theta^m a \\
\vdots & \vdots & & \vdots \\
& & &
\end{bmatrix}
\cdot
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$\omega =$ exponent for matrix multiplication

Reduces to **Polynomial linear algebra** in degree $O(n\delta)$: $\widetilde{O}(n^{\omega+1}\delta)$ [Zhou Labahn Storjohann 12]

Best known **complexity** for all specific problems except closure properties

Solving pseudo-Krylov systems

Assume $T = XM^{-1}Y \in \mathsf{K}(x)^{n\times n}$ with $\Delta = \det M$ and degree $\delta = \deg \Delta$

Degree of the column denominator:

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Target complexity: $\widetilde{O}(n^{\omega}\delta)$

Width of T

Smith normal form:

$$VMU = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_h \end{bmatrix} \qquad V, U \in \mathsf{K}[x]^{h\times h} \text{ unimodular, } \gamma_i \mid \gamma_{i+1}$$

$$\gamma_i = \text{gcd of all minors of dimension } i \text{ of } M$$

Generically, $\gamma_i = 1$ for $i < h$ and $\gamma_h = \det M = \Delta$.

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$$\Delta M^{-1} = U \begin{bmatrix} \Delta & & & \\ & \ddots & & \\ & & \Delta & \\ & & & 1 \end{bmatrix} V, \text{ and } \Delta T = \Delta X M^{-1} Y = uv^t \mod \Delta$$

T has width 1

Exploit the structure for T of width 1

$\Delta = \det M, \delta = \deg \Delta$

T admits a rank 1 decomposition

$$T = XM^{-1}Y = \frac{uv^t}{\Delta} + W, \quad u, v \in \mathsf{K}[x]^n_{<\delta}, W \in \mathsf{K}[x]^{n\times n}_{<\delta}$$

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Schur complement trick: $K \cdot \eta = 0 \iff \exists z, \begin{bmatrix} D & A \\ -N & F \end{bmatrix} \cdot \begin{bmatrix} z \\ \eta \end{bmatrix} = 0$

Polynomial linear algebra in degree δ!

New algorithm

A **unified** and **fast** algorithm solving pseudo-Krylov systems for T of width 1

On input matrix $T = XM^{-1}Y \in \mathsf{K}(x)^{n\times n}$, $a \in \mathsf{K}[x]^n$, and $m \geq 1$

1. Compute vectors u, v
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2. Build the polynomial matrices F, N, D, A
3. Compute $\begin{bmatrix} Z \\ H \end{bmatrix}$ a minimal basis of $\ker \begin{bmatrix} D & A \\ -N & F \end{bmatrix}$
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$\to \widetilde{O}\left(n^\omega \delta\right)$ [Zhou Labahn Storjohann 12]

Choice of m

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  Ex.: **Least** Common Left Multiple, **Minimal** order telescoper

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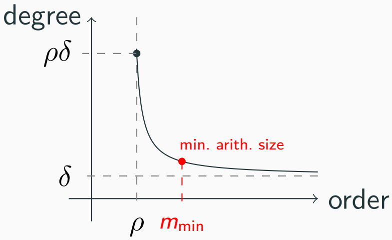

[Chen Jaroschek Kauers Singer 13] [G. 26]

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- Higher order LDE $m = \rho + \sigma$: $\dim\ker K = \sigma + 1$, there exists L s.t.

$$\mathrm{ord}(L) = m \quad \deg_x(L) = O(\frac{m}{\sigma+1}\delta)$$

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Yet for **Closure by product** and its extensions, *To be continued*
the matrices T involved do not admit a rank 1 decomposition

Computation of the realisation

$\Delta = \det M, \delta = \deg \Delta$

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$$\overline{F} = \begin{bmatrix} f_1 & \cdots & f_m \end{bmatrix}, \quad \overline{Q} = \begin{bmatrix} Q_1 & \cdots & Q_m \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ & \ddots & \vdots \\ & & \alpha_{m,m} \end{bmatrix}$$

Differential Toeplitz structure

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Similar reasoning on the Q_i's leads to

$$\overline{Q} = \overline{N} \cdot \overline{D}^{-1}$$

Thus,

$$N = \begin{bmatrix} 0 & \overline{N} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \\ & \overline{D} \end{bmatrix}^{-1}$$

$$K = \begin{bmatrix} a & \overline{F} \end{bmatrix} + \begin{bmatrix} 0 & \\ & \overline{Q} \end{bmatrix} \cdot \begin{bmatrix} 0 & \\ & \overline{A} \end{bmatrix} = F + Q \cdot A$$

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Differential Toeplitz structure

Conclusion

- **Unified** & **fast** algorithm for pseudo-Krylov systems for generic $T = XM^{-1}Y$
- Frequently yields a fast(er) algorithm for generic inputs in applications
- Convenient to implement using existing softwares
- Direct analogue for recurrences

Perspectives

- What can be done when no rank 1 decomposition of T exists?
- Handle more general classes of functions for Creative telescoping

  [Bostan Chyzak Lairez Salvy 2018, Brochet Salvy 2024]
- Efficient implementation of our approach
  tested on families of examples coming from applications

Thank you for your attention!

Example 2: Algeqtodiffeq

Series $C(x)$ root of polynomial $P(x, y) = p_e(x)y^e + \cdots + p_1(x)y + p_0(x)$

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Cockle's algorithm: Compute $V_\ell \in \mathrm{K}(x)[y]/(P)$ such that $C^{(\ell)}(x) = V_\ell(x, C(x))$

[Bostan Chyzak Salvy Lecerf Schost 07] $V_\ell(x,y) = v_{0,\ell}(x) + \cdots + v_{e-1,\ell}(x) \cdot y^{d-1}$

LDE obtained as a linear relation between the V_ℓ's

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$$= (\partial_x V_\ell + V_1 \cdot \partial_y V_\ell)(x, C(x))$$

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Series $C(x)$ root of polynomial $P(x,y) = p_e(x)y^e + \cdots + p_1(x)y + p_0(x)$

Cockle's algorithm: Compute $V_\ell \in \mathrm{K}(x)[y]/(P)$ such that $C^{(\ell)}(x) = V_\ell(x, C(x))$

[Bostan Chyzak Salvy Lecerf Schost 07] $V_\ell(x,y) = v_{0,\ell}(x) + \cdots + v_{e-1,\ell}(x) \cdot y^{d-1}$

LDE obtained as a linear relation between the V_ℓ's

$$\begin{aligned}
C^{(\ell+1)}(x) &= \left(C^{(\ell)}(x)\right)' = (V_\ell(x, C(x)))' \\
&= \frac{\partial V_\ell}{\partial x}\Big|_{y=C(x)} + C'(x)\frac{\partial V_\ell}{\partial y}\Big|_{y=C(x)} \\
&= (\partial_x V_\ell + V_1 \cdot \partial_y V_\ell)\,(x, C(x))
\end{aligned}$$

$$\begin{aligned}
0 &= P(x, C(x)) \\
0 &= P_x(x, C(x)) + C'(x) P_y(x, C(x)) = 0 \\
C'(x) &= -\frac{P_x(x, C(x))}{P_y(x, C(x))} \text{ and } V_1 = -\frac{P_x}{P_y} \mod P
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$$\begin{cases} V_{\ell+1} & = (\partial_x + T) \cdot V_\ell, \quad V_0 = y \\ T\colon v & \mapsto -\frac{P_x}{P_y} \cdot \partial_y(v) \mod P \end{cases}$$

Algeqtodiffeq $T = XM^{-1}Y$

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$$\begin{tikzcd}
v \in \mathcal{A}_e \arrow[rr, bend left, "T"] \arrow[d, "Y"] & & Tv = U \in \mathcal{A}_e \\
w = -P_x \partial_y(v) \in \mathcal{A}_{2e-1} \arrow[r, bend right, "M^{-1} = \mathrm{Syl}^{-1}(P, P_y)"'] & \begin{array}{c}(U, V) \in \mathcal{A}_e \times \mathcal{A}_{e-1} \\ \text{s.t. } w = UP_y + VP\end{array} \arrow[ur, "X"] &
\end{tikzcd}$$

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Assumption: T admits a rank one decomposition $T = uv^t/\Delta + W$

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