

A unified approach for degree bound estimates of linear differential operators

ISSAC 2025, Guanajuato, Mexico

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July 30th 2025

D-finite functions

$f(x) \in k[[x]]$ solution of

$$a_r(x)f^{(r)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) = 0, \quad a_i(x) \in k[x]$$

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Ex (LCLM): Given L_1, L_2 s.t. $L_1(f) = 0, L_2(g) = 0 \rightarrow$ Find L s.t. $L(f + g) = 0$.

LCLM of differential operators

$$L_1 = (x^2 + 1) \partial^2 - (x + 2) \partial - 3 \text{ and } L_2 = x^2 \partial^2 - (x + 3) \partial - 2$$

Compute $L = L_1 \oplus L_2$

a minimal order operator s.t. $L(\alpha) = L(\alpha_1 + \alpha_2) = 0$ for all α_1, α_2 s.t. $L_1(\alpha_1) = L_2(\alpha_2) = 0$

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$$\alpha = \alpha_1 + \alpha_2$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \eta_0 \end{bmatrix} \stackrel{?}{=} 0$$

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$$\alpha = \alpha_1 + \alpha_2 \quad \alpha' = \alpha'_1 + \alpha'_2$$
$$\alpha'' = \frac{3}{x^2 + 1} \alpha_1 + \frac{x + 2}{x^2 + 1} \alpha'_1 + \frac{2}{x^2} \alpha_2 + \frac{x + 3}{x^2} \alpha'_2$$

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Solve the linear system and get $L = \eta_4 \cdot \partial^4 + \eta_3 \cdot \partial^3 + \eta_2 \cdot \partial^2 + \eta_1 \cdot \partial + \eta_0 \in k[x] \langle \partial \rangle$

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One specific instance of a class of algorithms

A unified algorithmic scheme

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Focus on [several problems](#):

- Closure properties (LCLM, Symmetric product) [\[Stanley 1980\]](#) [\[van der Hoeven 2016\]](#)
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- Computation of a differential equation satisfied by an algebraic function
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Goal: Prove tight degree bounds in **all specific problems** at once

Adopt a **unified viewpoint** for bounds and algorithms

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Known bounds on $\deg \eta$?

- General arguments (col. by col.) $\rightarrow$ weak bounds
- *Ad hoc* arguments for special T $\rightarrow$ tight bounds

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Contributions

Key observation: In *all specific problems*, a structure inherited from

$$T = XM^{-1}Y$$

with X, M, Y polynomial matrices and $\det M$ *small*

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Let $T = XM^{-1}Y$ and $\delta = \deg \det M$. (+ technical assumptions in ISSAC paper)

Then, there exists a solution with $\deg \eta_i \in O(n\delta)$.

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Unified approach for bounds:

- Exhibit a *small* realisation $T = XM^{-1}Y$ (i.e. with *minimal* $\delta = \deg \det M$)
- Retrieve or improve the best known bound

Bounds in specific problems

Our unified approach catches the bounds!

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	Previous Bound	Our Bound	Matrix of the Problem
LCLM $\oplus$	$ds^2r + o(ds^2r)$ [BCLS 12]	$ds^2r + o(ds^2r)$	$T = \text{Diag}(C_1, C_2)$
SymProd $\otimes$	$O(dr^{2s})$ [Kauers 14]	$O(dr^{2s-1})$	$T = C_1 \otimes I_{r_2} + I_{r_1} \otimes C_2$
AlgeqtoDiffeq	$4d_y^2d_x + o(d_y^2d_x)$ [BCLSS 07]	$2d_y^2d_x + o(d_y^2d_x)$	$T: a \bmod P \mapsto -\partial_y(a)P_x/P_y \bmod P$
Hermite	$2d_y^2d_x + o(d_y^2d_x)$ [BCCL 10]	$2d_y^2d_x + o(d_y^2d_x)$	$T: a \bmod Q \mapsto -\text{herm}(Q_x a/Q^2)$

Denominators of minors of *pseudo*-Krylov matrices

$T = XM^{-1}Y \in k(x)^{n \times n}$, and $\Delta = \det M$, $\delta = \deg \Delta$,

$$K = \begin{bmatrix} a & \cdots & \theta^\rho a \end{bmatrix}$$

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For $s_1 \leq \cdots \leq s_r$ and $K = \begin{bmatrix} \theta^{s_1} a & \cdots & \theta^{s_r} a \end{bmatrix}$.

Any $r \times r$ minor m of K has denominator dividing Δ^{s_r} .

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Naive expansion:

$$m = \frac{\cdots}{\Delta^{s_1 + \cdots + s_r}}$$

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Key Proposition

For $s_1 \leq \cdots \leq s_r$ and $K = \begin{bmatrix} \theta^{s_1} a & \cdots & \theta^{s_r} a \end{bmatrix}$.

Any $r \times r$ minor m of K has denominator dividing Δ^{s_r} .

Technical tool:

Determinantal denominators [Coppel 74]

Naive expansion:

$$m = \frac{\cdots}{\Delta^{s_1 + \cdots + s_r}}$$

Warm-up without differentiation: the classical Krylov case

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No such matrix factorisation in the differential case

Remark: Gives tight estimates on the size of the minimal polynomial of T

Pseudo-Krylov case (improved proof)

$$K = \begin{bmatrix} \theta^{s_1} u_1 & \theta^{s_2} u_2 & \cdots & \theta^{s_r} u_r \end{bmatrix} \quad \theta = \partial + T \quad T = XM^{-1}Y \quad \Delta = \det M \quad u_i \in k[x]^n$$

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By multilinearity of det and induction: Minor of $K_1 = \frac{\cdots}{\Delta^{s_r-1}}$

$$\implies \text{Minor of } K = \frac{\cdots}{\Delta^{s_r}} \quad (\text{by Cauchy-Binet} + \text{multilinearity})$$

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- Non minimal operators (ex: CLM instead of LCLM) *Order-degree curves*

Thank you for your attention!