Fast algorithms for D-finite functions by matrix fraction descriptions

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joint work with Alin Bostan, Bruno Salvy, Gilles Villard

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D-finite functions / P-finite sequences

$f(x) \in \mathrm{K}[[x]]$ solution of

$$a_r(x)f^{(r)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) = 0$$

$$L(f) = 0 \text{ with } L = a_r\partial^r + \cdots + a_1\partial + a_0$$

$\boldsymbol{u} = (u_n) \in \mathrm{K}^{\mathbb{N}}$ solution of

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$r =$ order of L, $\quad d = \max(\deg a_i) =$ degree of L

Equation as data structure

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Examples:

$$e^{\arctan(x)} \quad \{(1+x^2)f' - f = 0, f(0) = 1\}$$

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$\{(n+2)^3 A_{n+2} - (34n^3 + 153n^2 + 229n + 115)A_{n+1} + (n+1)^3 A_n, A_0 = 1, A_1 = 3\}$$

$$A(x) = \sum_{n=0}^{\infty} A_n x^n$$

$$\{x^2(x^2 - 34x + 1)f^{(3)} + x(6x^2 - 153x + 3)f'' + (7x^2 - 112x + 1)f' + (x - 5)f = 0, f(0) = 1\}$$

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Goal: Efficient algorithms for computing LDEs and LREs

Typical algorithmic problems

1. **Effective closure properties**
   Given L_1, L_2 s.t. $L_1(f_1) = L_2(f_2) = 0$
   Compute L s.t. $L(f) = 0$
   where $f = f_1 \diamond f_2$ and $\diamond \in \{+, \times\}$

$\cos(x) + e^{\arctan(x)}$ satisfies

$$a(x)y^{(3)} + b(x)y'' + a(x)y' + b(x)y = 0,$$
$$a(x) = (x^2 + 1)(x^4 + 2x^2 - 2x + 2),$$
$$b(x) = -x(x^3 + 8x - 6)$$

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Catalan generating series $C(x) = 1 + xC(x)^2$

$$2C(x) + 2(-1 + 5x)C'(x) + x(4x - 1)C''(x) = 0$$

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   Find an LDE satisfied by $I(x) = \int_\gamma f(x,t)dt$
   for f D-finite

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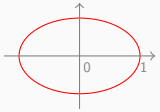
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Perimeter of an ellipse

$$p(e) = 2\int_{-1}^{1} \sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx$$



$$(e-e^3)p''(e) + (e^2-1)p'(e) + ep(e) = 0$$

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Algorithm

For $\ell = 0, 1, \ldots$
Compute $\partial_x^\ell(f)$ "modulo"

1. $L_1(f_1) = L_2(f_2) = 0$
2. $\langle P \rangle$
3. the *telescoping ideal* of f

until a linear relation is found

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Pseudo-Krylov(θ, a, m)

Input: $\theta = \partial + T$ or $T \cdot \sigma$, $T \in \mathrm{K}(x)^{n \times n}$, $a \in \mathrm{K}[x]^n$

Output: $\eta = (\eta_0, \ldots, \eta_m) \in \mathrm{K}[x]^{m+1} \setminus \{0\}$ s.t.:

$$\eta_0 \cdot a + \eta_1 \cdot \theta a + \cdots + \eta_m \cdot \theta^m a = 0$$

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Pseudo-Krylov system

Questions

- What is the size of the output operator? Tight degree bounds for η_i?
- How fast can we solve pseudo-Krylov systems?

Example: AlgeqtoDiffeq

Theorem [Abel 1827, Cockle 1861]

An algebraic series $C(x)$ is D-finite.

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Let $P(x, C(x)) = 0$ with $P \in \mathsf{K}[x, y]$ square-free wrt y. $(d, r) = (\deg_x(P), \deg_y P)$
Differentiating yields

$$P_x(x, C(x)) + C'(x)P_y(x, C(x)) = 0$$
$$C'(x) = -\frac{P_x(x, C(x))}{P_y(x, C(x))}$$

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Modular inverse $V = 1/P_y \bmod P$

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By induction: $C^{(\ell)} \in \mathsf{span}_{\mathsf{K}(x)}\left(1, C, \ldots, C^{r-1}\right) \rightarrow$ confined in a finite dimensional vector space

This proof is an algorithm!

Cockle's algorithm

Cockle's algorithm: Compute $V_\ell \in \mathrm{K}(x)[y]/(P)$ such that $C^{(\ell)}(x) = V_\ell(x, C(x))$

$V_\ell(x, y) = v_{0,\ell}(x) + \cdots + v_{r-1,\ell}(x) \cdot y^{r-1}$

LDE obtained as a linear relation between the V_ℓ's

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$$\begin{cases} V_{\ell+1} & = (\partial_x + T) \cdot V_\ell, \quad V_0 = y \\ T : v & \mapsto -\frac{P_x}{P_y} \cdot \partial_y v \mod P \end{cases}$$

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$$T = \begin{bmatrix} 0 & \frac{4}{27x^2+40x+16} & \frac{6x}{(27x^2+40x+16)(x+1)} \\ 0 & -\frac{4+9x}{27x^2+40x+16} & \frac{14x+8}{(27x^2+40x+16)(x+1)} \\ 0 & -\frac{3x}{27x^2+40x+16} & -\frac{18x^2+20x+8}{(27x^2+40x+16)(x+1)} \end{bmatrix}$$

$$\Delta = \left(27x^2 + 40x + 16\right)(x+1) = \mathrm{Res}(P, P_y)$$

Solving pseudo-Krylov systems: the direct approach

Let $T = \frac{1}{\Delta} \cdot B \in \mathrm{K}(x)^{n \times n}$, $B \in \mathrm{K}[x]^{n \times n}$, $\Delta \in \mathrm{K}[x]$, degree $d_T := \max(\deg \Delta, \deg B)$

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Ex.: **AlgeqtoDiffeq** $\deg \eta = O(n d_T) = O(r^2 d)$ **vs.** $O(r^3 d)$

with $n = r$, $d_T = O(rd)$ [Bostan Chyzak Salvy Lecerf Schost 07]

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Questions

- How to explain the degree gap?
- How to exploit it algorithmically?

Examples of results for AlgeqtoDiffeq

Random input $P(x, y)$ of bi-degree (d, r) over $\mathsf{K} = \mathbb{F}_p$ a 40-bit prime field, timings in seconds

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$d = r$	Cramer bd	New bd	Actual deg	Direct Pseudo-Krylov	**New Algo**
8	4,320	960	856	0.030	0.025
16	67,456	7,936	7,472	1.056	0.505
24	338,400	27,072	25,992	9.396	3.390
32	1,064,448	64,512	62,560	42.87	12.48
44	3,789,720	168,432	164,692	302.2	56.06
52	7,380,568	278,512	273,260	789.0	128.2

r	d	Cramer bd	New bd	Actual deg	Direct Pseudo-Krylov	**New Algo**
4	1,000	70,000	28,000	25,988	0.513	0.636
4	8,000	560,000	224,000	207,988	5.048	6.289
8	1,000	540,000	120,000	113,944	6.413	5.844
8	8,000	432,000	96000	911,944	73.34	60.66
16	1,000	4,216,000	496,000	481,760	111.1	60.54
16	2,000	8,432,000	992,000	963,760	265.5	133.4

Tight degree bounds

Matrix fraction description (MFD) [Kailath 1980]

Let $R \in \mathrm{K}(x)^{n \times n}$, suppose R proper i.e., $\lim_\infty R < \infty$

$$R = Q^{-1}P, \quad Q, P \in \mathrm{K}[x]^{n \times n}$$

McMillan degree: an invariant of R

$$\deg_M(R) \coloneqq \min\{\deg(\det Q) \mid R = Q^{-1}P, Q, P \in \mathrm{K}[x]^{n \times n}\}$$

minimal arithmetic size of a MFD of R: $O(n \deg_M(R))$

MFD for T

In **specific problems**, $T = (\Delta I_n)^{-1} \cdot B$ can be compressed to a *small* MFD $T = Q^{-1}P$

$$\deg_M(T) = \deg(\det Q) \ll nd_T$$

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Example: AlgeqtoDiffeq

$$T = 1/\Delta \cdot \begin{bmatrix} 0 & 4x+4 & 6x \\ 0 & -(x+1)(4+9x) & 14x+8 \\ 0 & -3x(x+1) & -18x^2-20x-8 \end{bmatrix} = \begin{bmatrix} 1+3x & 1 & 1 \\ 5 & -9x-7 & -7 \\ 10x+4 & 4+x & -3x \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Delta = \left(27x^2+40x+16\right)(x+1)$

Here $d_T = \deg(\Delta) = 3$ and $\deg_M(T) = \deg(\det Q) = 3 < nd_T = 9$

MFDs explain the gap

Theorem [G. 26]

1. $\exists\ \eta \neq 0$ s.t. $K\eta = 0$ and $\deg(\eta) \leq \deg_M(K)$
2. Pseudo-Krylov matrix $K = \begin{bmatrix} a & \theta a & \cdots & \theta^m a \end{bmatrix}$ satisfies $\deg_M(K) \leq n \deg_M(T)$

- (1.) is a special case of the **duality** between rational matrices and their kernel [Verghese van Dooren Kailath 1979]
- Proves tight degree bounds and explains the gap $\deg \eta = O(n \deg_M(T)) \ll n^2 d_T$

Proof of $\deg_M(K) \leq m \deg_M(T)$

Let $\theta = \partial + T = \partial + Q^{-1}P$

$$K = \begin{bmatrix} \theta a & \cdots & \theta^m a \end{bmatrix} = Q^{-1} \cdot \begin{bmatrix} Q\theta a & \cdots & Q\theta^m a \end{bmatrix}$$

Proof of $\deg_M(K) \leq m \deg_M(T)$

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Gauge transformation: $\theta_{[Q]} = Q\theta Q^{-1} = \partial + T_{[Q]}$ with $T_{[Q]} = (P - Q')Q^{-1}$

$$\implies \deg_M(T_{[Q]}) \leq \deg_M(T)$$

$$K = Q^{-1} \cdot \begin{bmatrix} u & \theta_{[Q]} u & \cdots & \theta_{[Q]}^{m-1} u \end{bmatrix} = Q^{-1} \cdot K_{[Q]}$$

with $u = Q\theta a \in \mathrm{K}[x]^n$

Proof of $\deg_M(K) \leq m \deg_M(T)$

Let $\theta = \partial + T = \partial + Q^{-1}P$

$$K = \begin{bmatrix} \theta a & \cdots & \theta^m a \end{bmatrix} = Q^{-1} \cdot \begin{bmatrix} Q\theta a & \cdots & Q\theta^m a \end{bmatrix}$$

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$$K = \begin{bmatrix} a & \theta a & \cdots & \theta^m a \end{bmatrix} = Q^{-1} \cdot \begin{bmatrix} Qa & Q\theta a & \cdots & Q\theta^m a \end{bmatrix}$$

Gauge transformation: $\theta_{[Q]} = Q\theta Q^{-1} = \partial + T_{[Q]}$ with $T_{[Q]} = (P - Q')Q^{-1}$

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$$\deg_M(K) \leq \deg_M(T) + \deg_M(K_{[Q]}) \leq m \deg_M(T)$$

The proof directly adapts for $\theta = T\sigma = Q^{-1}P\sigma$

Choice of m

$$K \cdot \eta = \begin{bmatrix} a & \theta a & \cdots & \theta^m a \end{bmatrix} \cdot \begin{bmatrix} \eta_0 \\ \eta_1 \\ \vdots \\ \eta_m \end{bmatrix} = 0$$

Choice of m

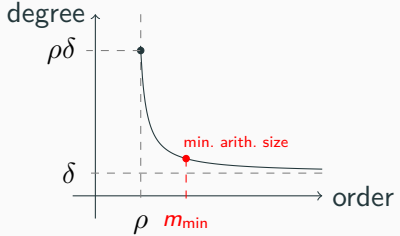
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- Minimal order LDE $m = \rho$: $\dim \ker K = 1$
  Ex.: **Least** Common Left Multiple, **Differential resolvent**, **Minimal** telescoper

$$\operatorname{ord}(L) = \rho \leq n \quad \deg_x(L) = O(\rho\delta) \quad \delta := \deg_M(T)$$

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[Chen Jaroschek Kauers Singer 13] [G. 26]

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- Higher order LDE $m = \rho + \sigma$: $\dim \ker K = \sigma + 1$, there exists L s.t.

$$\operatorname{ord}(L) = m \quad \deg_x(L) = O(\frac{m}{\sigma+1}\delta)$$

Algorithms

Computing MFDs

Theorem

$K = \begin{bmatrix} a & \theta a & \cdots & \theta^m a \end{bmatrix}$ satisfies $\deg_M(K) \leq n \deg_M(T)$

$\Longrightarrow$ $\exists$ compressed MFD $\boxed{K = D^{-1}N}$

Remark: $\ker K = \ker N$!

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Theorem

$K = \begin{bmatrix} a & \theta a & \cdots & \theta^m a \end{bmatrix}$ satisfies $\deg_M(K) \leq n \deg_M(T)$

$\Longrightarrow \exists$ compressed MFD $\boxed{K = D^{-1}N}$ **Remark:** $\ker K = \ker N$!

Remaining algorithmic question

- How to compute a compressed MFD of K efficiently?
- How to turn the previous proof into an efficient algorithm?

Divide and Conquer scheme:

Let $m = 2\ell$

$$K_m = \begin{bmatrix} \theta a & \cdots & \theta^{\ell} a & \theta^{\ell+1} a & \cdots & \theta^{m} a \end{bmatrix} = \begin{bmatrix} K_\ell & \theta^{\ell}(K_\ell) \end{bmatrix}$$

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- Recursive call: $K_\ell = D_1^{-1} N_1$

Divide and Conquer scheme:

Let $m = 2\ell$

$$K_m = \begin{bmatrix} D_1^{-1}N_1 & \theta^\ell(D_1^{-1}N_1) \end{bmatrix} = D_1^{-1} \cdot \begin{bmatrix} N_1 & D_1\theta^\ell(D_1^{-1}N_1) \end{bmatrix}$$

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- Describe $\widetilde{K} = D_1\theta^\ell(D_1^{-1}N_1) = D_2^{-1}N_2$

Divide and Conquer scheme:

Let $m = 2\ell$

$$K_m = D_1^{-1} \cdot \begin{bmatrix} N_1 & D_2^{-1} N_2 \end{bmatrix} = (D_2 D_1)^{-1} \cdot \begin{bmatrix} D_2 N_1 & N_2 \end{bmatrix} = D^{-1} N$$

- Recursive call: $K_\ell = D_1^{-1} N_1$
- Describe $\widetilde{K} = D_1 \theta^\ell (D_1^{-1} N_1) = D_2^{-1} N_2$

Description in the recurrence case

$\theta = T \cdot \sigma = Q^{-1}P \cdot \sigma$

$\theta^\ell = (T\sigma(T)\cdots\sigma^{\ell-1}(T)) \cdot \sigma^\ell = T^{(\ell)} \cdot \sigma^\ell$

Description in the recurrence case

$\theta = T \cdot \sigma = Q^{-1}P \cdot \sigma$ $\qquad$ $\theta^\ell = (T\sigma(T)\cdots\sigma^{\ell-1}(T)) \cdot \sigma^\ell = T^{(\ell)} \cdot \sigma^\ell$

Recursive scheme with common denominator:

$$\begin{cases} T^{(\ell)} &= D_1^{-1}W_1 \\ K_\ell &= D_1^{-1}N_1 \end{cases} \qquad \begin{cases} T^{(2\ell)} &= D_1^{-1}W_1D_1(x+\ell)^{-1}W_1(x+\ell) \\ \theta^\ell(K_\ell) &= D_1^{-1}W_1D_1(x+\ell)^{-1}N_1(x+\ell) \end{cases}$$

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A description $D_2^{-1}W_2 = W_1D_1(x+\ell)^{-1}$ yields

$$\begin{cases} T^{(2\ell)} &= D^{-1}W \\ K_{2\ell} &= D^{-1}N \end{cases}$$

with $D = D_2D_1$, $W = W_2W_1(x+\ell)$ and
$N = \begin{bmatrix} D_2N_1 & W_2N_1(x+\ell) \end{bmatrix}$

Description in the recurrence case

$\theta = T \cdot \sigma = Q^{-1}P \cdot \sigma$ $\qquad$ $\theta^\ell = (T\sigma(T) \cdots \sigma^{\ell-1}(T)) \cdot \sigma^\ell = T^{(\ell)} \cdot \sigma^\ell$

Recursive scheme with common denominator:

$$\begin{cases} T^{(\ell)} & = D_1^{-1} W_1 \\ K_\ell & = D_1^{-1} N_1 \end{cases} \qquad \begin{cases} T^{(2\ell)} & = D_1^{-1} W_1 D_1(x+\ell)^{-1} W_1(x+\ell) \\ \theta^\ell(K_\ell) & = D_1^{-1} W_1 D_1(x+\ell)^{-1} N_1(x+\ell) \end{cases}$$

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$$\begin{bmatrix} D_2 & W_2 \end{bmatrix} \cdot \begin{bmatrix} W_1 \\ -D_1(x+\ell) \end{bmatrix} = 0$$

$D_2^{-1}W_2$ obtained by shifted minimal nullspace basis computation [ZLS 12]

Description in the differential case

$\theta = \partial + T = \partial + Q^{-1}P$

No such structural property on the powers of θ

$$K_m = \begin{bmatrix} \theta a & \cdots & \theta^\ell a & \theta^{\ell+1} a & \cdots & \theta^m a \end{bmatrix} = \begin{bmatrix} K_\ell & \theta^\ell(K_\ell) \end{bmatrix}$$

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Transformed pseudo-linear map

$\widetilde{K} = \begin{bmatrix} \theta_\ell u & \cdots & \theta_\ell^\ell u \end{bmatrix}$, with $\theta_\ell = \partial + T_\ell = \partial + Q_\ell^{-1}P_\ell$

Gauge transformation: $\theta_{\ell+1} = Q_\ell \cdot \theta_\ell \cdot Q_\ell^{-1}$

$T_{\ell+1} = Q_{\ell+1}^{-1}P_{\ell+1} = (P_\ell - \partial(Q_\ell))Q_\ell^{-1}$ obtained by shifted minimal nullspace basis computation!

Conclusion

- **Unified** & **fast** algorithms for pseudo-Krylov systems
- Applied to:

  Effective Closure Properties / AlgeqtoDiffeq / Creative telescoping for bivariate rational functions
- Work for computing minimal **and** non-minimal operators
- Behave well in practice
- Convenient to implement using existing softwares (Flint, PML)

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- Work for computing minimal **and** non-minimal operators
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Perspectives

- Handle more general classes of functions for Creative telescoping
  - Hypergeometric sequences / Hyperexponential functions

    [Bostan Chen Chyzak Li Xin 13, Chen Huang Kauers Li 15, Bostan Dumont Salvy 16]
  - General D-finite functions [Bostan Chyzak Lairez Salvy 18, van der Hoeven 20, Brochet Salvy 24]
- Cost analysis in the differential case

Thank you for your attention!