Channel assignment revisited through 1-extendability of graphs

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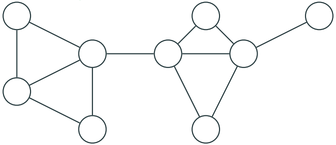
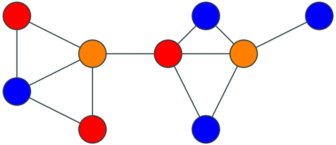
Context

Input : A conflict graph G and a number of channels (colors) k

Output : Is it possible to give a channel to each vertex such that there is no conflict in a channel, or equivalently can we find a proper k-coloring of the graph ?

Example





Wi-Fi network

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Question : What property should verify each color class to :

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Each color class has to be **1-extendable**.

Definition

A *Maximum Independent Set* (MIS) of a graph G is a set of pairwise independent vertices of maximum size.

[1] Soung Chang Liew, et al. "Back-of-the-Envelope Computation of Throughput Distributions in CSMA Wireless Networks," in 2009 IEEE International Conference on Communications, pp. 1-6, 2007.
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Wi-Fi network

Definition
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Previous work[1] proved that, under saturation, the performance of a node v in a Wi-Fi network is

$$p_v \sim \frac{\text{nb of MIS of G containing v}}{\text{nb of MIS of G}}$$

Thus, $p_v > 0$ if and only if v belongs to at least one MIS.

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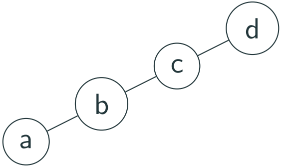
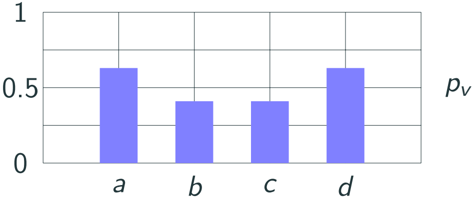
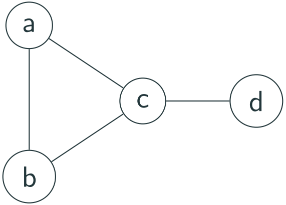
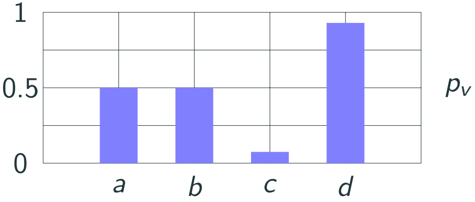
1-extendability and ns-3 simulations

Definition (Berge '78)
A graph G is 1-*extendable* if each vertex belongs to an MIS.









1-extendable k-partition

Definition
Given a graph G and an integer k, a 1-extendable k-partition of G is a partition $V_1, ..., V_k$ of the vertex set $V(G)$ such that $G[V_i]$ is 1-extendable for all $1 \leqslant i \leqslant k$.

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We call the 1-extendable chromatic number, $\chi_{1\text{-ext}}(G)$, the minimum integer k such that G has a 1-extendable k-partition.

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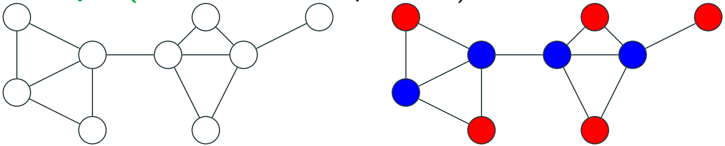
Example (A 1-extendable 2-partition)



First results

Complexity results

Theorem (Bergé, Busson, Feghali, Watrigant 2022)
Testing 1*-extendability is NP-hard.*

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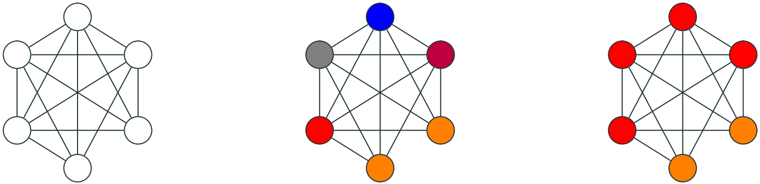
Testing 1*-extendability is* Θ_2^p*-complete, where* $\Theta_2^p = P^{NP[\log]}$.

Theorem
Deciding if a graph has 1*-extendable chromatic number at most* k
Θ_2^p*-hard for every fixed* $k \geqslant 1$.

Theorem
For any graph G with n vertices, $\chi_{\text{1-ext}}(G) \leqslant 2\sqrt{n}$.



proper 5-coloring

1-extendable 2-coloring

Upper bound on $\chi_{\text{1-ext}}$

Lemma
For any graph G, $\chi_{\text{1-ext}}(G) \leqslant \alpha(G)$.

Proof.
If $\alpha(G) = 1$, then G is a clique and $\chi_{\text{1-ext}}(G) = 1$. If $\alpha(G) > 1$, let S be the set of vertices of G that are in an MIS. Notice that :

- $G[S]$ is 1-extendable ;
- $\alpha(G - S) \leqslant \alpha(G) - 1$.

By induction hypothesis, $\chi_{\text{1-ext}}(G - S) \leqslant \alpha(G) - 1$ and use one color for S. □

Upper bound on $\chi_{\text{1-ext}}$

Theorem
For any graph G with n vertices, $\chi_{\text{1-ext}}(G) \leqslant 2\sqrt{n}$.

Proof.
While $\alpha(G) > \sqrt{n}$, extract an MIS S, use one color for S and continue with $G - S$.

If $\alpha(G) \leqslant \sqrt{n}$, use $\alpha(G)$ colors with the previous lemma. □

Algorithmic results

Goal

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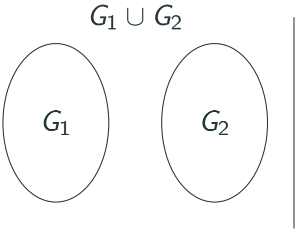
Main tool : Modular decomposition of the graph.

Modular decomposition

In 1967, Gallai proved that any graph G can be obtained by one of the following three operations :

- disjoint union of two graphs G_1 and G_2, written $G_1 \cup G_2$;



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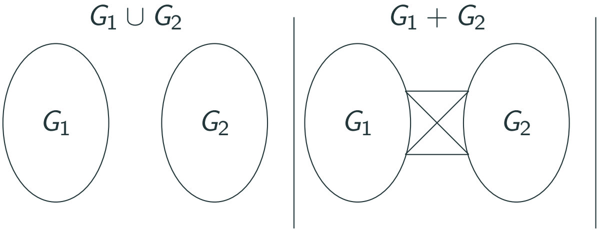

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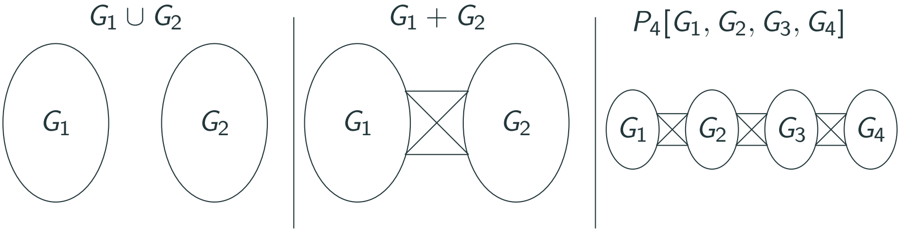
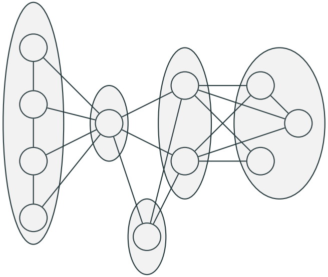
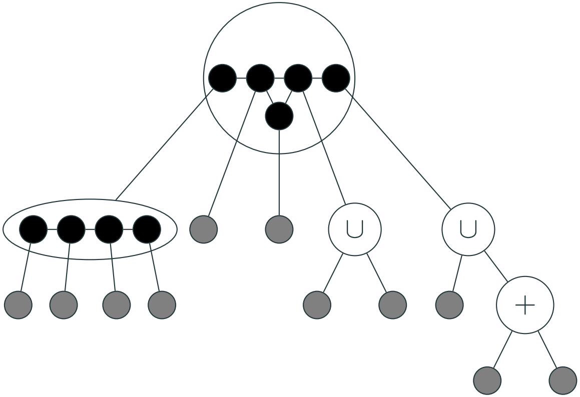
- disjoint union of two graphs G_1 and G_2, written $G_1 \cup G_2$;
- complete sum of two graphs G_1 and G_2, written $G_1 + G_2$;
- substitution of all the m vertices of a “prime” graph H with m graphs $G_1, ..., G_m$, written $H[G_1, ..., G_m]$.







Definition
The *modular-width of G* ($mw(G)$) is smallest number of vertices in a prime graph along the decomposition.

A *cograph* is a graph of modular-width 0.

1-extendable cographs

Proposition
Given two graphs G_1 and G_2,

1. $G_1 \cup G_2$ *is* 1*-extendable iff both* G_1 *and* G_2 *are* 1*-extendable ;*
2. $G_1 + G_2$ *is 1-extentable iff both* G_1 *and* G_2 *are 1-extendable and* $\alpha(G_1) = \alpha(G_2)$.

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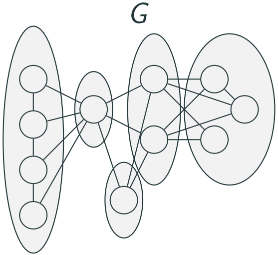
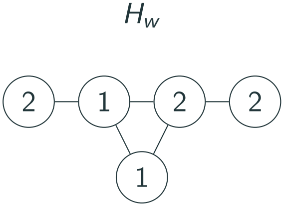
Idea : This characterization will be good for the design of a recursive algorithm, can we extend the substitution case ?

Recursive characterization of 1-extendability

Notation :

- H : representative graph of G (i.e. the first node of the modular decomposition)
- H_w : weighted representative graph (i.e. H with the weight function corresponding to the independence number of each modules).





Theorem
A graph G is 1-extendable if and only if all its maximal strong modules are 1-extendable and if its weighted representative graph is 1-extendable.

Theorem
Deciding whether a graph G is 1*-extendable can be done in time* $2^{O(mw(G))} \cdot n$.

Algorithms

Theorem
Deciding whether a graph G is 1-extendable can be done in time $2^{O(mw(G))} \cdot n$.

Theorem
Deciding whether a graph G with n vertices and independence number α has 1-extendable chromatic number at most k can be done in time $\alpha^{O(mw(G)k)} \cdot n$.

Conclusion and further research

Other results and open questions

Theorem
For any cograph G, $\chi_{1\text{-ext}}(G) \leqslant \log_2(\alpha(G)) + 1$.

- On cographs : algorithm in quasi-polynomial time ($O(n^{\log(n)})$) to compute $\chi_{1\text{-ext}}$, it is possible to do it in polynomial-time ? (open even in complete multipartite graphs)

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THANKS

Partition of cographs

Theorem
For any cograph G, $\chi_{\textit{1-ext}}(G) \leqslant \log_2(\alpha(G)) + 1$.

Idea of the proof.
Find a partition of $V(G) = V_1 \sqcup V_2$ with :

- $G[V_1]$ 1-extendable ;
- $\alpha(G[V_2]) \leqslant \alpha(G)/2$

$\square$

Lemma

Lemma

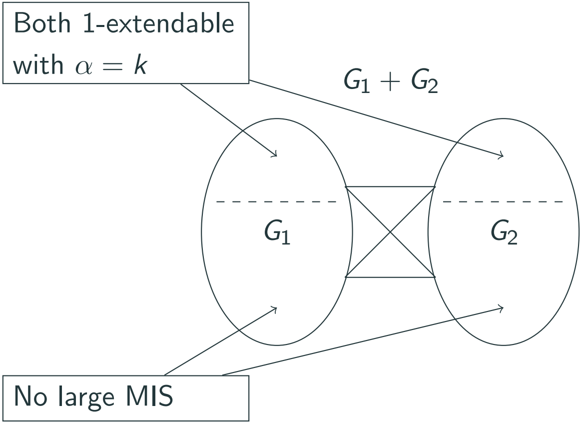
For any cograph $G = (V, E)$ and any $k \in \{0, ..., \alpha(G)\}$, there exists a partition of the vertices into two subsets V_1 and V_2 such that

- $G[V_1]$ *is* 1*-extendable ;*
- $\alpha(G[V_1]) = k$ *;*
- $\alpha(G[V_2]) \leqslant \max(k-1, \alpha(G) - k)$

Proof of the theorem.
Apply the lemma for $k = \frac{\alpha(G)}{2}$, $G[V_1]$ 1-extendable and $\alpha(G[V_2]) \leq \frac{\alpha(G)}{2}$. Continue recursively with $G[V_2]$. □

Proof of the lemma, induction case $G = G_1 + G_2$

Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on (G_1, k) and (G_2, k).



Proof of the lemma, induction case $G = G_1 \cup G_2$

Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on (G_1, k_1) and (G_2, k_2), where

$$k_1 = k\frac{\alpha(G_1)}{\alpha(G_1)+\alpha(G_2)} \qquad k_2 = k\frac{\alpha(G_2)}{\alpha(G_1)+\alpha(G_2)} \qquad k_1 + k_2 = k$$