A structural description of Zykov and Blanche Descartes graphs

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Introduction

χ-boundedness

- $\chi(G)$: chromatic number
- $\omega(G)$: clique number
- For any graph G, $\chi(G) \geqslant \omega(G)$.

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Perfect graphs	
Chordal graphs	
Interval graphs	
Cographs	
P_t-free	
Bounded clique-width	

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Triangle-free graphs with arbitrarily large chromatic number

Theorem
For any $k \geqslant 1$*, there exists a graph* G *with* $\omega(G) = 2$ *and* $\chi(G) \geqslant k$*.*

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Many different proofs :

- Zykov's construction ($\sim$ 1950)
- Blanche Descartes' (or Tutte's) construction ($\sim$ 1950)
- Mycielski's construction (1955)
- Erdős' random graph (1959)
- Burling's construction (1965)
- Shift graphs (Erdős, Hajnal, 1968)
- ...
- Twin-Cuts (Bonnet, Bourneuf, Duron, Geniet, Thomassé, Trotignon, 2024)

Zykov graphs

- Z_1 is a single vertex
- For $k \geqslant 1$, constuct Z_{k+1} as follows :
  - Take the disjoint union of Z_1,.., Z_k
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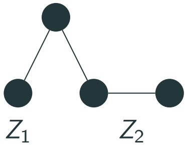

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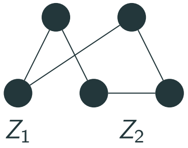

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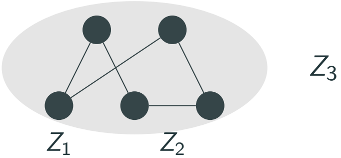

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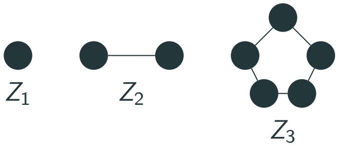

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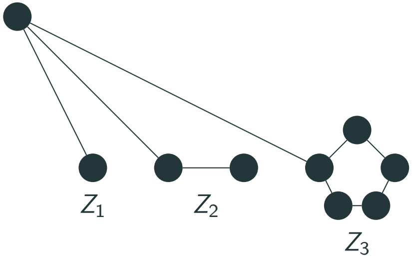

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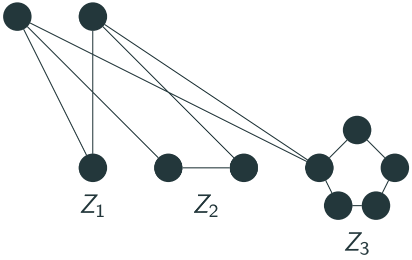

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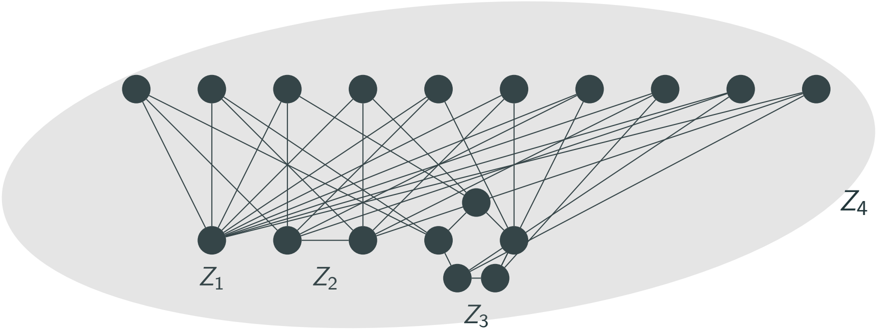

Zykov graphs

Theorem (Zykov, 1950)

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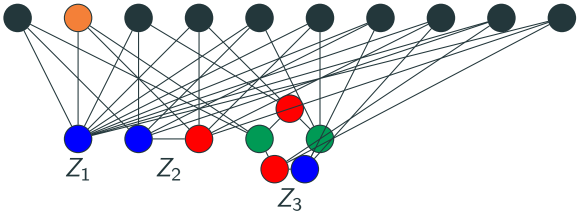

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Objective : Better desciption of Zykov graphs

Structural description of Zykov graphs and applications

Main result

Definition
A *splitting stable set* of a graph G is a subset of vertices $S \subseteq V(G)$ such that

- S is a stable (independent) set
- for every vertex $v \in S$ and connected component C of $G \setminus S$, v has at most one neighbor in C

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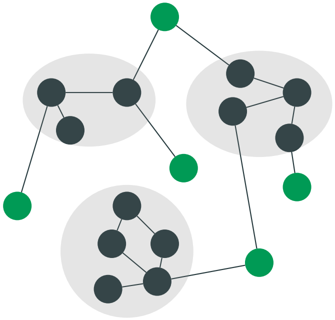

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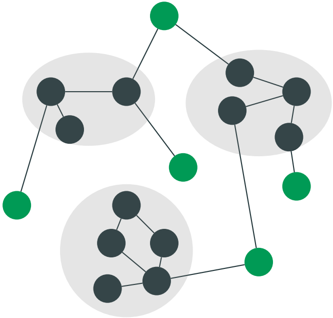

Theorem (M., Thomassé, Trotignon, Watrigant)
A graph G is a Zykov graph if and only if all induced subgraphs of G contain a non-empty splitting stable set.

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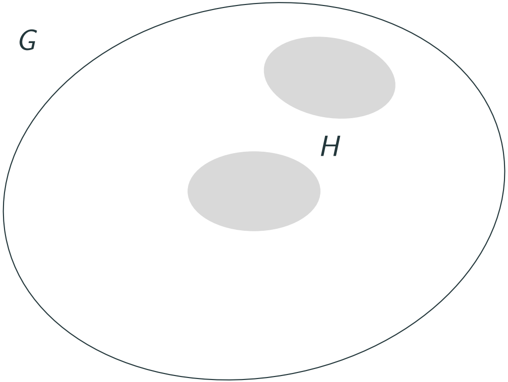

H induced subgraph of G

Goal : finding a non-empty splitting stable set in H $\square$

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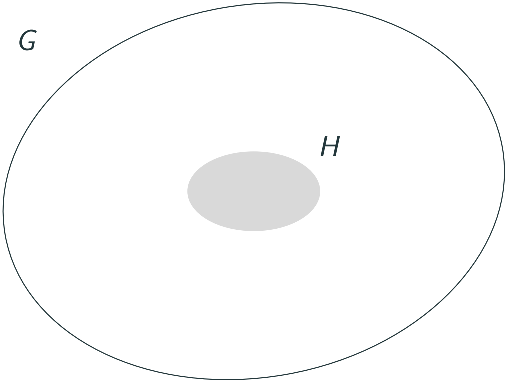

Restrict H to one connected component

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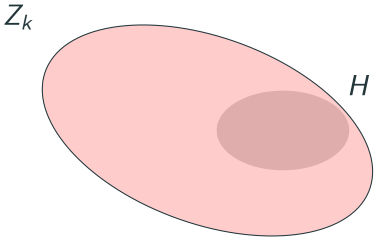

H is a Zykov graph (since G is Zykov)
Consider k minimum such that H is an induced subgraph of Z_k.

□

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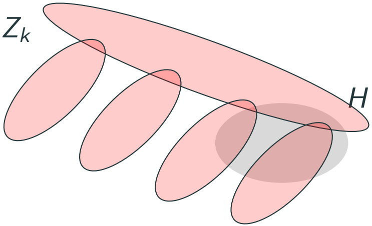

H has a non-empty intersection with the maximum splitting stable set of Z_k.

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Proof idea of $\Leftarrow$.
By induction on the number of vertices.

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Proof idea of $\Leftarrow$.
By induction on the number of vertices.

- Take A a non-empty splitting splitting set of G.
- By induction hypothesis, all ℓ the connected component of $G \setminus A$ are induced subgraph of some Z_k.
- G is an induced subgraph of $Z_{k+\ell}$.

$\square$

Application 1 : Zykov or Non-Zykov ?

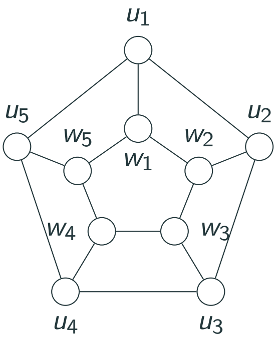

Figure 1: Graph F

Proposition
The graph F is not a Zykov graph.

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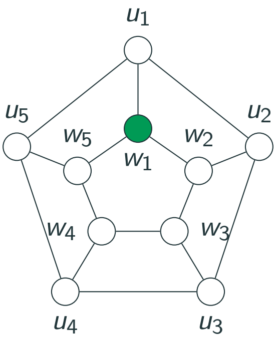

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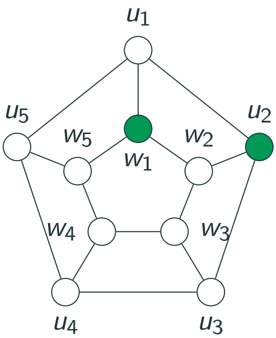

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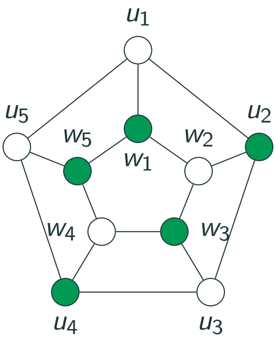

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All twin-cut and Blanche-Descartes graphs are Zykov graphs.

Theorem (M., Thomassé, Trotignon, Watrigant)
There exist non-Zykov graphs of arbitrarily large girth.

Application 2 : Complexity results

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Theorem (M., Thomassé, Trotignon, Watrigant)
MAXIMUM INDEPENDENT SET *and* 3-COLORING *are NP-complete on Zykov graphs.*

The case of Blanche-Descartes graphs

Blanche Descartes graphs

- D_1 is a single vertex ;
- For $k \geqslant 1$, construct D_{k+1} as follows :
    - Take a stable set S of $k(n-1)+1$ vertices where $n = |V(D_k)|$;
    - For each n-tuple of S, add a copy of D_k and a matching between the n-uple and the copy.

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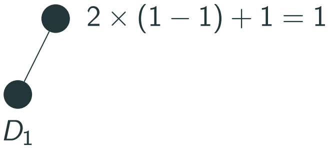

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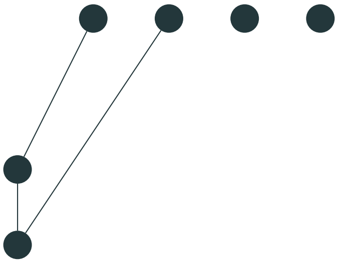

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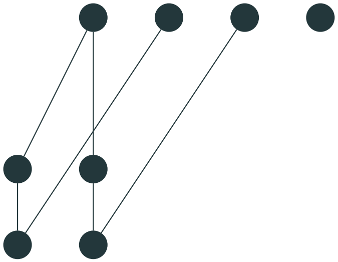

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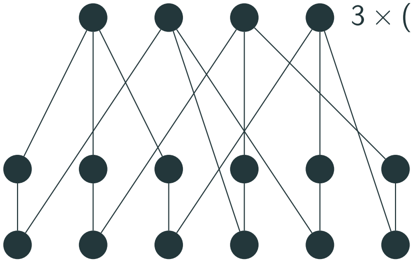

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A *strong splitting stable set* of a graph G is a subset of vertices $S \subseteq V(G)$ such that

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- every vertex has at most one neighbor in each connected component of $G \setminus S$
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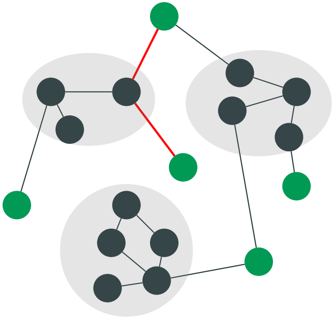

Characterisation of Blanche Descartes graphs

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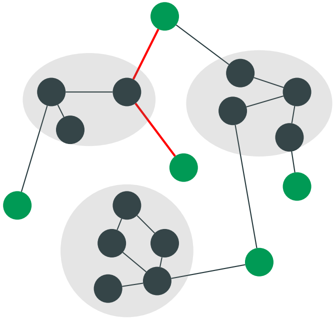

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Theorem (M., Thomassé, Trotignon, Watrigant)
Recognizing Blanche Descartes graphs is NP-complete.

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Question 2 Let $\mathcal{C}$ be a hereditary class of graphs with arbitrarily large chromatic number.

Call $\mathcal{C}$ *minimal* if $\forall H \in \mathcal{C}, \exists c_H$, $\forall G \in \mathcal{C}$, if G is H-induced free, then $\chi(G) \leqslant c_H$.

Know minimal classes :

- complete graphs
- Burling graphs (Abrishami, Briański, Davies, Du, Masaříková, Rzażewski, Walczak '25+)

Does there exist a minimal class contained within the class of Zykov graphs (respectively, Blanche-Descartes graphs)?

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