Beyond recognizing well-covered graphs
WG 2024

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Background

Complexity of recognizing 1-extendable graphs

Chordal graphs

Open problems

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- Chvátal and Slater, and independently Sankaranarayana and Stewart : recognizing well-covered graphs is coNP-complete.

Some historical points : $\mathrm{W_k}$ graphs

- 1975 : Staples introduced the concept of $\mathrm{W_k}$ graphs.

Definition
A graph is $\mathrm{W_k}$ if, and only if, for any k disjoint independent sets $(A_i)_{1 \leqslant i \leqslant k}$, there exist a set $(B_i)_{1 \leqslant i \leqslant k}$ of disjoint MIS such that $A_i \subseteq B_i$ for any $1 \leqslant i \leqslant k$.

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Some historical points : k-extendable

- 1994 : Dean and Zito introduced the concept of k-extendable graphs.

Definition
A graph is k-extendable if any independent set of size k is contained in an MIS.

- (k-extendable for all $k \geqslant 1$) = well-covered
- Notation : a graph is E_s if k-extendable for any $1 \leqslant k \leqslant s$.
- Nested sets:

$$\mathrm{E}_1 \supseteq \mathrm{E}_2 \supseteq \cdots \supseteq \mathrm{W}_1 \supseteq \mathrm{W}_2 \supseteq \cdots$$

- Is it easier to recognize well-covered graphs in 1-extendable graphs ?

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- Is it easier to recognize well-covered graphs in 1-extendable graphs ?
- **Global question :** What are the relative complexities of this hierarchy ?

Results

- Complete overview of the relative complexities :

Question / Input	E_1	E_s	W_1	W_k
Arbitrary	NP-hard	?	coNP-c	coNP-c
E_1	-	?	?	?
E_{s-1}	-	?	?	?
W_1	-	-	-	?
W_{k-1}	-	-	-	?
Chordal	?	?	L	?

Previous result

Our contribution

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Our contribution

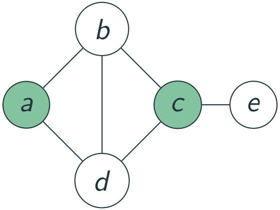
- Complete view on chordal graphs : for instance structural characterization of 1-extendable chordal graphs.

Complexity of recognizing 1-extendable graphs

Practical motivation for 1-extendability

- A graph is **1-extendable** iff every vertex belongs to an MIS.
- In a Wi-Fi network, if an access point does not belong to an MIS of the conflict graph, the throughput is close to zero.

1-extendable



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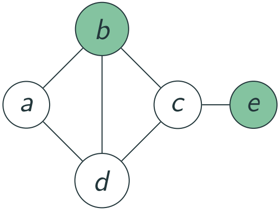

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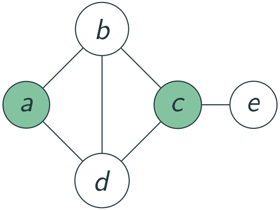

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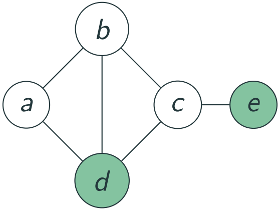

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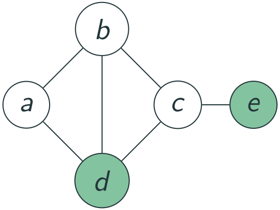

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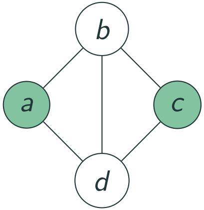
not 1-extendable



Theorem (Bergé, Feghali, Busson and Watrigant, 2023)
Recognizing 1-extendable graphs is NP-hard, even on unit disk graph.

1-extendability is in Θ_2^p

- No polynomial certificate for testing 1-extendability ;
- Θ_2^p : class of problems sovable in polynomial time with a logarithmic call to a $\textsc{Sat}$ oracle;

Theorem
The problem of recognizing 1*-extendable graphs is* Θ_2^p*-complete.*

Proof of membership in Θ_2^p*:*

Algorithm to decide if G is 1-extendable :

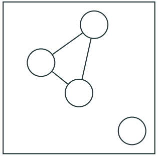
1. Compute $\alpha(G)$ with $log(n)$ call to a SAT oracle.
2. Decide if every vertex belongs to an independent set of size $\alpha(G)$ using a unique SAT oracle.

Reduction from MIS EQUALITY which is Θ_2^p-complete.

MIS EQUALITY
Input : Two n-vertex graphs G and H.
Question : $\alpha(G) = \alpha(H)$?

Let $\pi(G, H)$ be the graph :

G



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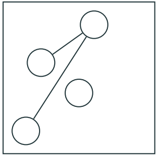

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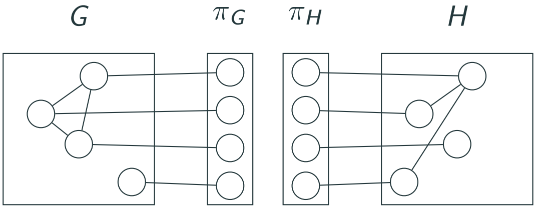

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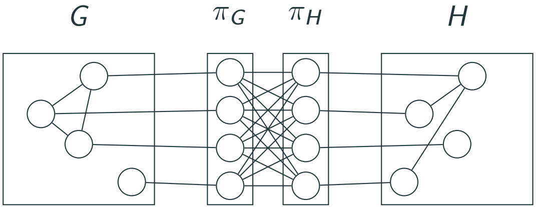

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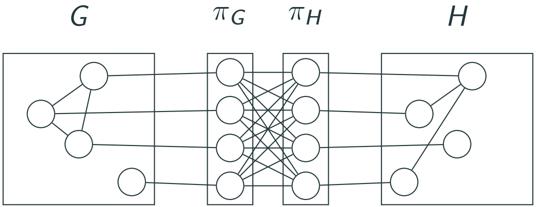


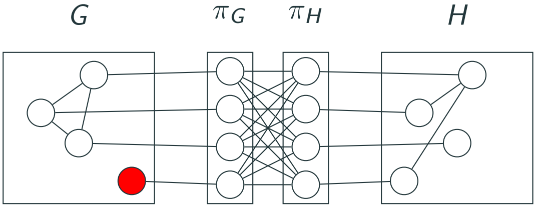
<u>Claim 1</u> : $\alpha(\pi(G, H)) = n + \max(\alpha(G), \alpha(H))$.

<u>Claim 2</u> : $\alpha(G) = \alpha(H)$ if, and only if $\pi(G, H)$ is 1-extendable.

Proof of Claim

<u>Claim 2</u> : $\alpha(G) = \alpha(H)$ if, and only if $\pi(G, H)$ is 1-extendable.

$\boxed{\Rightarrow}$ If $\alpha(G) = \alpha(H)$, let $v \in V(\pi(G, H))$,



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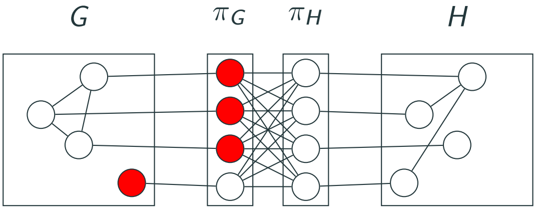

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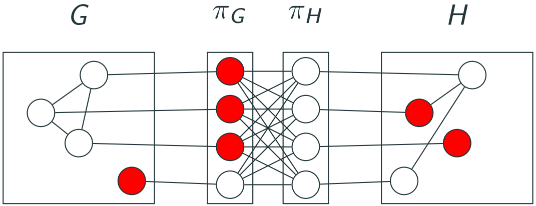

Thus $\pi(G, H)$ is 1-extendable.

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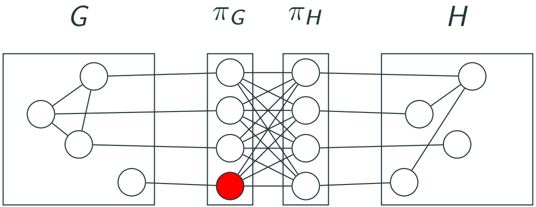
Let $v \in \pi_G$, and S an MIS of $\pi(G, H)$ that contains v.



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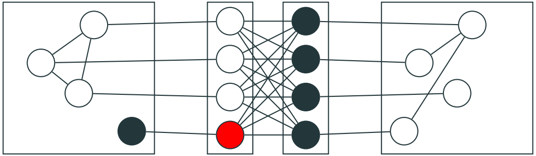

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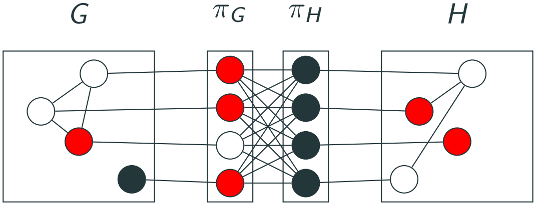


We have $|S \cap V(H)| = \alpha(H)$ and $|S \cap (V(G) \cup \pi(G))| = n$, and thus

$$\alpha(\pi(G, H)) = n + \alpha(H)$$

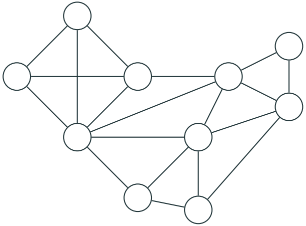
By a symmetric argument, $\alpha(\pi(G, H)) = n + \alpha(G)$, finally $\alpha(G) = \alpha(H)$.

Chordal graphs

Theorem (Prisner, Topp, Vestergaard, 1996)
A chordal graph G is well-covered if, and only if there exists a partition of $V(G)$ into simplices.

Theorem
Let G be a chordal graph. G is 1-extendable iff there is a partition of $V(G)$ into $\alpha(G)$ parts such that each of them is a maximal clique in G.

Example



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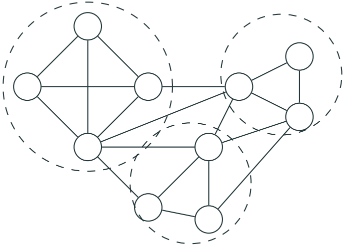

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THANKS