1-extendable partition of graphs

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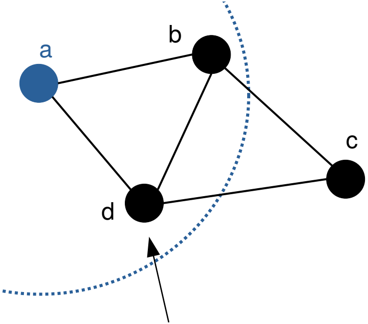
Context

Wi-Fi Network :



$\mathcal{S}(G)$: set of independent sets of G.

p_v : Probability of access of node v.

$$p_v = \frac{\sum_{S \in \mathcal{S}(G), v \in S} \theta^{|S|}}{\sum_{S \in \mathcal{S}(G)} \theta^{|S|}}$$

where $\theta >> 1$ if a "physical parameter" [1].

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Wireless Networks

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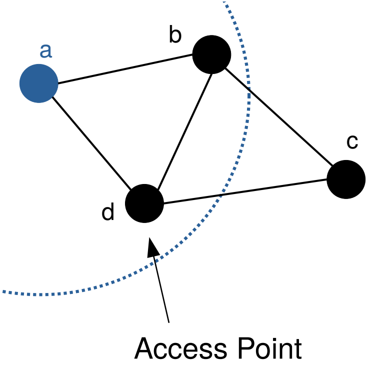

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Example

$$p_a = \frac{\theta^2 + \theta}{\theta^2 + 4\theta} \qquad p_b = \frac{\theta}{\theta^2 + 4\theta}$$

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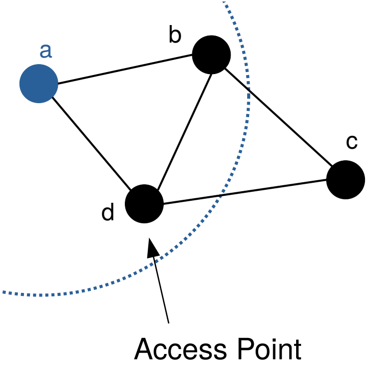

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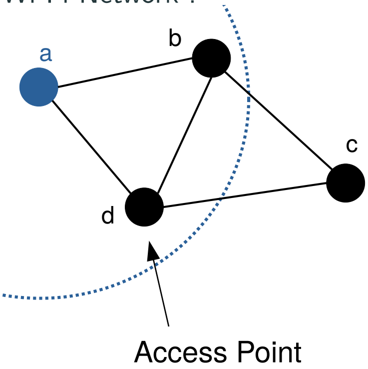

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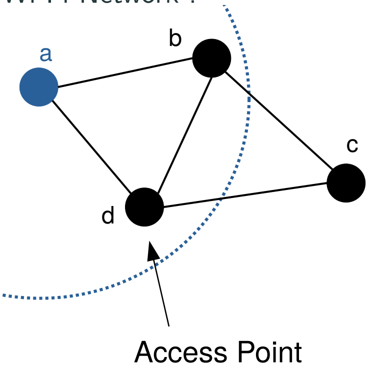

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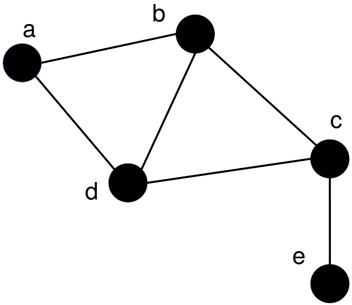
BAD

What’s the difference between a good and a bad network ?

Definition
A graph $G = (V, E)$ is 1-**extendable** if any vertex belongs to an MIS.

Example



A 1-extendable graph.

If $G = (V, E)$ is 1-extendable, for any $v \in V$, $\tilde{p_v} > 0 \longrightarrow$ Minimal fairness, Good

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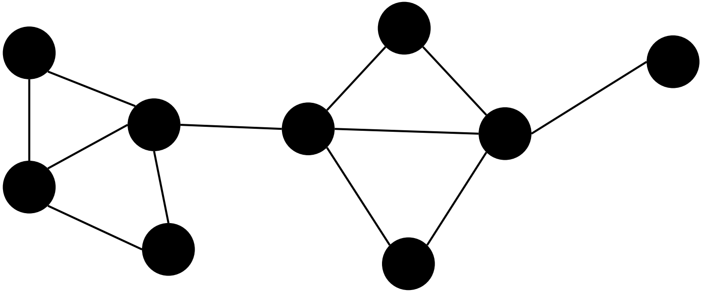

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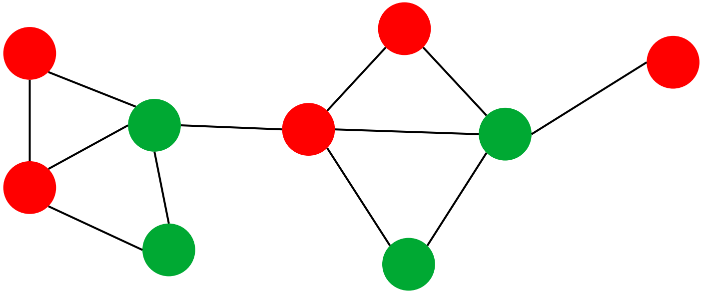

1-extendable and well-covered graphs

Definition (Berge 78)
A graph G is 1-*extendable* if each vertex belongs to an MIS.

Theorem (Bergé, Busson, Feghali, Watrigant 2022)
Testing 1*-extendability is NP-hard, even on unit disk graph.*

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1-Extendable Partition

Input : A graph $G = (V, E)$ and an integer k.

Question : Can we find a partition $V = V_1 \cup ... \cup V_k$ such that $G[V_i]$ is 1-extendable for any $1 \leqslant i \leqslant k$?

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Theorem
1-EXTENDABLE k-PARTITION *is NP-hard for any fixed* k*.*

Notations

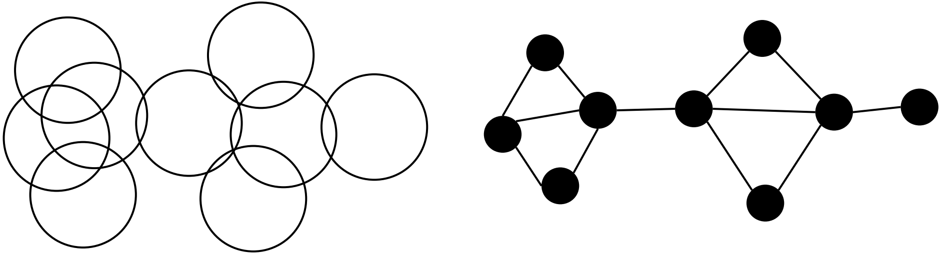
$\chi_{1\text{-ext}}(G)$: smallest integer k such that G has a partition into k 1-extendable induced subgraphs.

Structural results

Unit disk graphs : Model for wireless networks

Definition
A graph $G = (V, E)$ is a *unit disk graph* if there exists a mapping $f : V \to \mathbb{R}^2$ such that $uv \in E$ if, and only if, $\|f(u) - f(v)\| \leqslant 1$.

Theorem
For any unit disk graph G, $\chi_{1\text{-}ext}(G) \leqslant 7$.



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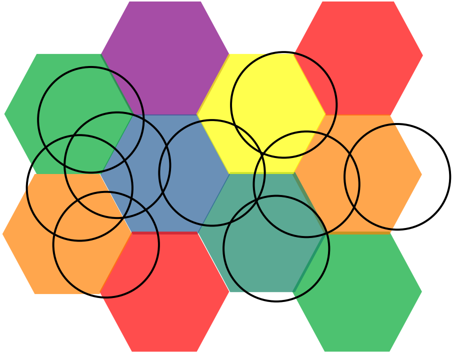
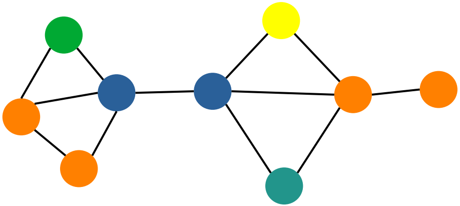




Theorem
For any graph G with n vertices, $\chi_{\text{1-ext}}(G) \leqslant 2\sqrt{n}$.

Lemma
For any graph G, $\chi_{\text{1-ext}}(G) \leqslant \alpha(G)$.

Proof.
If $\alpha(G) = 1$, then G is a clique and $\chi_{\text{1-ext}}(G) = 1$. If $\alpha(G) > 1$, let S be the set of vertices of G that are in an MIS. Notice that :

- $G[S]$ is 1-extendable ;
- $\alpha(G - S) \leqslant \alpha(G) - 1$.

By induction hypothesis, $\chi_{\text{1-ext}}(G - S) \leqslant \alpha(G) - 1$ and use one color for S. □

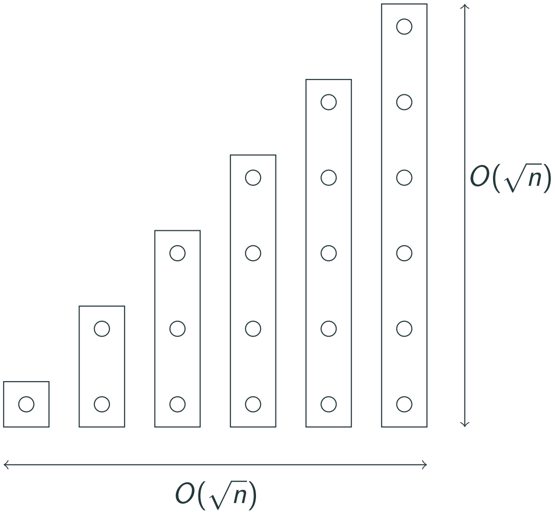
Theorem
For any graph G with n vertices, $\chi_{1\text{-}ext}(G) \leqslant 2\sqrt{n}$.

Proof.
If $\alpha(G) > \sqrt{n}$, extract an MIS S, use one color for S and recursively color $G - S$.

If $\alpha(G) \leqslant \sqrt{n}$, use $\alpha(G)$ colors with the previous lemma $\square$

Lower bound

Is $O(\sqrt{n})$ optimal ? Consider the following complete multipartite graph G_n :



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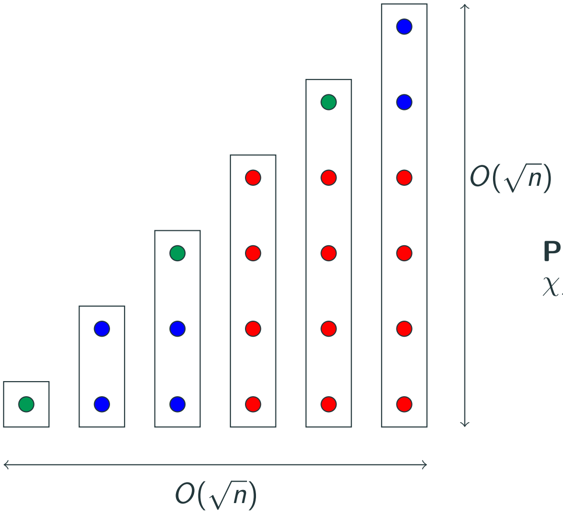


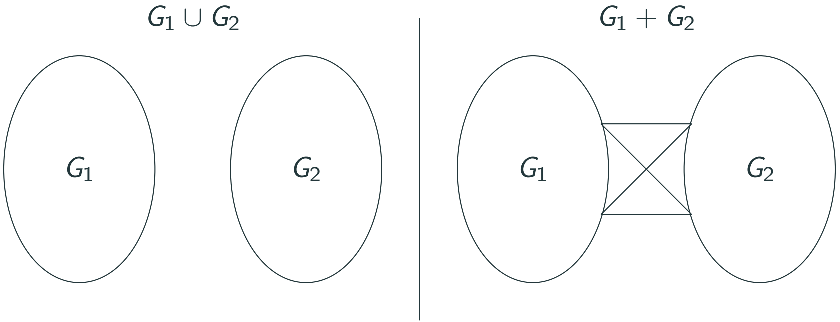
Proposition
$\chi_{\textit{1-ext}}(G_n) = \Theta(\log(n))$.

Cographs

Cographs

A *cograph* is defined recursively as follows :

- A graph with a single vertex is a cograph.
- If G_1 and G_2 are both cographs, then $G_1 \cup G_2$ and $G_1 + G_2$ are cographs.



1-extendable cographs

Proposition
Given two graphs G_1 and G_2,

1. $G_1 \cup G_2$ *is* 1*-extendable iff both* G_1 *and* G_2 *are* 1*-extendable ;*
2. $G_1 + G_2$ *is 1-extentable iff both* G_1 *and* G_2 *are 1-extendable and* $\alpha(G_1) = \alpha(G_2)$*.*

Partition of cographs

Theorem
For any cograph G, $\chi_{\textit{1-ext}}(G) \leqslant \log_2(\alpha(G)) + 1$.

Idea of the proof.
Find a partition of $V(G) = V_1 \sqcup V_2$ with :

- $G[V_1]$ 1-extendable ;
- $\alpha(G[V_2]) \leqslant \alpha(G)/2$

$\square$

Lemma

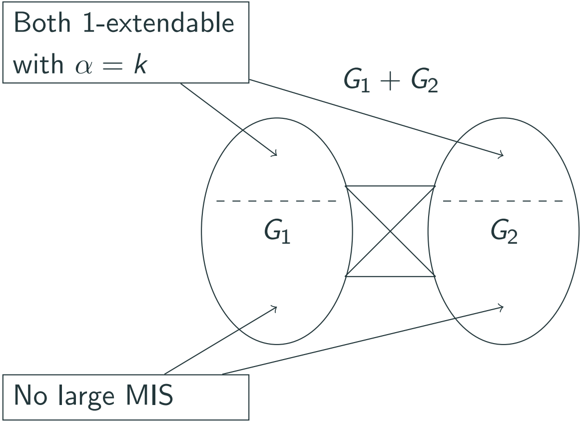
For any cograph $G = (V, E)$ and any $k \in \{0, ..., \alpha(G)\}$, there exists a partition of the vertices into two subsets V_1 and V_2 such that

- $G[V_1]$ *is 1-extendable ;*
- $\alpha(G[V_1]) = k$ *;*
- $\alpha(G[V_2]) \leqslant \max(k-1, \alpha(G) - k)$

Proof of the theorem.
Apply the lemma for $k = \frac{\alpha(G)}{2}$, $G[V_1]$ 1-extendable and $\alpha(G[V_2]) \leq \frac{\alpha(G)}{2}$. Continue recursively with $G[V_2]$. □

Proof of the lemma, induction case $G = G_1 + G_2$

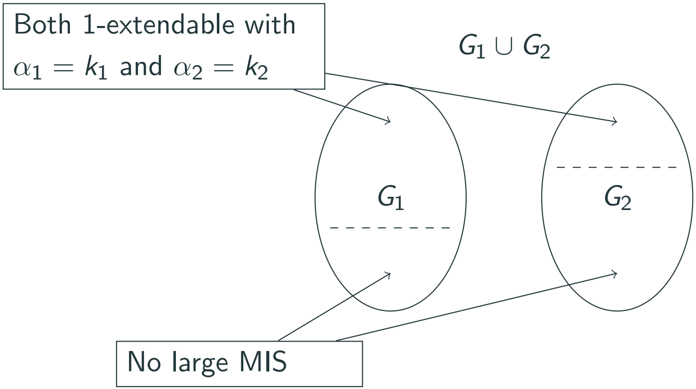
Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on (G_1, k) and (G_2, k).



Proof of the lemma, induction case $G = G_1 \cup G_2$

Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on (G_1, k_1) and (G_2, k_2), where

$$k_1 = k\frac{\alpha(G_1)}{\alpha(G_1)+\alpha(G_2)} \qquad k_2 = k\frac{\alpha(G_2)}{\alpha(G_1)+\alpha(G_2)} \qquad k_1 + k_2 = k$$



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Corollary
The 1-Extendable Partition *problem can be solved in time $\mathcal{O}(n^{poly(\log(n))})$ (quasi-polynomial), and thus is not NP-Hard on cographs, unless the ETH is false.*

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THANKS