Subcoloring of (unit) disk graphs

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Context

Subcoloring of unit disk graphs

Complexity results

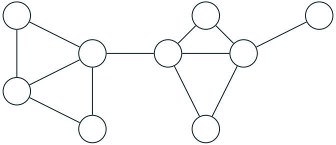
Subcoloring of disks graphs

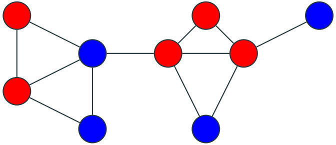
Context

Definitions :

- A *k-subcoloring* of a graph G is a partition of $V(G)$ into $V_1, ..., V_k$ such that $G[V_i]$ is a disjoint union of cliques for all $1 \leqslant i \leqslant k$.

Example





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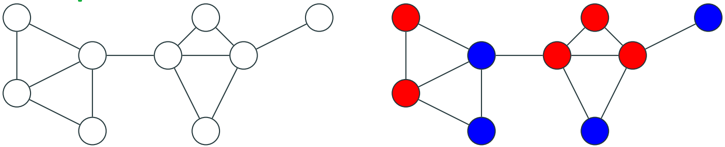

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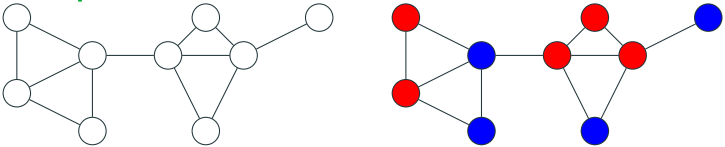

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- $\chi_s(G)$: minimum k such that G has a k-subcoloring.
- k-Subcoloring : Given a graph G, does $\chi_s(G) \leqslant k$ hold ?

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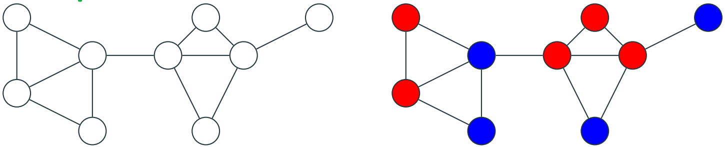

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Theorem (Stacho, '2008)
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- What is the complexity of k-Subcoloring on interval graphs when k **is part of the input**?
- What is the complexity of k-Subcoloring on co-comparability graphs? **NP-complete when** $k \geqslant 3$

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Subcoloring of unit disk graphs

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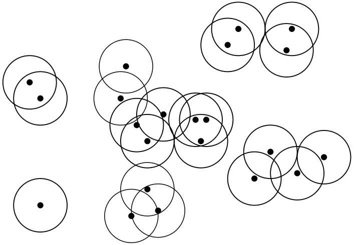
A unit disk graph is the intersection graph of unit disks on the plane.



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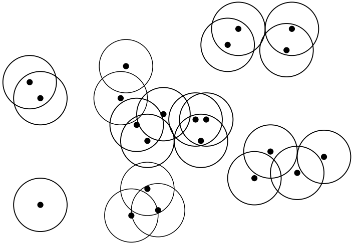

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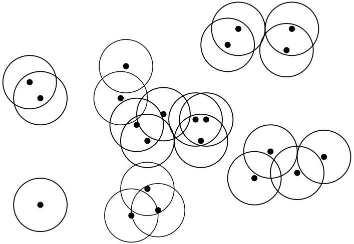

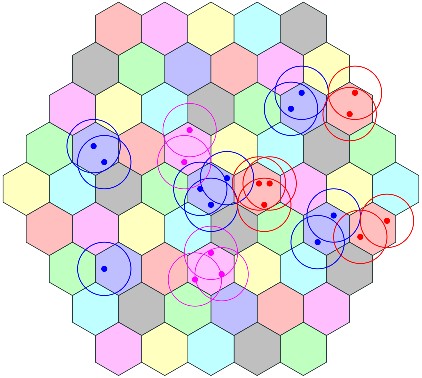

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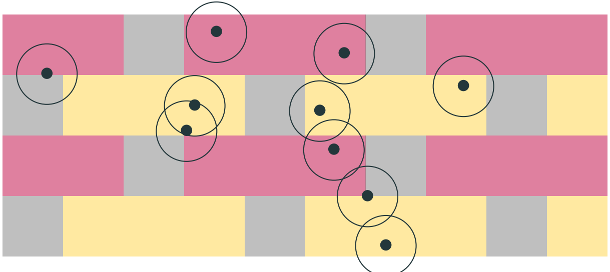

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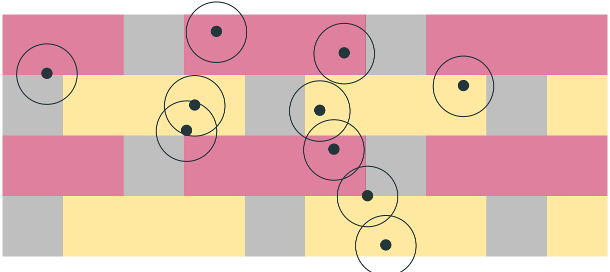

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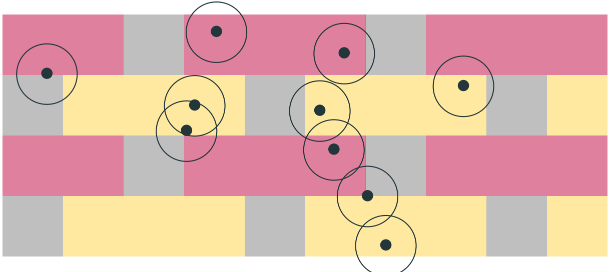
3. If $\chi_s(G) = 2$, we can return a 6-subcoloring, otherwise return a 7-subcoloring.



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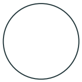
Subcoloring of disks graphs

How large can be $\chi_s(G)$ for disk graphs ? Construction of Broersma et al.



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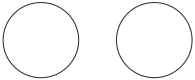

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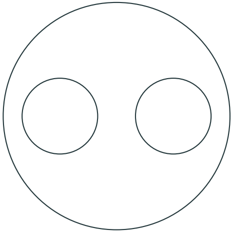

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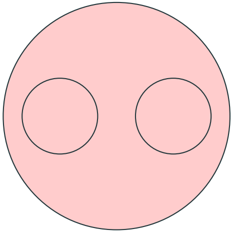

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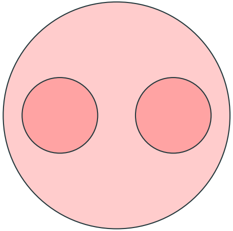

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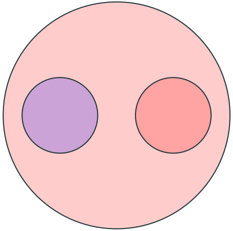

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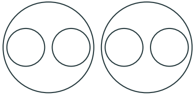

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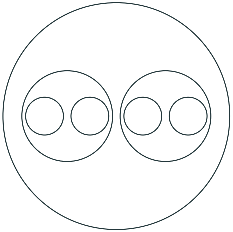

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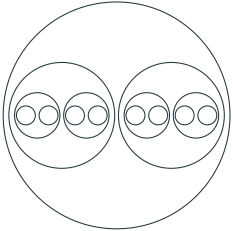

$\chi_s(G) = 4$

Theorem
For every $n \geqslant 1$, there exists a n-vertex disk graph such that $\chi_s(G) \geqslant \log_2(n)$.

Polylogarithmic upper bound

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For any n-vertex disk graph with $n \geqslant 1$, $\chi_s(G) = O(\log^3(n))$.

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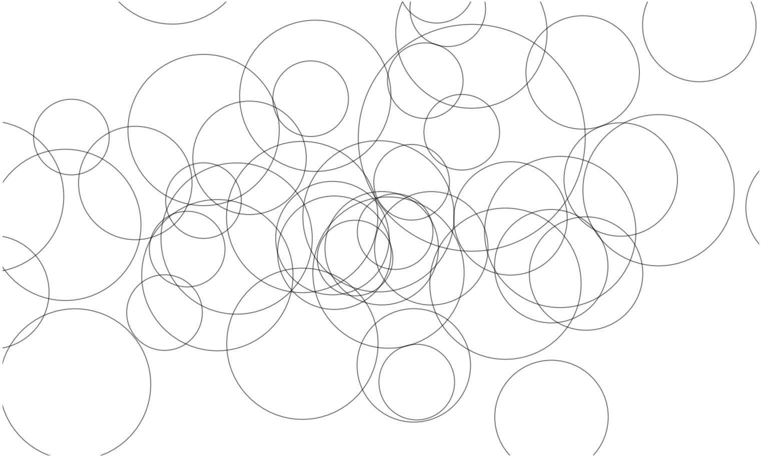
Proof using Divide-And-Conquer.

Decomposition of disk graphs



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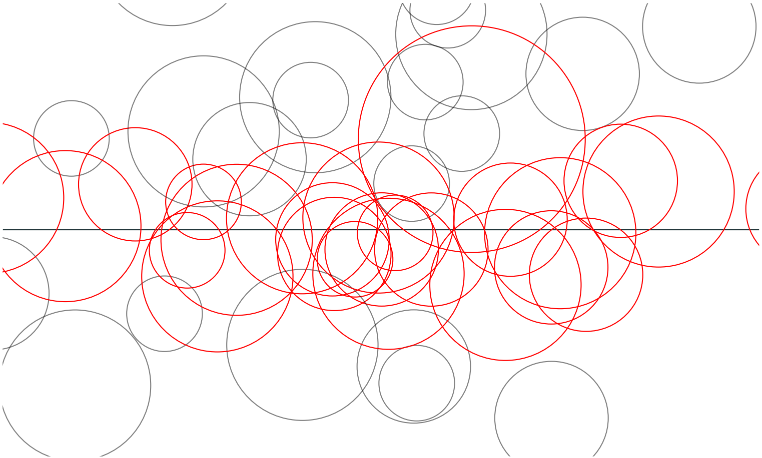

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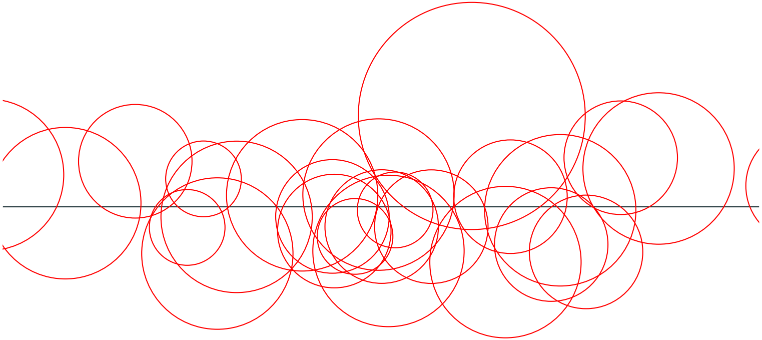

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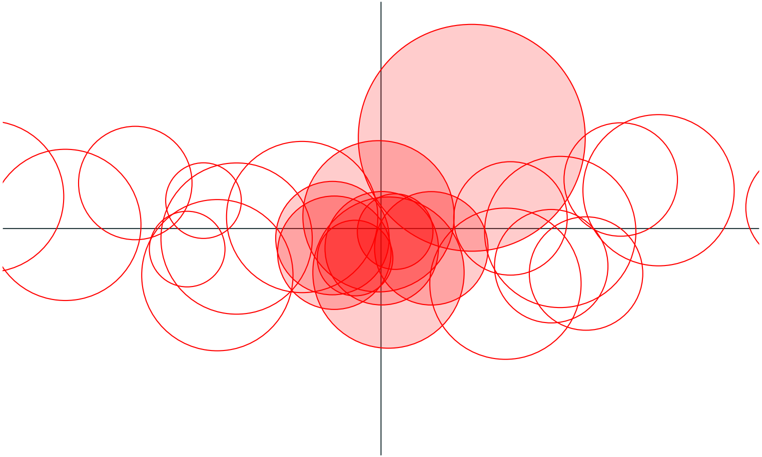

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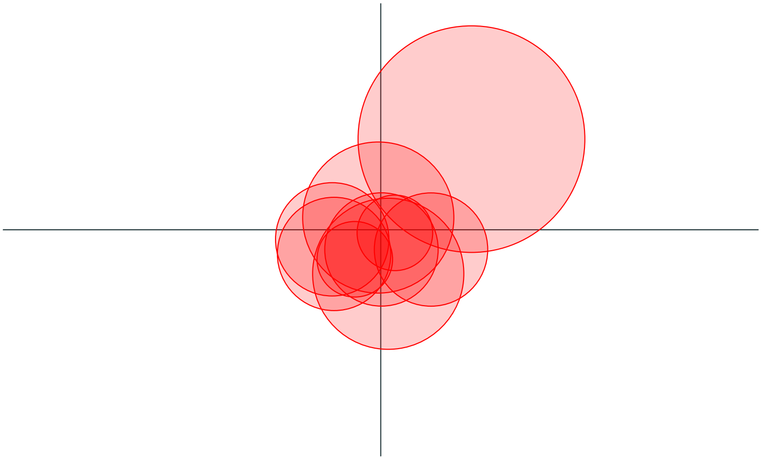
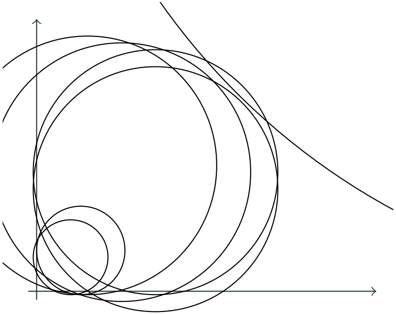


Definition
A disk graph G is a Δ-disk graph if it has a disk representation where :

- each disk center has positive coordinate ;
- no disk cross the origin ;
- each disk intersects both axis.



Lemma
Each disk graph can be partionned into $O(\log^2(n))$ disjoint union of Δ-disk graphs.

Lemma
For any n-vertex Δ-disk graphs G, $\chi_s(G) = O(\log_2(n))$.

Theorem
For any n-vertex disk graph with $n \geqslant 1$, $\chi_s(G) = O(\log^3(n))$.

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THANKS