Subexponential and Parameterized Mixing Time of Glauber Dynamics on Independent Sets

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Introduction

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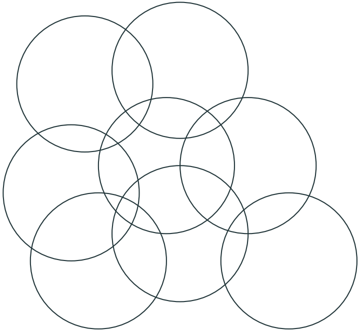
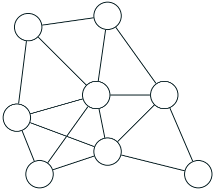
Independent set

Example: Unit disk graph





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An **independent set** in a graph G is a set of vertices $I \subseteq V(G)$ pairwise non-adjacent.

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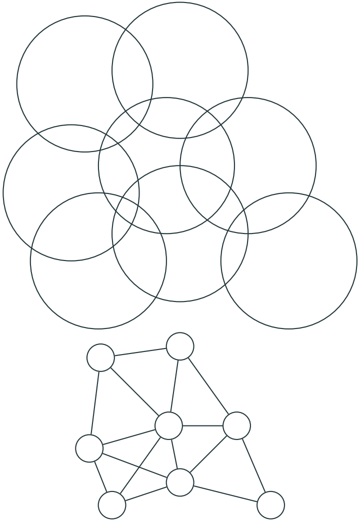

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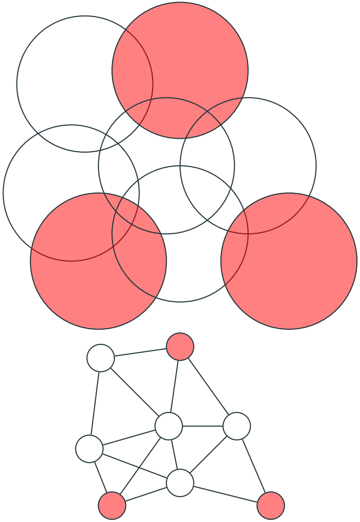

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Theorem (Clark et al. '90)
Finding a maximum independent set is NP-complete, even on unit disk graphs.

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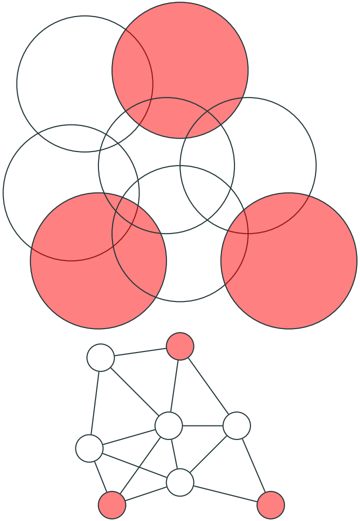

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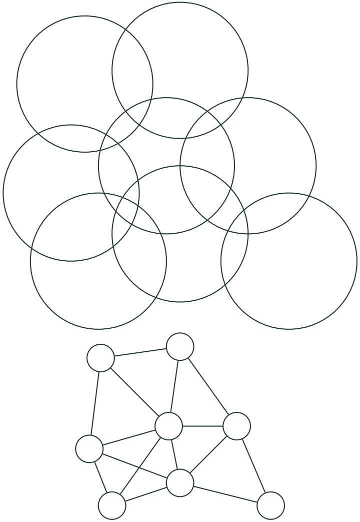

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Theorem (De Berg et al, '18)
Finding a maximum independent set in unit disk graphs can be solved in $2^{O(\sqrt{n})}$ time.

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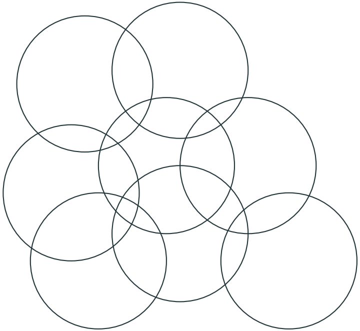

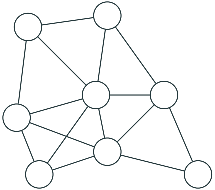

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Theorem (De Berg et al, '18)
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Any unit disk graph has a balanced separator consisting of $O(\sqrt{n})$ cliques.

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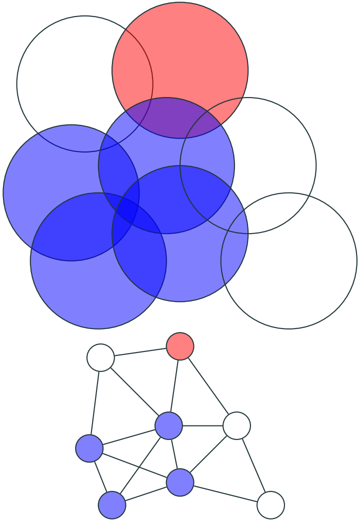

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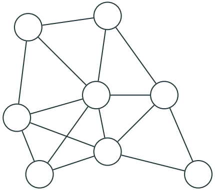

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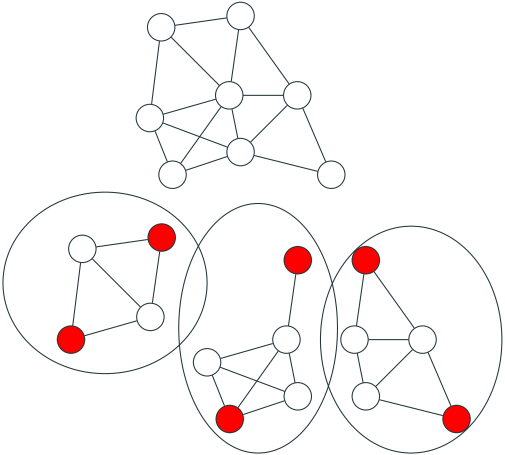

Parameterized Algorithms : Tree independence number

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In the example, tree-$\alpha(G) = 2$ and $\text{tw}(G) = 4$.

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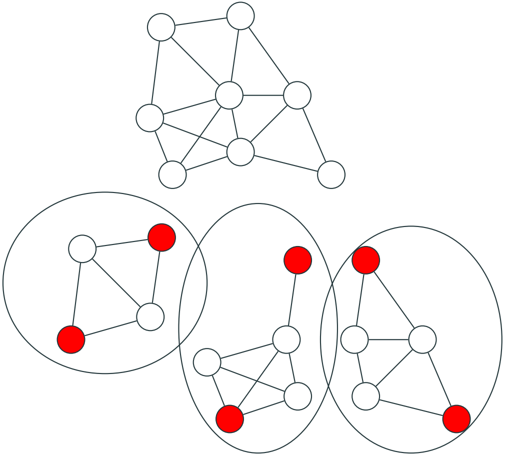

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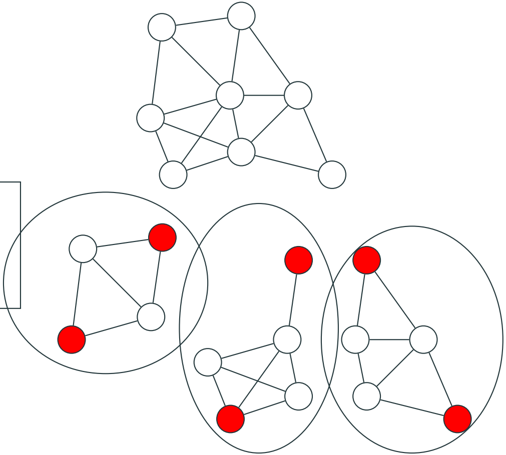

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Corollary
Finding a maximum independent set in unit disk graphs can be done in $n^{O(\sqrt{n})}$ time.

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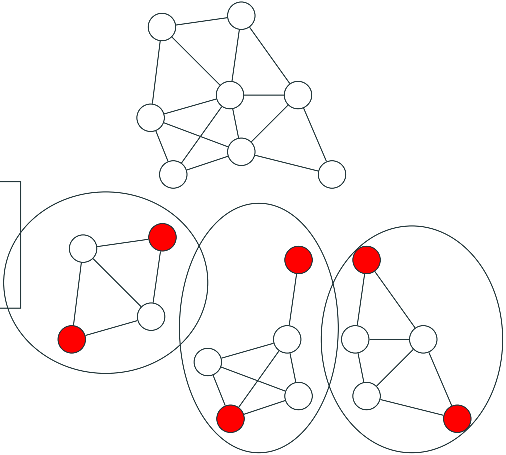

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Hardcore Model and Glauber Dynamics

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where Z is the number of independent sets of G.

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If t is large enough, the probability that the result is an independent set is I is exactly $\pi_G(I)$.

Question: Why people are interested about this distribution ?

- Stationary behavior of particles systems (like Ising model, Potts model, ...) ;
- Stationary behavior of saturated Wi-Fi networks as well.

Mixing Time

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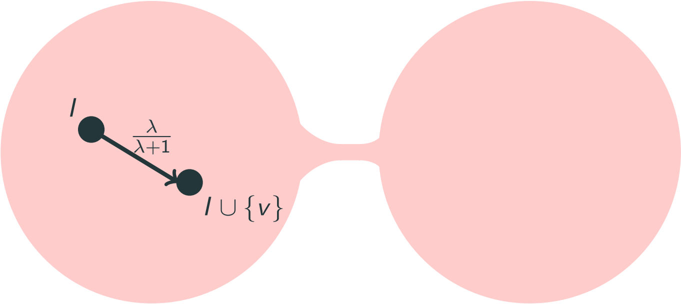
- The **mixing time** $\tau_{G,\lambda}$ is the number of iterations needed to be close enough to π_G.
- Up to some polynomial, $\tau_{G,\lambda}$ is the **congestion** of the reconfiguration graph of independent sets :



Small weighted cut (reconfiguration graph) = large mixing time

Upper bound on mixing time

Parameterized mixing times

Theorem (Dyer, Greenhill, Müller, '19)
For any graph G and $\lambda \geqslant 1$,

- $\tau_{G,\lambda} = (n\lambda)^{O(\textit{path-}\alpha(G))}$;
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The result of Dyer, Greenhill and Müller does not capture the width of the tree decomposition.

Parameterized mixing times

Theorem (M. '25)
For any graph G and $\lambda > 0$,

- $\tau_{G,\lambda} = (b\lambda)^{O(\textit{path-}\alpha(G))} \cdot n^{O(1)}$;
- $\tau_{G,\lambda} = (n\lambda)^{O(\textit{tree-}\alpha(G))\ln(b)}$;

where b is the minimum width of a path (resp. tree) decomposition of independence number path-$\alpha(G)$ (resp. tree-$\alpha(G)$).

- Capture also the width of the tree decomposition ;

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- Capture also the width of the tree decomposition ;
- Brings together the result of Dyer, Greenhill, Müller, '19 with a result of Bezáková and Sun '20 on **chordal graphs** and Eppstein and Frishberg '20 on **bounded treewidth**.

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- Match the complexity of the best deterministic algorithm.
- Generalize to other geometric graphs (dimension $d \geqslant 3$ for instance).

Lower bound

Open question

Open question of Dyer, Greenhill and Müller '19 : is $(n\lambda)^{O(\text{tree-}\alpha(G))\ln(n)}$ tight for bounded tree independence number ?

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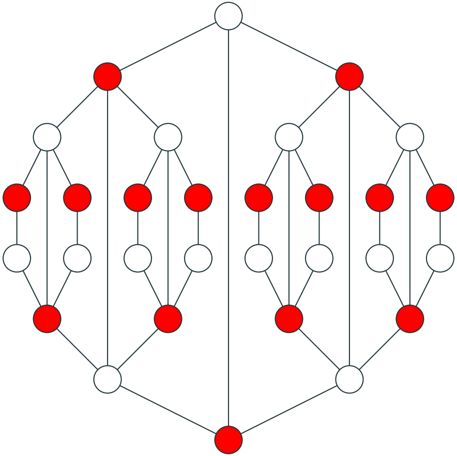
Theorem (M. '25)
For any $t \geqslant 1$, there exist infinitely many graphs G such that tree-$\alpha(G) = 2t$ and for any $\lambda \geqslant 1$,

$$\tau_{G,\lambda} \geqslant (\lambda n)^{ct\ln(n)}$$

where $n = |V(G)|$ and c is a constant.

Sketch of the proof

Consider the following graph G_h.



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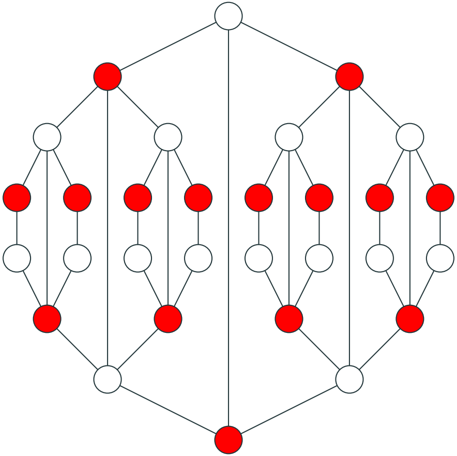

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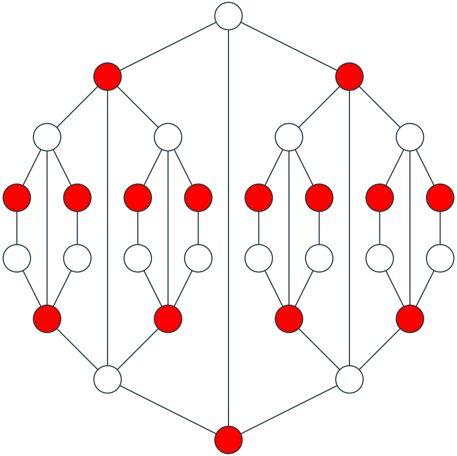

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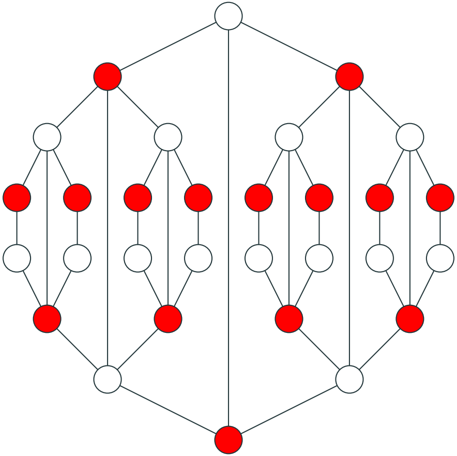

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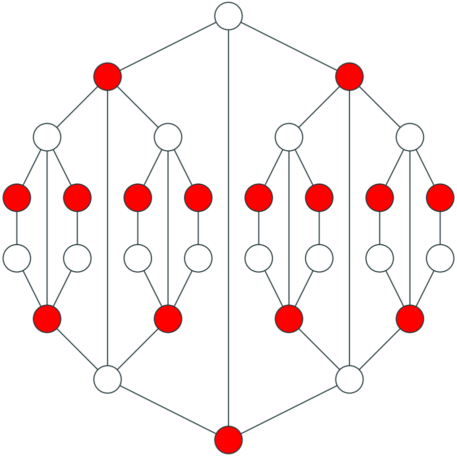

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Blow up each vertex by the disjoint union of t cliques of size n.

Conclusion

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THANKS