Subcoloring of (unit) disk graphs

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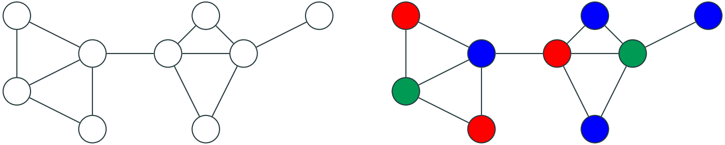
Context

Definitions :

- A *k-coloring* of a graph G is a partition of $V(G)$ into $V_1, \ldots, V_k$ such that $G[V_i]$ is an independent set for all $1 \leqslant i \leqslant k$.

Example



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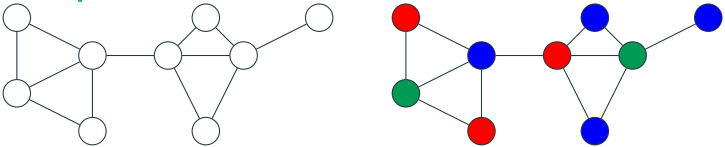

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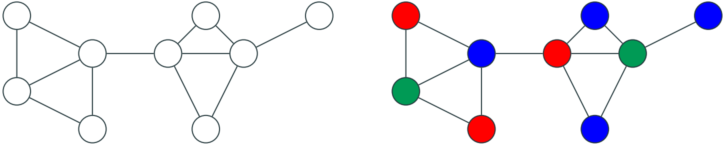

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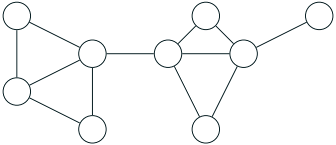

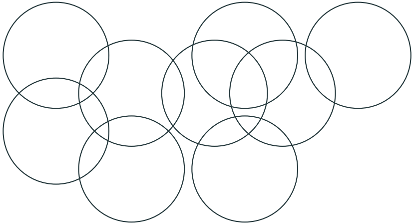

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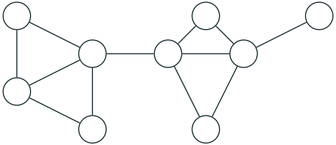

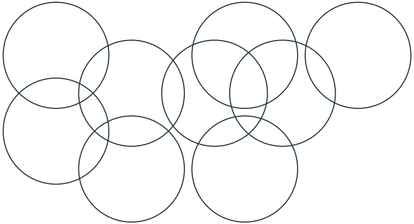

Theorem (Gräf, Stumpf and Weißenfels, '98)
3-COLORING *is NP-complete on unit disk graphs and admits a 3-approximation algorithm.*

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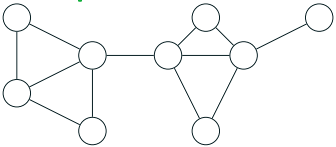

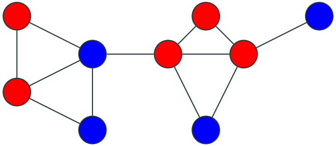

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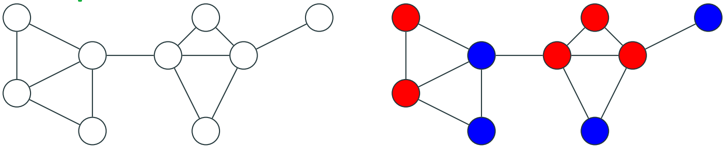

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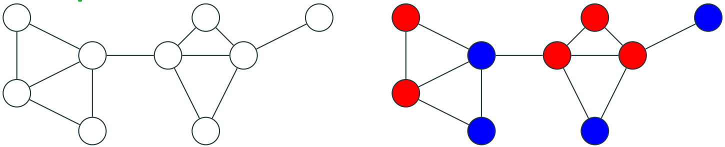

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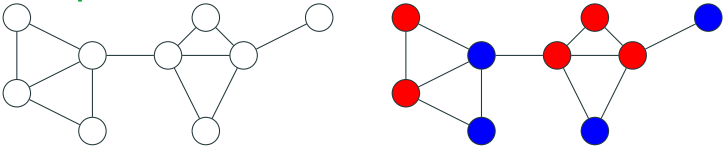

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Theorem (Stacho, '2008)
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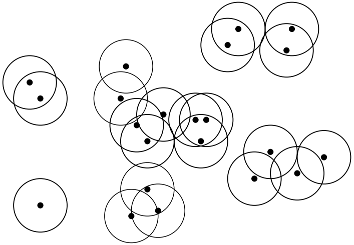
Subcoloring of unit disk graphs

Observation
For any unit disk graph G, $\chi_s(G) \leqslant 7$.



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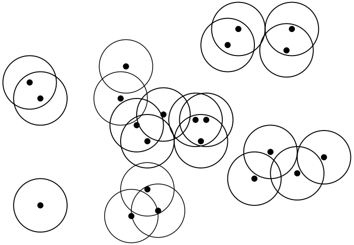

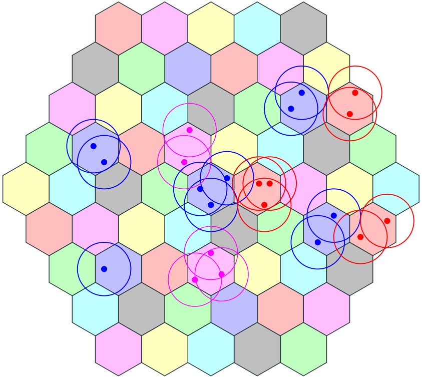

Remark
Open whether 7 is the optimal, best lower bound is 4.

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There exists a 3.5*-approximation algorithm for* Subcoloring *on unit disk graphs.*

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Check if $\chi_s(G) = 1$. If $\chi_s(G) \geqslant 2$, return a 7-subcoloring. □

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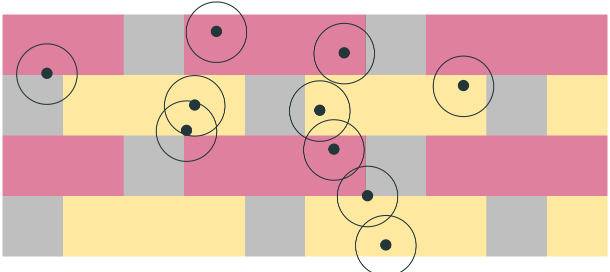

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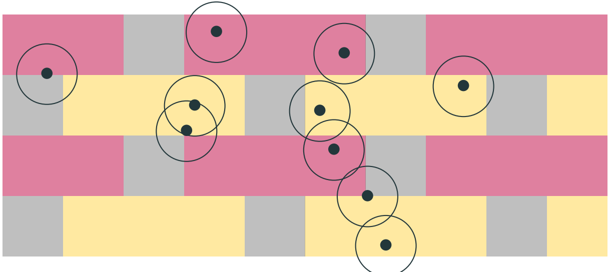

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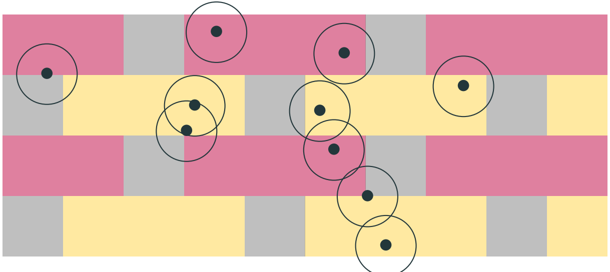
3. If $\chi_s(G) = 2$, we can return a 6-subcoloring, otherwise return a 7-subcoloring.



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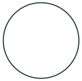
Subcoloring of disks graphs

How large can be $\chi_s(G)$ for disk graphs ? Construction of Broersma et al.



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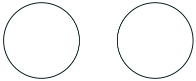

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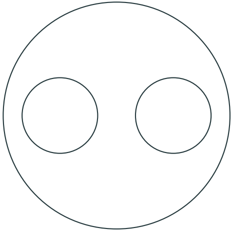

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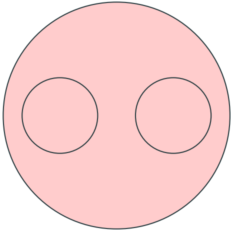

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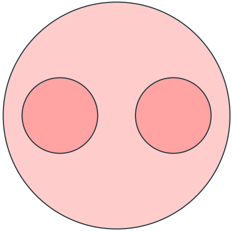

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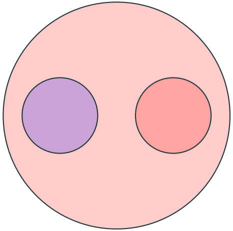

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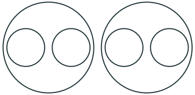

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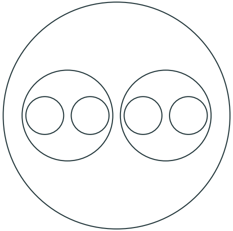

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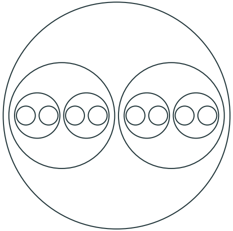

$\chi_s(G) = 4$

Theorem (Broersma, Fomin, Nešetřil, and Woeginger, '2002)
For every $n \geqslant 1$, there exists a n-vertex disk graph such that
$\chi_s(G) \geqslant \log_2(n)$.

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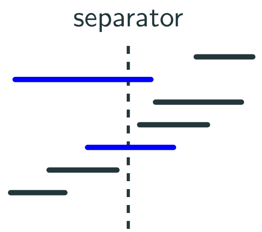

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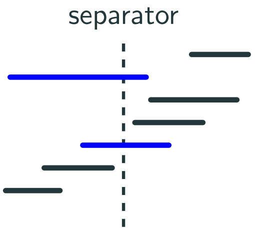

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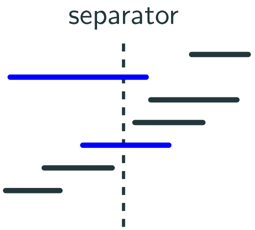
Conclude by induction. □



Polylogarithmic upper bound

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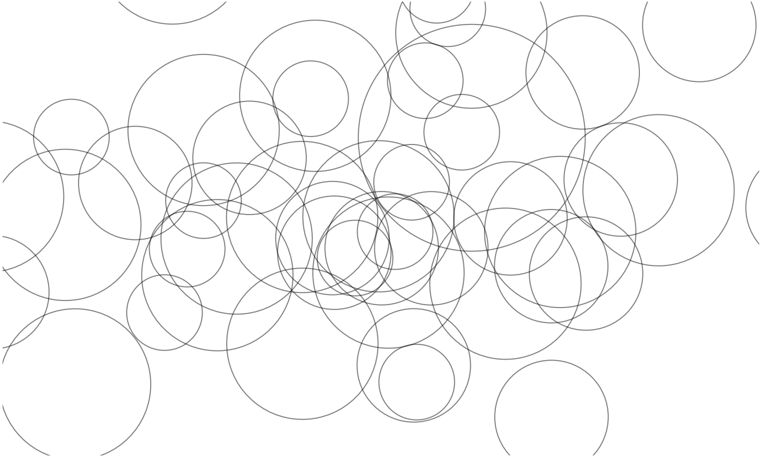
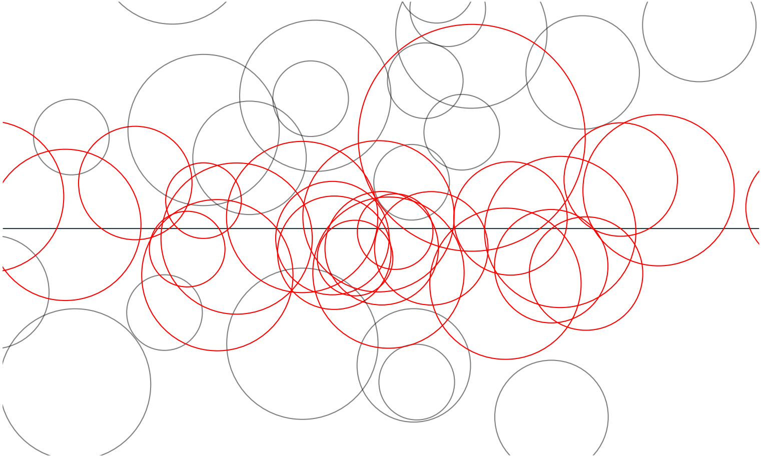
Proof using Divide-And-Conquer.

Decomposition of disk graphs





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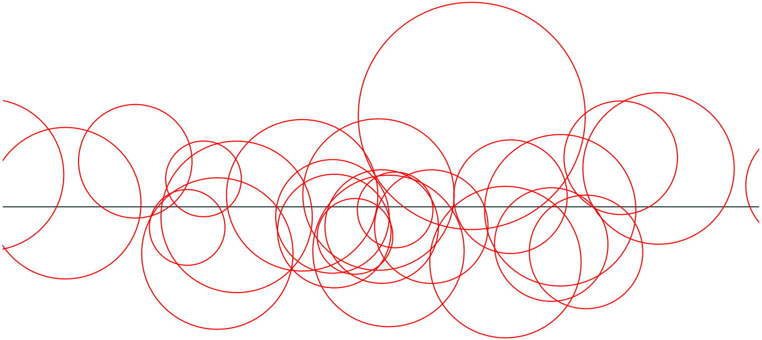

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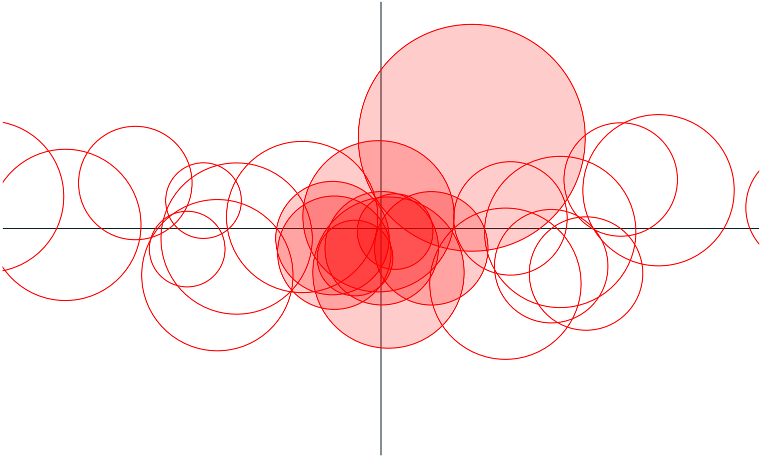

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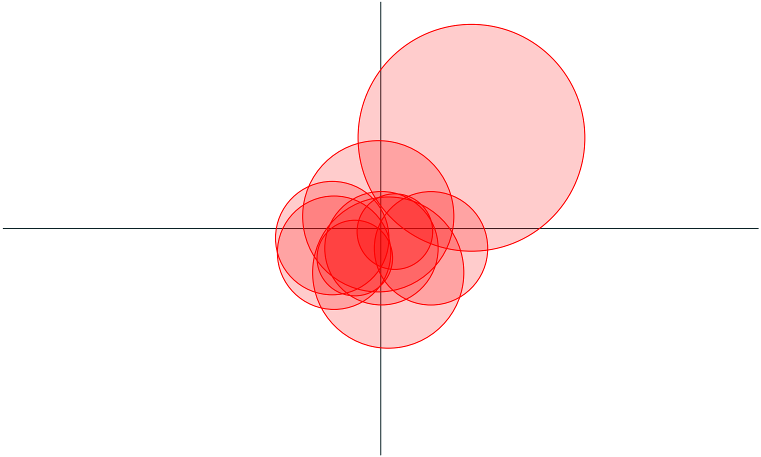
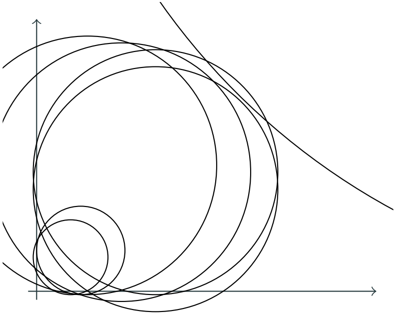


Definition
A disk graph G is a Δ-disk graph if it has a disk representation where :

- each disk center has positive coordinate ;
- no disk cross the origin ;
- each disk intersects both axis.



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There exists a $O(\log^2(n))$*-approximation algorithm for* SUBCOLORING *on* n*-vertex disk graphs.*

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THANKS