The dynamics of conformal Hamiltonian flows on locally symplectic manifolds (1)

Simon Allais & Marie-Claude Arnaud

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Motivation and setting

- On a symplectic manifold, it happens that some dynamics don't preserve the symplectic form but alter it up to some conformal factor (example: damped pendulum).
- Place this in a broader context: let M be a closed manifold. We define on the set $\Lambda^2(M)$ of 2-forms on M an equivalence relation $\sim$ by $\omega_1 \sim \omega_2$ if $\omega_1 = a\omega_2$ for some positive function a.
- We endow M with an equivalence class $\mathcal{C}$ for $\sim$ such that for every $x \in M$, there exists a neighbourhood U of x and $\omega \in \mathcal{C}$ such that $\omega_{|U}$ is symplectic.
- Then a dynamics $f : M \circlearrowleft$ is conformally symplectic (conformal) if it preserves this equivalence class. In other words, for every $\omega \in \mathcal{C}$, there exists a non vanishing function $a : M \to \mathbb{R}_+^*$ such that $f^*\omega = a.\omega$.

A result of Libermann

In 1958, Libermann proved the following result.

If (M, ω) *is a connected symplectic manifold such that* $\dim M \geqslant 4$*, if* $a : M \to \mathbb{R}$ *is such that* $a.\omega$ *is also symplectic, then* a *is constant.*

If now M is a connected manifold such that $\dim M \geqslant 4$ endowed with an equivalence class $\mathcal{C}$ as described before, if $\omega \in \mathcal{C}$, we know that locally there exists $a : U \to \mathbb{R}_+^*$ such that $a\omega$ is closed. Because of Libermann result, $\log a$ is unique up to an additive constant and so $\eta = -d \log a$ defines a global closed 1-form such that

$$0 = d(a\omega) = a(d\omega - \eta \wedge \omega).$$

Locally symplectic manifolds

Definition

(M, ω, η) is a locally symplectic manifold (LSM) if

- η is a closed 1-form;
- ω is a non-degenerate 2-form such that $d_\eta \omega = 0$ where

$$d_\eta \omega := d\omega - \eta \wedge \omega$$

  is the de Rham-Lichnerowicz derivative of ω.

The 1-form η is called the Lee form.

- If $\omega' = e^u \omega \in \mathcal{C}'$, if $\eta' = \eta + du$, then $d_{\eta'} \omega' = 0$.
- Two locally conformally symplectic manifolds (M, ω, η) and (M, ω_1, η_1) are gauge equivalent if there exists $u : M \to \mathbb{R}$ such that $(M, \omega_1, \eta_1) = (M, e^u \omega, \eta + du)$.
- When $\eta = du$ is exact, then (M, ω, η) is gauge equivalent to a symplectic manifold $(M, \omega' = e^{-u} \omega, 0)$.

Conformal symplectic dynamics

Definition

Let (M, ω, η) be a locally symplectic manifold. Then $f : M \circlearrowleft$ is conformally symplectic (or conformal) if there exists a function $a : M \to \mathbb{R}_+^*$ such that $f^*\omega = a\omega$.

- If (M, ω', η') is gauge equivalent to (M, ω, η) the two LSM have the same conformally symplectic diffeomorphisms.
- When $(M, \omega, 0)$ is symplectic and compact, its conformally symplectic diffeomorphisms are symplectic.

Conformal Hamiltonian dynamics (1)

Definition

Let (M, ω, η) be a conformal symplectic manifold (CSM) and $H : M \to \mathbb{R}$ be a function. The conformal Hamiltonian vector field X associated to H is defined by

$$i_X\omega = d_\eta H = dH - H\eta.$$

Then the Hamiltonian flow (φ_t) is conformal. If we define for every $x \in M$ and $t \in \mathbb{R}$ the winding number of x at time t by

$$r_t(x) := \int_0^t \eta(X(\varphi_s(x)))ds$$

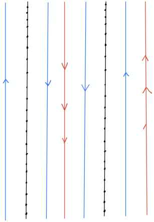
then $\varphi_t^*\omega = e^{r_t}\omega$ and $H \circ \varphi_t = e^{r_t}H$.

Conformal Hamiltonian dynamics (2)

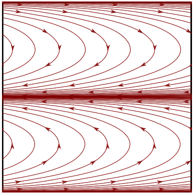
- If you change (ω_1, η_1) into $(\omega_2, \eta_2) = (e^u \omega_1, \eta_1 + du)$ that is gauge equivalent, you obtain the same set of Hamiltonian vector fields.
- More precisely, if $K = e^u H$, then the Hamiltonian vector field of H for (ω_1, η_1) is the Hamiltonian vector field of K for (ω_2, η_2).
- When H doesn't vanish, using a change of gauge, we are reduced to the case where $H = 1$. The conformal associated vector field is called the Lee vector field.
- When η is not exact, there is a bijection between the set of Hamiltonians and the set of Hamiltonian vector fields.

An example: on surfaces

- Every symplectic surface endowed with a closed 1-form defines a conformal structure.
- We endow $\mathbb{T}^2$ with the conformal structure $(\eta = d\theta_1, \omega = d\theta_1 \wedge d\theta_2)$ and look at the Dynamics of the following Hamiltonians in the fundamental domain $[0, 1]^2$



$H(\theta_1, \theta_2) = \sin(2\pi\theta_1)$



$H(\theta_1, \theta_2) = \sin(2\pi\theta_2)$

Conservative and dissipative sets

We always assume that (M, ω, η) is a closed CSM, that $H : M \to \mathbb{R}$ is a Hamiltonian with vector-field X and flow (φ_t).
A point $x \in M$ is

- either (positively) dissipative when $\lim_{t\to+\infty} |r_t(x)| = +\infty$, $x \in \mathcal{D}_+$;
- or (positively) conservative, $x \in \mathcal{C}_+$.

Theorem

Up to a set with zero Lebesgue measure, the set of positively recurrent points coincides with $\mathcal{C}_+$. The ω-limit set of every point in $\mathcal{D}_+$ is in $\{H = 0\}$ and if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, every neighbourhood of $\omega(x) = A$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. Hence A is infinite.

Ideas of proof

Statement:

Up to a set with zero Lebesgue measure, the set of positively recurrent points coincides with $\mathcal{C}_+$.

- we first remark that almost every point of $\{H = 0\}$ is trivially recurrent and in $\mathcal{C}_+$: every point of the subset $\{H = 0\} \cap \{dH = 0\}$ is fixed by the dynamics whereas $\{H = 0\} \cap \{dH \neq 0\}$ is negligible.
- Because $\varphi_t^* \omega = e^{r_t} \omega$ and $H \circ \varphi_t = e^{r_t} H$, the first return map to

$$\mathcal{H}_k := \left\{ x \in M \mid |H(x)| \geqslant \frac{1}{k} \right\}.$$

preserves the mesure $\frac{\omega^n}{H^n}$ hence almost every point is $\mathcal{C}_+$ is positively recurrent.

Ideas of proof

Statement:

The ω-limit set of every point in $\mathcal{D}_+$ is in $\{H = 0\}$ and if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, every neighbourhood of $\omega(x) = A$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. Hence A is infinite.

- By definition, if $x \in \mathcal{D}_+$, then $\lim_{t\to+\infty} |r_t(x)| = +\infty$. As $H \circ \varphi_t = e^{r_t} H$ and H is bounded on the compact manifold M, we deduce that either $H(x) = 0$ and then $H \circ \varphi_t(x) = 0$ or $H(x) \neq 0$ and $\lim_{t\to+\infty} r_t(x) = -\infty$, which implies that $\lim_{t\to+\infty} H \circ \varphi_t(x) = \lim_{t\to+\infty} e^{r_t} H(x) = 0$.
- Hence , the ω-limit set of every point in $\mathcal{D}_+$ is in $\{H = 0\}$.
- Moreover, when $H(x) \neq 0$, then $\lim_{t\to+\infty} r_t(x) = -\infty$ gives the last assertion.

Examples

- A fixed point is always conservative;
- when x is a non-critical periodic point, then
  - when $x \in \mathcal{C}_+$, the first return map to a Poincaré section preserves a closed 2-form and a foliation into (local) hypersurfaces;
  - when $x \in \mathcal{D}_+$, then $H(x) = 0$ and the first return map to a Poincaré section alters a certain closed 2-form up to a constant factor that is different from 1;
- every attractor intersects $\{H = 0\}$, has non-trivial homology and almost every point in its basin of attraction that doesn't belong to the attractor is in $\mathcal{D}_+$.

Past and future

Proposition

Up to a set with zero Lebesgue measure, the set of positive recurrent points coincides with the set of negatively recurrent points.

But it can happen that these two sets are different.

Conservative dynamics

Definition

The dynamics is (positively) conservative when $\mathcal{C}_+ = M$. It is (positively) dissipative when $\mathcal{C}_+$ has zero volume.

Proposition

When $\{H = 0\}$ *has a neighbourhood* V *such that for every loop* $\gamma : \mathbb{T} \to V$, $\int_\gamma \eta = 0$, *then* (φ_t) *is conservative,*

This contains the case when H doesn't vanish. Hence there exists non-empty C^0-open sets of C^2 Hamiltonians such that the associated CH flows are conservative.

Some ideas of proofs

Proposition

When $\{H = 0\}$ *has a neighbourhood* V *such that for every loop* $\gamma : \mathbb{T} \to V$, $\int_\gamma \eta = 0$, *then* (φ_t) *is conservative,*

We prove that the map $(t, x) \mapsto r_t(x)$ is bounded.

- Close to $\{H = 0\}$, η has a primitive and the variations of this primitive are bounded;
- When the endpoints of an arc are far from $\{H = 0\}$ the variations of $\log |H|$ are bounded and so also the variation of r_t along orbits that begin and end far from $\{H = 0\}$.
- For other case, we decompose into shorter arcs.

Lee flow

The Lee vector-field L is associated to $H = 1$;

$$i_L \omega = -\eta.$$

Up to a change of gauge and time, every CH flow that is associated to a non-vanishing Hamiltonian is a Lee flow.
We have examples of Lee flows

- that are transitive ; this is different from the Hamiltonian symplectic case, where the level sets of H are preserved;
- that have no periodic orbits ; Weinstein conjecture in the contact setting and Arnol'd conjecture in the symplectic setting assert the existence of periodic orbits. This example emphasizes one difference between the CH and the Reeb flows as well as the symplectic Hamiltonian flows.
- Constructions of these examples use the twisted symplectization (see Simon talk).

Question

Question:

In dimension larger than 2, can a Lee flow be minimal?

Let $(a, b) \in \mathbb{R}^2$ such that a and b are rationally independent. Then $(\mathbb{T}^2, \eta = ad\theta_1 + bd\theta_2, \omega = d\theta_1 \wedge d\theta_2)$ is a conformally symplectic manifold and its Lee vector-field is then

$$b\partial_{\theta_1} - a\partial_{\theta_2}.$$

and the Lee flow is minimal.

Dissipative flows

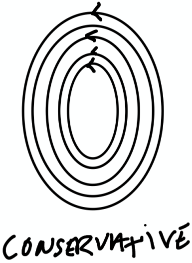
- We have examples of dissipative conformal Hamiltonian flow , with one normally hyperbolic attractor that is a Lagrangian submanifold, one normally hyperbolic repulsor that is also a Lagrangian submanifold and the remaining part of the manifold that is filled with heteroclinic connections.
- This gives a C^1 open set of conformal Hamiltonian flows that are dissipative.

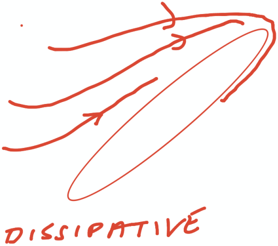
Mixed behaviours

There also exist C^1-open sets of conformal Hamiltonian flow such that both $\mathcal{C}_+$ and $\mathcal{D}_+$ have positive Lebesgue measure. This happens when there is a normally hyperbolic periodic attractor and one non degenerate local minimum of $e^{-\theta}H$ where θ is a local primitive of η.

Indeed

$$d(e^{-\theta}H)X = e^{-\theta}\big(dH - H\eta\big)(X) = \omega(X,X) = 0.$$





There are C^1 open sets of such dynamics.

Thank you for your attention.