Variational properties of Twist maps and Multi-dimensional Birkhoff billiards

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based on joint works with
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Introduction

Dynamics of multi-dim Twist maps and smooth Convex billiards is highly non-trivial.

For example, it is not known if multi-dim Convex billiard can be ergodic.

Are there invariant hypersurfaces in the phase space?

M. Berger proved - no caustics if the table is different from ellipsoid.

We study remarkable closed invariant set $\mathcal{M}$ of *locally maximizing orbits*. For Twist maps of $T^*\mathbb{T}^d$, $\mathcal{M}$ contains invariant Lagrangian graphs and Aubry-Mather sets. For billiards in $2D$ the set $\mathcal{M}$ is crucial for Birkhoff conjecture, the measure of $\mathcal{M}$ is positive and can be estimated from above effectively. In multi-dim convex billiards-no results for the measure of $\mathcal{M}$.

Variational construction of orbits:
1. Morse and Hedlund: Variational theory of geodesics, Class A.
2. E. Hopf, L. Green: Geodesics with no conjugate points.
3. Aubry and Mather: Variational construction of orbits.
4. Percival, MacKay and MacKay, Meiss, Stark: Variational properties for converse KAM.
5. B.-MacKay: Symplectic twist maps without conjugate points.
6. Marie-Claude Arnaud: Geometry of Green bundles...

The goal today:

1. To give effective geometric criterion for orbits of Twist map to be locally maximizing.
2. To prove that the set $\mathcal{M}$ of all locally maximizing orbits does not depend on the choice of the generating function.
3. To apply to multi-dimensional Birkhoff billiards.

Twist symplectic maps

Let us recall the definitions.

Definition

Let $T : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$. Then T is an *exact symplectic twist map* if it can be lifted to a diffeomorphism $(q,p) \mapsto (Q(q,p), P(q,p))$ of $\mathbb{R}^{2d}$ which satisfies:

1. $Q(q+m,p) = Q(q,p) + m$ for all $m \in \mathbb{Z}^d$.
2. For every q the map $p \mapsto Q(q,p)$ is a diffeomorphism.
3. The 1-form $PdQ - pdq = dS(q,Q)$ ($S : \mathbb{R}^{2d} \to \mathbb{R}$ is called *generating function*) satisfying $S(q+m, Q+m) = S(q,Q)$ for all $m \in \mathbb{Z}^d$.
   In particular the matrix $\partial_{12}S$ is non-singular (Twist condition)

Example

1. Standard like map: $S(q,Q) = \frac{1}{2}||Q-q||^2 + V(q)$, V is a $\mathbb{Z}^d-$periodic function.

2. The billiard ball map in higher dimension is a twist map acting on a ball bundles over a sphere

Definition

Let $N \subseteq \mathbb{R}^d$ be diffeomorphic to $\mathbb{S}^{d-1}$. Let $\pi : M \to N$ be a ball bundle over N, where $M \subseteq T^*N$ is a neighborhood of the zero section of T^*N: $\forall x \in N$ the fiber $N_x = \pi^{-1}(x) \subset T^*_x N$ is diffeomorphic to a ball.

1. A diffeomorphism $T : M \to M$ is called a *twist map* of M, if it is a symplectomorphism, and it satisfies the *twist condition*: for all $x \in N$, the map $\pi \circ T \mid_{N_x} : N_x \to N$ is a diffeomorphism from the interior of the ball N_x to $N \setminus \{x\}$.
2. A *generating function* for T is a function $S : N \times N \setminus \Delta \to \mathbb{R}$ such that $\forall x \neq y \in N$ and $u \in N_x$, $v \in N_y$ $\boxed{T(x,u) = (y,v) \iff u = -S_1(x,y) \text{ and } v = S_2(x,y),}$ where subindices $1, 2$ stand for the differentials wrt x, y. We equip N with a Riemannian metric induced from $\mathbb{R}^d$, then we can view the second order derivatives of S as linear operators:

$$S_{12}(x,y) : T_yN \to T_xN, \quad S_{21}(x,y) : T_xN \to T_yN,$$
$$S_{11}(x,y) : T_xN \to T_xN, \quad S_{22}(x,y) : T_yN \to T_yN.$$

where the operators S_{12}, S_{21} are conjugate to each other, and S_{11}, S_{22} are self-adjoint. In this setting, the twist condition is the requirement that the operators S_{12}, S_{21} are non-degenerate.
Configurations are extremals of the Variational principle:

$$\sum_{n=-\infty}^{\infty} S(q_n, q_{n+1}).$$

We study the class $\mathcal{M}$ of locally maximizing orbits, which we call *m-orbits* such that the configurations $\{q_n\}$ for which every finite subsegment $\{q_n\}_{n=N}^{M}$ is a local maximum of the truncated action functional,

$$F_{M,N}(x_M, ..., x_N) = S(q_{M-1}, x_M) + \sum_{i=M}^{N-1} S(x_i, x_{i+1}) + S(x_N, q_{N+1}).$$

second variation of F_{MN} has the following block form:

$$W_{MN} = \delta^2 F_{MN} = \begin{pmatrix} a_M & b_M & 0 \cdots & 0 & 0 \\ b_M^* & a_{M+1} & b_{M+1} \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{N-2}^* & a_{N-1} & b_{N-1} \\ 0 & \cdots & 0 & b_{N-1}^* & a_N \end{pmatrix},$$

where the operators a_n, b_n are given by

$$a_n = S_{11}(q_n, q_{n+1}) + S_{22}(q_{n-1}, q_n),\ b_n = S_{12}(q_n, q_{n+1}),\ b_n^* = S_{21}(q_n, q_{n+1}),$$

where b_k^* denotes the dual linear map to b_k.
(Operator or Matrix-)Jacobi equation for the discrete case:

$$b_{n-1}^T \xi_{n-1} + a_n \xi_n + b_n \xi_{n+1} = 0.$$

The class $\mathcal{M}$

The set $\mathcal{M}$ is a closed invariant set, containing in particular all invariant Lagrangian graphs, Aubry-Mather sets.

Theorem 1 [B-MacKay][B-Tsodikovich]. *Let T be an exact symplectic twist map of $T^*\mathbb{T}^{d-1}$. Then $\{q_n\}$ is an m-configuration iff there exists a Jacobi field of non-singular matrices $J_n, n \in \mathbb{Z}$, such that all the matrices*

$$X_n = -H_{12}(q_n, q_{n+1})J_{n+1}J_n^{-1},$$

are symmetric negative definite.

In case N is diffeomorphic to $\mathbb{S}^{d-1}$, and M is a ball bundle over N. Suppose that T is an exact symplectic twist map of M with a generating function H. Then we require operator Jacobi field $J_n : E \to T_{q_n}N$ of non-degenerate operators, for which $X_n = -H_{12}(q_n, q_{n+1})J_{n+1}J_n^{-1}$ are all negative definite endomorphisms of $T_{q_n}N$ (here E is an arbitrary $d-1$ dimensional space).

Construction of J_n is the discrete realization of the ideas by E.Hopf and L.Green started in [Bialy, MacKay].

Geometric criterion

Theorem 2 [B-MacKay][B-Tsodikovich].
An orbit $\{(q_n, p_n)\}$ *is an m-orbit of* T*, iff there exists a field of Lagrangian subspaces* L_n *such that:*

1. *Transversality:* L_n *is transversal to the vertical subspace* V_n*.*
2. *Invariance:* $L_{n+1} = dT_{(q_n,p_n)}(L_n)$*.*
3. *Inequality:* $\alpha_n \underset{V_n}{<} L_n \underset{V_n}{<} \beta_n$*,*

where α_n*,* β_n *are the image and preimage (respectively) of the vertical subspace:* $\alpha_n = dT(V_{n-1})$*,* $\beta_n = dT^{-1}(V_{n+1})$*.*

Let $\gamma \subset U$ be a Lagrangian subspace in symplectic space U. Choose symplectic coordinates (q, p) on U such that $\gamma = \{q = 0\}$. If a Lagrangian $L \pitchfork \gamma$, then $L = \{p = Aq\}$, for a symmetric matrix A.

If L_1, $L_2 \pitchfork \gamma$ are two Lagrangian subspaces that are transversal to γ, then we set $\boxed{L_1 \underset{\gamma}{<} L_2 \;\Leftrightarrow\; A_1 < A_2}$. This order does not depend on symplectic coordinates as long as γ has the description $\{q = 0\}$.

Theorem 1 $\Rightarrow$ Theorem 2

Write H for the generating function of T. One can check that α_n and β_n are the graphs of the self adjoint endomorphisms $H_{22}(q_{n-1}, q_n)$, $-H_{11}(q_n, q_{n+1})$, respectively. Let $\{(q_n, p_n)\}$ be an m-configuration. Then by Theorem 1, there exists a Jacobi field J_n for which $X_n = -b_n J_{n+1} J_n^{-1}$ is negative definite. Define the subspace L_n to be the graph of the self adjoint endomorphism

$$W_n = -H_{11}(q_n, q_{n+1}) + X_n.$$

From Jacobi equation we get that X_n satisfies the recursive relation

$$X_{n+1} = a_{n+1} - b_n^* X_n^{-1} b_n.$$

This implies that L_n is an invariant subspace field. Since X_n is negative definite, then $W_n < -H_{11}$, and hence $L_n \underset{V_n}{<} \beta_n$.

Also, we can write, thanks to the recursive relation of X_n,

$$W_n = -H_{11} + a_n - b_{n-1}^* X_{n-1}^{-1} b_{n-1} = H_{22}(q_{n-1}, q_n) - b_{n-1}^* X_{n-1}^{-1} b_{n-1},$$

and as a result $H_{22} < W_n$ so we also have $\alpha_n \underset{V_n}{<} L_n$.

Two generating functions

Suppose that T is an exact symplectic twist map with respect to two generating functions, H_1 and H_2. Write $\mathcal{M}_{H_1}$, for the set swept by m-orbits of H_1, and $\mathcal{M}_{H_2}$ for the set swept by m-orbits of H_2.

Geometric Assumption GA

- Suppose that at every point of $\mathcal{M}_{H_1} \cup \mathcal{M}_{H_2}$ there exists a homotopy of Lagrangian subspaces V_t connecting V^{H_1} and V^{H_2}, such that for all t, the subspace V_t is transversal to all four subspaces α^{H_1}, β^{H_1}, α^{H_2}, β^{H_2}.

Theorem 3 [B-Tsodikovich]. *Suppose that H_1 and H_2 are two generating functions of an exact symplectic twist map T. If H_1 and H_2 satisfy the Geometric Assumption, then the set of m-orbits coincide:*

$$\mathcal{M}_{H_1} = \mathcal{M}_{H_2}.$$

Connection with Tonelli Hamiltonians

Bernard and Mazzucchelli, Sorrentino proved that if a Tonelli Hamiltonian remains Tonelli after an exact symplectic change of variables, then the Aubry, Mañe, and Mather sets are the same for both Hamiltonians.

By Moser: every exact symplectic twist map of a cylinder is the time-one map for some Tonelli Hamiltonian. This suggests that in the two dimensional case, the result on Tonelli Hamiltonians and our results may be connected.

It is also known that a higher dimensional twist map that satisfies a "symmetric" twist condition is also a time-one map of a Tonelli Hamiltonian. Such interpolation result is not available for Birkhoff billiards in high dimension.

Our approach is direct, meaning that we deal directly with the discrete system, and do not rely on interpolation by Tonelli Hamiltonians.

Two generating functions for Birkhoff billiards

Let $\Sigma \subseteq \mathbb{R}^d$ be a smooth, strictly convex hypersurface. Consider the billiard motion in Σ: When the particle hits the boundary, it reflects according to the classical law of geometric optics.

The phase space $\mathcal{L}$ of the billiard map is the space of oriented lines in $\mathbb{R}^d$ that intersect the hypersurface Σ.

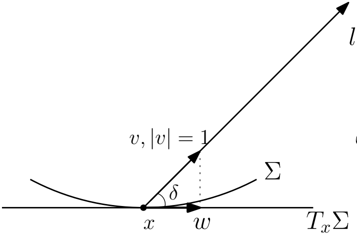
The space $\mathcal{L}$ can be seen as a ball bundle over a hypersurface N diffeomorphic to a sphere in two different ways. Thus $\mathcal{L}$ inherits from T^*N two sets of symplectic coordinates.

L-coordinates: Let $x \in \Sigma$ and $w \in T_x\Sigma, |w| < 1$. The pair (x, w) determines the oriented line which passes through x and has direction $v = w - \sqrt{1 - |w|^2} n_x$ where n_x is the outer normal to Σ at x. In these coordinates, wdx is a primitive of the canonical symplectic form of $T^*\Sigma$. Thus, in this description, $\mathcal{L}$ is the unit ball bundle of $N = \Sigma$. The generating function in those coordinates is

$$L : (\Sigma \times \Sigma \setminus \Delta) \to \mathbb{R}, \quad L(x, y) = |x - y|.$$

This generated function was studied and used by Birkhoff.

Symplectic coordinates



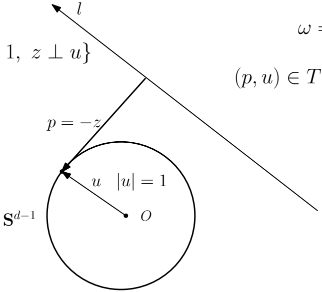
$\Sigma \subset \mathbf{R}^d$

$\mathcal{L} \cong B^*\Sigma$

$\omega = dw \wedge dx$

$w = Proj(v), |w| < 1$

$l = \{z + ut, \ |u| = 1, \ z \perp u\}$



$\omega = dp \wedge du$

$(p, u) \in T^*\mathbf{S}^{d-1} \cong \mathcal{L}$

S-coordinates Assume that the origin is inside Σ. Given an oriented line in $\mathbb{R}^d$, we can associate to it a point in $T^*\mathbb{S}^{d-1}$ in the following way: take the unit vector direction of the line, $u \in \mathbb{S}^{d-1}$, and take $z \in \{u\}^\perp \cong T^*_u\mathbb{S}^{d-1}$ to be the intersection point of the line with $\{u\}^\perp$ and set $p = -z$.

Then the form pdu is also a primitive of the canonical symplectic form on $T^*\mathbb{S}^{d-1}$. This symplectic form coincides with the one described for the L-coordinates. Thus in this case we have $N = \mathbb{S}^{d-1}$, and the ball N_u, for $u \in \mathbb{S}^{d-1}$ is (the negative of) the projection of Σ on the hyperplane $\{u\}^\perp$. The generating function for the billiard map in those coordinates is $S : (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \setminus \Delta) \to \mathbb{R}$ given by

$$\boxed{S(u_1, u_2) = \left\langle G^{-1}\left(\frac{u_1 - u_2}{|u_1 - u_2|}\right), u_1 - u_2 \right\rangle = h(n)|u_1 - u_2| = 2h(n)\sin\delta}$$

where $u_1 \neq u_2 \in \mathbb{S}^{d-1}$ and $G : \Sigma \to \mathbb{S}^{d-1}, n = (u_1 - u_2)/|u_1 - u_2|$ denotes the Gauss map of Σ.

Ref: [Suris2016] for ellipsoids,
general case [Bialy-Mronov] in $2d$, and in [Bialy] for $d > 2$.

Application for Billiards.

Theorem 4 [B-Tsodikovich].
Let $\Sigma \subseteq \mathbb{R}^d$ be a C^2-smooth convex hypersurface of positive curvature. Let L and S denote the two generating functions (as above) for Birkhoff billiard in Σ. Then:

$$\mathcal{M}_L = \mathcal{M}_S.$$

In the 2-dimensional case, Theorem 4 has very important applications to effective version of rigidity and Birkhoff-Poritsky conjecture.

We use the machinery of **wave fronts**, which provides a convenient way to think about Lagrangian subspaces in $\mathcal{L}$ and to verify the **GA** condition for the generating functions L, S.

Ref: Sinai-Chernov, Arnold.

Wave fronts

Let ℓ be an oriented line in $\mathbb{R}^d$ and $L \subset T_\ell \mathcal{L}$ be a Lagrangian subspace. Take a germ of Lagrangian sumanifold Y tangent to L at ℓ. Then Y determines a Lagrangian submanifold K of $T\mathbb{R}^d$ swept by unit vectors tangent to the lines of Y.

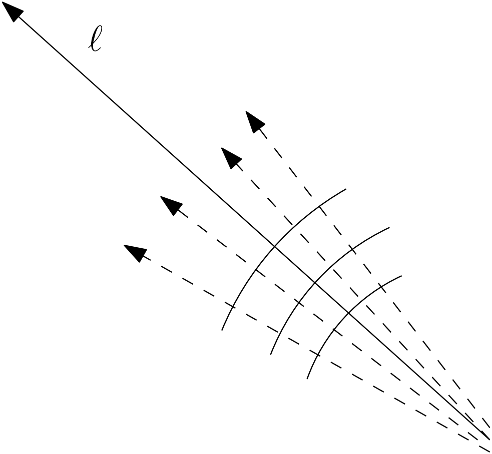
Take a point (q_0, p_0) at the lift of ℓ where K is a graph:

$$K = \{(q, f(q)), q \in \mathbb{R}^d, |f(q)| = 1\}.$$

Since L is Lagrangian, then the distribution of the planes $f^\perp$ is Frobenius integrable. The leaves, called fronts, are $\perp$ to ℓ. Thus the curvature operators of the fronts at the points of ℓ are well defined by L.

Moreover $L_1 \pitchfork L_2$ iff the difference of the curvature operators on $\ell^\perp$ has a trivial Kernel.

Propagation of fronts and their curvatures without collisions (free flight) is governed by Huygens principle, as for the collisions–by Sinai-Chernov formula.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Verification of GA for billiards

Now we are ready to verify the hypotheses of Geometric Assumption for the generating functions L, S which we described for billiards.

Fix an oriented line $\ell = x + \mathbb{R}u \in \mathcal{L}$, $x \in \Sigma$, $|u| = 1$. We construct now the required homotopy V_s of the Lagrangian subspaces of $T_\ell \mathcal{L}$ as follows:

The vertical subspace of the L coordinates at $T_\ell \mathcal{L}$ corresponds to rays emanating from the point x, fronts are spheres centered at x. The vertical subspace of the S coordinates at $T_\ell \mathcal{L}$ corresponds to rays that are parallel to ℓ, fronts are the hyperplanes $\ell^\perp$.

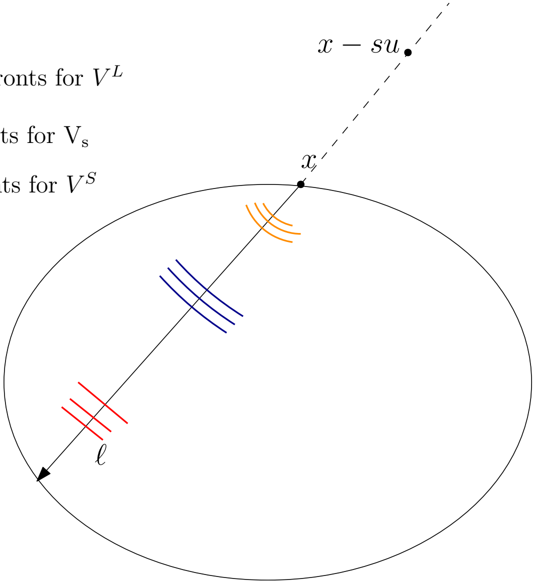
At the level of fronts, we define the homotopy moving the center of the sphere from x to $x - su$ where $s \in (0, \infty)$. The leaf of the foliation that passes through x is then a sphere of radius s centered at $x - su$. In terms of Lagrangian subspaces, the subspace V_s is just tangent space at ℓ of the normal bundle to the sphere of radius s centered at $x - su$, for $s \in (0, \infty)$. This extends continuously when $s \to 0$ to the subspace V^L, and when $s \to \infty$ to the subspace V^S.

Homotopy of wave fronts

Orange – fronts for V^L

Blue – fronts for $\mathrm{V_s}$

Red – fronts for V^S



Partial order $\underset{\gamma}{<}$ and the index form Q

Partial order $\underset{\gamma}{<}$ is related to the Maslov index form Q (Arnold 1985):

Let $\alpha \pitchfork \beta$ be two Lagrangian subspaces of U (but might not be transversal to γ). Then any $u \in U$ can be uniquely written as $u = u_1 + u_2$ for $u_1 \in \alpha$ and $u_2 \in \beta$, and set the quadratic form $Q[\alpha, \beta]$ by: $\boxed{Q[\alpha, \beta](u) = \omega(u_1, u_2).}$

Proposition

1. *Suppose that $\alpha \pitchfork \beta$ are Lagrangian subspaces transversal to γ. Then*
   $$\boxed{\alpha \underset{\gamma}{<} \beta \Longleftrightarrow Q[\alpha, \beta] \,|_\gamma > 0.}$$
2. *Let $\alpha \pitchfork \beta$, and L are Lagrangian subspaces transversal to γ, and $\alpha \underset{\gamma}{<} \beta$.*

   Then: $\boxed{\alpha \underset{\gamma}{<} L \underset{\gamma}{<} \beta \Longleftrightarrow Q[\alpha, \beta] \,|_L < 0.}$
3. *Let $\alpha \pitchfork \beta$ and $\{\gamma_t\}_{t \in [0,1]}$ be a homotopy of Lagrangian subspaces, such that $\forall t \in [0,1]$, $\gamma_t \pitchfork \alpha$ and $\gamma_t \pitchfork \beta$.*
   If $Q[\alpha, \beta]$ is positive definite on γ_0, then it is positive definite on γ_t for all $t \in [0,1]$.

Proof of Theorem 3

Suppose that $\{(q_n, p_n)\}$ is an m-orbit with respect to H_1. By Theorem 2 here exists an invariant field of Lagrangian subspaces L along the orbit which is transversal to V^{H_1}, and for which

$$\alpha^{H_1} \underset{V^{H_1}}{<} L \underset{V^{H_1}}{<} \beta^{H_1}.$$

In particular, $\alpha^{H_1} \underset{V^{H_1}}{<} \beta^{H_1}$, and hence by Proposition

$$\boxed{Q[\alpha^{H_1}, \beta^{H_1}]\ |_{V^{H_1}} > 0, \text{ and } \forall t \in [0,1] \Rightarrow Q[\alpha^{H_1}, \beta^{H_1}]\ |_{V_t} > 0.}$$

Also it follows from item 2 of Proposition: $\boxed{Q[\alpha^{H_1}, \beta^{H_1}]\ |_L < 0.}$
Hence $\forall t, L \pitchfork V_t \Rightarrow L \pitchfork V^{H_2}$, and then also $L \pitchfork \alpha^{H_2}$ and $L \pitchfork \beta^{H_2}$.

We shall prove the inequality $L \underset{V^{H_2}}{<} \beta^{H_2}$, the other one is analogous.

We start with the inequality $\alpha^{H_1} \underset{V^{H_1}}{<} L$. By Proposition we have

$$Q[\alpha^{H_1}, L] \mid_{V^{H_1}} > 0.$$

Since $V_t \pitchfork L$ and $V_t \pitchfork \alpha^{H_1}$, we get from item 3 of Proposition:

$$\forall t \Rightarrow Q[\alpha^{H_1}, L] \mid_{V_t} > 0,$$

and in particular, $Q[\alpha^{H_1}, L] \mid_{V^{H_2}}$ is positive definite. This claim is invariant under the action of a symplectomorphism, so we can apply T^{-1} to all three subspaces, and see that $Q[V^{H_1}, L] \mid_{\beta^{H_2}}$ is also positive definite. Now we shift cyclically [Arnold 1985], to get that

$$Q[L, \beta^{H_2}] \mid_{V^{H_1}} > 0.$$

Using Proposition again we conclude that this inequality persists through the homotopy, and hence

$$Q[L, \beta^{H_2}] \mid_{V^{H_2}} > 0.$$

Hence by Proposition we finally conclude that $L \underset{V^{H_2}}{<} \beta^{H_2}$.

Idea of the proof of Theorem 1

1. Suppose that $\{q_n\}$ is an m-configuration. Fix an integer k, and suppose that $\{\xi_n^{(k)}\}_{n\in\mathbb{Z}}$ is a matrix Jacobi field with $\xi_{k-1}^{(k)} = 0$, $\xi_k^{(k)} = I$. Then $\xi_n^{(k)}$ is invertible for all $n \geq k$ (analogous to the Riemannian case). It then follows $\forall n \geq k$ the matrices $A_n^{(k)} = -b_n \xi_{n+1}^{(k)} \left(\xi_n^{(k)}\right)^{-1}$ are well defined.
2. It is straightforward to verify that these matrices satisfy

$$A_k^{(k)} = a_k \text{ and } A_{n+1}^{(k)} = a_{n+1} - b_n^T \left(A_n^{(k)}\right)^{-1} b_n, \forall n \geq k.$$

Since a_n are symmetric, it follows that $A_n^{(k)}$ are also symmetric.
We need to show that these matrices are all negative definite. Take any $\eta \in \mathbb{R}^d$ and $m \geq k$ be any integer. For $n \geq k-1$, we set $\eta_n = \xi_n^{(k)} \left(\xi_m^{(k)}\right)^{-1} \eta$. These vectors are obtained by multiplying the constant vector $\left(\xi_m^{(k)}\right)^{-1} \eta$ by a matrix Jacobi field, so the result satisfies the vector Jacobi equation. In additional, it holds that $\eta_{k-1} = 0$, since $\xi_{k-1}^{(k)} = 0$, and also, $\eta_m = \eta$. Therefore we have from formula (7),

$$\delta^2 F_{k,m}(\eta_k, ..., \eta_m) = - < b_m \cdot \eta_{m+1}, \eta_m > = < A_m^{(k)} \eta, \eta > .$$

Thus negative definiteness of $A_m^{(k)}$ follows from that of $\delta^2 F_{k,m}$.

3. The sequence $A_n^{(k)}$ is monotone in $k \le n$. To show this, fixing k, we prove by induction on $n \ge k$ that $A_n^{(k)} - A_n^{(k-1)}$ is negative definite. For $n = k$, we have $A_k^{(k)} = a_k$ and $A_k^{(k-1)} = a_k - b_k^T \left(A_{k-1}^{(k-1)}\right)^{-1} b_k$, so the we get that $A_k^{(k)} - A_k^{(k-1)} = b_k^T \left(A_{k-1}^{(k-1)}\right)^{-1} b_k$ which is indeed negative definite. For the induction step, if $A_n^{(k)} - A_n^{(k-1)}$ is negative definite, then:

$$A_{n+1}^{(k)} - A_{n+1}^{(k-1)} = -b_n^T \left(\left(A_n^{(k)}\right)^{-1} - \left(A_n^{(k-1)}\right)^{-1} \right) b_n,$$

and since $A_n^{(k)}$, $A_n^{(k-1)}$ are also negative definite, then it follows that $A_{n+1}^{(k)} - A_{n+1}^{(k-1)}$ is also negative definite.

As a result, for fixed n, the sequence $A_n^{(k)}$ is a sequence of negative definite matrices, and it increases as k decreases to $-\infty$, so $\boxed{\exists \lim_{k \to -\infty} A_n^{(k)} =: X_n}$.

The limit X_n is negative semi-definite, but it must also be non-degenerate, since for all $k < n$ we have the recursive relation satisfied also by the limit X_n:

$$A_{n+1}^{(k)} = a_{n+1} - b_n^T \left(A_n^{(k)}\right)^{-1} b_n \Leftrightarrow (A_{n+1}^{(k)} b_n^{-1} - a_{n+1} b_n^{-1}) A_n^{(k)} = -b_n^T.$$

If we set J_0 to be an arbitrary invertible matrix, then the required $\{J_n\}$ can be defined recursively: $\begin{cases} J_{n+1} = -b_n^{-1} X_n J_n, & n \ge 0, \\ J_n = -X_n^{-1} b_n J_{n+1}, & n < 0. \end{cases}$

Open Questions

1. Can multi-dim convex billiard be ergodic?

2. Does there exist an example of convex billiard different from ellipsoid with an invariant hypersurface in the phase space?

3. Give an example where the set $\mathcal{M}$ has positive measure.

4. Are ellipsoids the only integrable convex multi-dim billiards?

5. Can billiards in Spheres/Ellipsoids be characterized in variational terms? (in terms of $\mathcal{M}$)?

Thank you!