Nodal count via topological persistence

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joint works with

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The corresponding eigenfunctions f_j,

$$\Delta f_j = \lambda_j f_j,$$

form an orthonormal basis in $L^2(M)$.

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Nodal pattern of an eigenfunction on $\mathbb{S}^2$ corresponding to an eigenvalue $\lambda = 17 \cdot 18$.
(*Picture credit: M. Levitin.)*

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Theorem (R. Courant, 1923)

A Laplace eigenfunction f_j has at most j nodal domains.

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4. higher topological invariants: Betti numbers m_r instead of m_0 (Arnold, 2005)

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Theorem (B.–Logunov–Sodin, 2020)

There exists a Riemannian metric g on a 2-torus and a sequence f_j of eigenfunctions of the Laplacian Δ_g, such that the functions $f_j + 1$ have infinitely many nodal domains.

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Idea: What if we ignore *small* oscillations?

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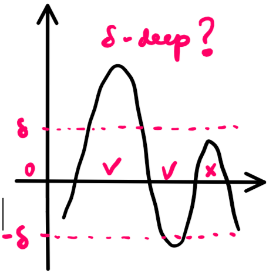

δ-deep nodal domains. (*Picture credit: E. Shelukhin.)*

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Our first main result shows that $m_0(f, \delta)$ is controlled by the appropriate Sobolev norms of f.

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Theorem (BP^{3}S^2, 2022)

Let $f \in W^{k,p}(M)$ *for* $k > \frac{n}{p}$*, where* $n = \dim M$*. Then for any* $\delta > 0$*,*

$$m_0(f, \delta) \leq C\delta^{-\frac{n}{k}} \|f\|_{W^{k,p}}^{\frac{n}{k}},$$

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This bound can be extended to higher *persistent Betti numbers* $m_r(f,\delta)$ to be discussed later.

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Theorem

Let $k > \frac{n}{2}$ be an integer. Then for any $\delta > 0$ and any $f \in \mathcal{F}_\lambda$ with $\|f\|_{L^2} = 1$,

$$m_0(f, \delta) \leq C\delta^{-\frac{n}{k}} (\lambda + 1)^{\frac{n}{2}} .$$

Remark

In two dimensions, versions of this result were proved by L. Polterovich–Sodin (2007) and I. Polterovich–L. Polterovich – Stojisavljević (2019).

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Theorem

For each $\delta > 0$ *one can find* $f \in \mathcal{F}_\lambda$ *with* $\|f\|_{L^2} = 1$, *such that*

$$m_0(f, \delta) \geq c \frac{(\lambda + 1)^{\frac{n}{2}}}{\max(1, \delta^2)} - 1.$$

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Theorem

Let $f_1, \ldots, f_l \in \mathcal{F}_\lambda$ be L^2-normalised linear combinations of eigenfunctions, $f = f_1 \cdot \cdots \cdot f_l$ and $k > n/2$ be an integer number. Then for any $\delta > 0$ and $\varepsilon > 0$,

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Thus up to the power $(\lambda + 1)^{\varepsilon}$, the upper bound is the same as in the previous case.

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- To sections of vector bundles, thus giving a possible answer to a question of V. Arnold (2003).
- To *persistent* Betti numbers of arbitrary degree:

$$m_r(f, \delta) = \dim \operatorname{Im} \left(H_r(\{|f| > \delta\}) \to H_r(M \setminus \mathcal{Z}_f) \right),$$

where H_r stands for the r-th homology group with coefficients in a field.

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$\mathcal{B}$ and $\mathcal{B}'$ are ε-*matched* if after erasing some bars of length $< 2\varepsilon$ the rest are in bijection up to an error of ε on the endpoints.

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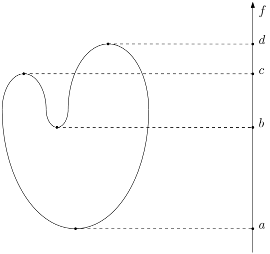
Bottleneck distance is given by

$$d_{bottle}(\mathcal{B}, \mathcal{B}') = \inf\{\varepsilon \mid \mathcal{B}, \mathcal{B}' \text{ are } \varepsilon\text{-matched}\}.$$

Example: barcode of a height function

$f : \mathbb{S}^1 \to \mathbb{R}$ is a height function on deformed circle given by:



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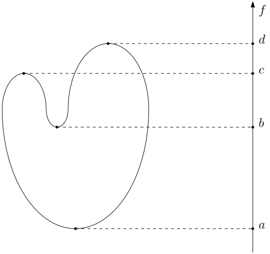

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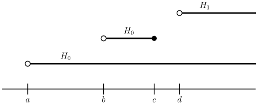

Barcode $\mathcal{B}(f)$. *(Picture credit: M. Levitin.)*

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Theorem (Stability theorem, Cohen-Steiner–Edelsbrunner–Harer, 2007)

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Stability theorem is a key feature of the theory.

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Why? Because:

$$m_0(f, \delta) \leq N_\delta(-|f|).$$

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In the theorem, $N_\delta(f)$ can be replaced by $N_\delta(-|f|)$ or by $N_\delta(|f|)$.

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- Nice behavior of N_δ under unions and stability theorem.

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Theorem (Morrey–Sobolev)

Let $\mathcal{P}_k(Q) \subset C^0(Q)$ denote the subspace of polynomials of degree $\leq k$. Then

$$d_{C^0}\left(f, \mathcal{P}_{k-1}(Q)\right) \leq C\left(\mathrm{Vol}(Q)\right)^{\frac{k}{n}-\frac{1}{p}} \|D^k f\|_{L^p(Q)}.$$

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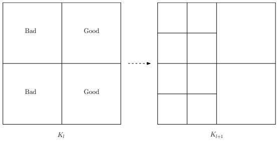

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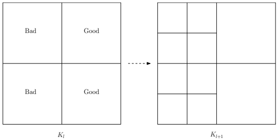

One can show that the total number κ of cubes in the end is bounded by

$$\kappa \le C_1 \delta^{-\frac{n}{k}} \left(\|D^k f\|_{L^p(Q)}\right)^{\frac{n}{k}} + C_2.$$

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Applying this fact together with a combinatorial argument that allows to control the number of non-disjoint unions we complete the proof.

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Milnor's bound, stability and the multiscale dyadic partition allows us to estimate the barcode counting function separately in each cube.

It remains to glue all the cubes together by using a Mayer-Vietoris type argument.

Fact: N_δ is additive for disjoint unions. In general, we have

$$N_{2\delta}\left(f|_{A_1\cup A_2}\right) \leq N_\delta\left(f|_{A_1}\right) + N_\delta\left(f|_{A_2}\right) + N_\delta\left(f|_{A_1\cap A_2}\right).$$

Applying this fact together with a combinatorial argument that allows to control the number of non-disjoint unions we complete the proof.

Theorem (BP³S², 2022)

Let $U \to V \to W$ be an exact sequence of persistence modules. Then

$$N_{2\delta}(V) \leq N_\delta(U) + N_\delta(W).$$

Thank you for your attention!