Subleading asymptotics of symplectic Weyl laws

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Classical Weyl law

(M^n, g) compact Riemannian manifold, possibly with boundary
eigenvalues of $-\Delta_g$ with Dirichlet boundary condition
$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots < \infty$
$N(\lambda) :=$ number of eigenvalues less than λ

Theorem (Weyl)

$$N(\lambda) = (2\pi)^{-n}\omega_n \,\mathrm{vol}(M)\lambda^{n/2} + E(\lambda) \quad \text{with} \quad E(\lambda) = o(\lambda^{n/2})$$

Theorem (Levitan, Avakumovic, Seeley)

$$E(\lambda) = O(\lambda^{(n-1)/2})$$

Remark: this is sharp for the round sphere

Theorem (Duistermaat-Guillemin, Ivrii)

If the set of closed geodesics has measure zero, then
$$E(\lambda) = -\frac{1}{4}(2\pi)^{1-n}\omega_{n-1}\,\mathrm{vol}(\partial X)\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2})$$

Remark: fails for round sphere

Embedded contact homology (ECH) Weyl law

$X \subset \mathbb{R}^4$ star-shaped domain $\quad \rightsquigarrow \quad$ ECH capacities

$$0 < c_1(X) \leq c_2(X) \leq \cdots < \infty$$

Spectrality property: For every k, we can find finitely many closed orbits $\gamma_i \subset \partial X$ such that $c_k(X) = \sum_i \mathcal{A}(\gamma_i)$

Theorem (Hutchings '10)

For all star-shaped domains $X \subset \mathbb{R}^4$ we have

$$c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + o(k^{1/2}) \qquad (k \to \infty).$$

Cristofaro-Gardiner-Hutchings-Ramos ('12): More general Weyl law for arbitrary contact 3-manifolds

Application: (Irie '15) A C^∞ generic Reeb flow on a closed 3-manifold has a dense set of periodic orbits.

Periodic Floer homology (PFH) Weyl law

Closed surface (Σ, ω) of area A, Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$
$\rightsquigarrow$ PFH spectral invariants $c_1(H), c_2(H), \cdots \in \mathbb{R}$

Theorem (CG-Prasad-Zhang, E.-Hutchings 2021)

For all Hamiltonians H we have

$$c_d(H) = dA^{-1} \int_{\mathbb{R}/\mathbb{Z} \times \Sigma} H dt \wedge \omega + o(d) \qquad (d \to \infty).$$

- Similar statement for area preserving diffeomorphisms
- Related Weyl law for link spectral invariants (CG-Humilière-Mak-Seyfaddini-Smith)

Applications: C^∞ closing lemma, Simplicity conjecture (CG-Humilière-Seyfaddini),. . .

Subleading asymptotics

For $X \subset \mathbb{R}^4$ star-shaped write $c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + e_k(X)$

Theorem (Hutchings '19)

We have $e_k(X) = O(k^{1/4})$ as $k \to \infty$.

- Slightly weaker bounds for general contact 3-manifolds by CG-Savale and Sun

Question: In all known examples $e_k(X) = O(1)$. Always true?

Theorem (Hutchings '19)

If X is a strictly convex or concave toric domain then

$$\lim_{k\to\infty} e_k(X) = -\frac{1}{2} Ru(X). \tag{1}$$

Counterexample: $Ru(B(a)) = 2a$ but
$\liminf_{k\to\infty} e_k(B(a)) = -3a/2$ $\qquad$ $\limsup_{k\to\infty} e_k(B(a)) = -a/2$
Question: Is (1) true for generic X?

Relationship with symplectic packing

ECH Weyl law $\quad c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + o(k^{1/2})$
Sketch of proof:
Step 1: true for ball ("direct" computation)
Step 2: true for disjoint unions of balls

$$c_k(\coprod_i X_i) = \max_{\sum_i k_i = k} \sum_i c_{k_i}(X_i)$$

Step 3: Let X be star-shaped, $\varepsilon > 0$ arbitrary. There exists disjoint union $B = \coprod_i B_i$ of finitely many balls such that

- $B \overset{s}{\hookrightarrow} X$
- $\mathrm{vol}(B) \geq \mathrm{vol}(X) - \varepsilon$

$\Rightarrow \quad c_k(X) \geq c_k(B) \geq 2((\mathrm{vol}(X) - \varepsilon)k)^{1/2} + o(k^{1/2})$
Step 4: For the reverse inequality consider a big ball $C \supset X$ and fill $C \setminus X$ by small balls

Relationship with symplectic packing

For (disjoint unions of) balls we have $e_k = O(1)$.
Question: Why does this proof not show $e_k(X) = O(1)$ for all star-shaped X?

Let B_n denote the disjoint union of n equal balls with total volume $\mathrm{vol}(B_n) = 1$. We have

$$\limsup_{k\to\infty} e_k(B_n) \longrightarrow -\infty \quad (n \to \infty)$$

If we can pack the full volume of X and $C \setminus X$ by finitely many balls, we get $e_k(X) = O(1)$.

Symplectic packing

Theorem (Gromov)

$\text{int}\, B^4(a) \coprod \text{int}\, B^4(a) \overset{s}{\hookrightarrow} \mathbb{C}P^2(1) \quad \Leftrightarrow \quad a \leq 1/2$

In particular: We can't pack more than half the volume by two equally sized balls.

Theorem (Packing stability, Biran '99)

Let (M^4, ω) be a closed rational symplectic 4-manifold. Then there exists N_0 such that for all $N \geq N_0$ the full volume of M can be packed by N equal balls.

Generalizations: higher dimension (Buse-Hind), irrational symplectic 4-manifolds (Buse-Hind-Opshtein)

Question: What about finite volume, open symplectic manifolds?

- true for balls, convex toric domains
- false in general (CG-Hind '23)

Question: What about compact symplectic manifolds with smooth boundary?

"New" ingredient

Consider $T^{2n} := (\mathbb{R}/\mathbb{Z})^{2n}$ with $\omega_0 = \sum_i dx_i \wedge dy_i$
For $\alpha \in \mathbb{R}^{2n}$ define rotation

$$R_\alpha : T^{2n} \to T^{2n} \quad R_\alpha(p) := p + \alpha$$

Theorem (Herman)

Suppose that α is Diophantine. Then for every $\varphi \in \mathrm{Ham}(T^{2n})$ sufficiently C^∞ close to id, there exists $\psi \in \mathrm{Ham}(T^{2n})$ such that

$$\varphi \circ R_\alpha = \psi R_\alpha \psi^{-1}.$$

Theorem (Banyaga)

Let (M, ω) be a closed symplectic manifold. Then $\mathrm{Ham}(M)$ is a simple group.

Proof of concept

Theorem (E. '23)

Let $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$ be a Hamiltonian on (S^2, ω). Then

$$c_d(H) = dA^{-1} \int_{\mathbb{R}/\mathbb{Z} \times S^2} H dt \wedge \omega + O(1).$$

(Best previously known error bound is $O(d^{1/2}.)$)

Thank you!