Weak KAM Theory on metric spaces

Roma, 23 May, 2023

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To Antonio, the Master of metric methods

Background on Weak KAM Theory on Lagrangian systems

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This second condition is equivalent to:

(ii') For every $K \geq 0$, we can find a finite constant $C(K)$ such that

$$L(x, v) \geq K\|v\|_x + C(K), \text{ for every } (x, v) \in TM.$$

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$\mathscr{A}_c = \emptyset$ for $c \neq c[0]$ and $\mathscr{A}_{c[0]} \neq \emptyset$

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where $u : M \to \mathbb{R}$ and $c \in \mathbb{R}$. A (viscosity) subsolution of (HJc) is a *Lipschitz* function $u : M \to \mathbb{R}$ such that $H(x, d_x u) \leq c$ for almost every $x \in M$.

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Generalization:
Weak KAM Theory on Metric Spaces

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We will denote by $\mathscr{S}_{\mathrm{sub}}(\phi)$, the set of ϕ-subsolutions.

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2) and 3), i.e. convexity and stability by adding a constant, follow routinely from the Definition of ϕ-subsolutions.

For 4), suppose that $u_i, i \in I$ is a family of subsolutions. We have

$$u_i(y) \leq u_i(x) + \phi(x, y), \text{ for all } x, y \in X \text{ and all } i \in I.$$

Thus

$$\inf_{i \in I} u_i(y) \leq \inf_{i \in I} u_i(x) + \phi(x, y), \text{ for all } x, y \in X.$$

Hence, since ϕ is finite everywhere, the inequality above implies either $\inf_{i \in I} u_i \equiv -\infty$ or $\inf_{i \in I} u_i$ finite everywhere.

Proof.

From the inequality $u(y) - u(x) \leq \phi(x, y)$, for all $x, y \in X$, satisfied by any ϕ-subsolution, we easily obtain

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In the sequel of this work we will assume that X has at least two points.

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Therefore $\phi^\eta(x_0, x_0) = 0$. Since $\eta > 0$, we indeed obtain $x_0 \in \mathscr{A}(\phi)$. □

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To show the equality $u(x) = \inf_{y \in \mathscr{A}(\phi)} u(y) + \phi(y, x)$, we apply the Theorem above to find $y_0 \in \mathscr{A}(\phi)$ such that

$$\begin{aligned} u(x) &= u(y_0) + \phi^{\infty}(y_0, x) \\ &\geq u(y_0) + \phi(y_0, x) \\ &\geq \inf_{y \in \mathscr{A}(\phi)} u(y) + \phi(y, x) \\ &\geq u(x). \quad \square \end{aligned}$$