Counting escape orbits

INDAM-Symplectic Dynamics

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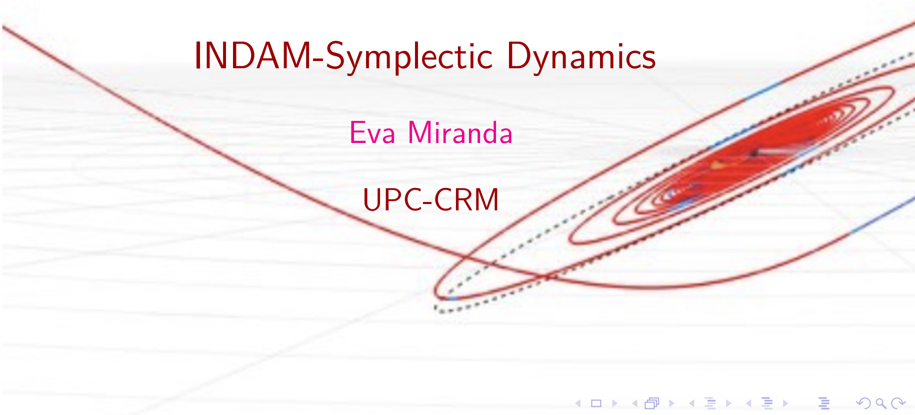


Escape orbits and Singular orbits

Singular periodic orbits are a particular case of *escape orbits* γ, $\gamma \subset M \setminus Z$ such that $\lim_{t\to\infty}\gamma(t) = p$ where p is an equilibrium point in Z (respectively $\lim_{t\to-\infty}\gamma(t) = p$) where Z exhibits some type of singularity.

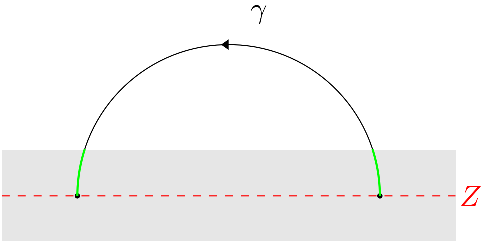


Figure: Singular periodic orbit vs. Escape orbits (in green)

A garden of singular orbits

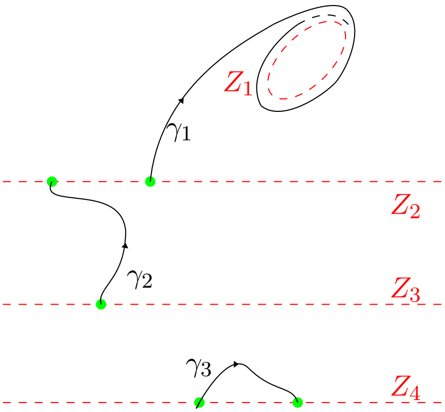
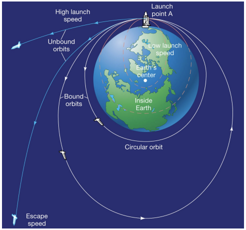
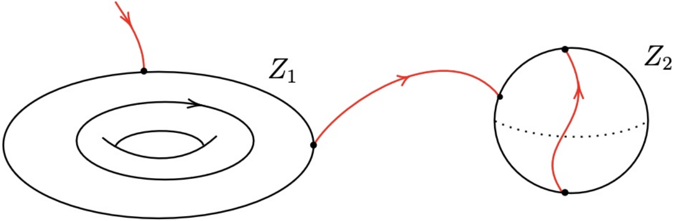


Figure: Different types of escape and singular periodic orbits: γ_1 is a generalized singular periodic orbit, γ_2, γ_3 are singular periodic orbits

Why escape orbits?





1. Some non-compact symplectic manifolds can be compactified as compact singular symplectic manifolds.
2. Singularities often appear as regularization transformations in celestial mechanics.
3. Singular symplectic manifolds model certain manifolds with boundary.

In all these situations periodic orbits may lead to escape orbits or singular periodic orbits in new singular scenarios.

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

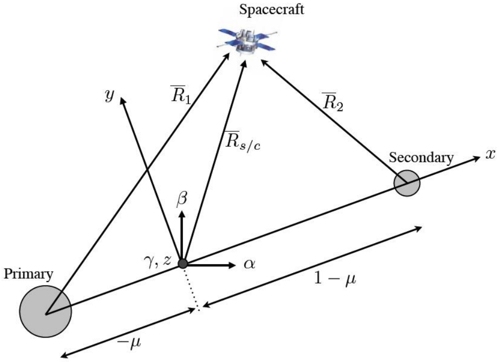


Figure: Circular 3-body problem

An example from Celestial Mechanics: Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|}$, with $q_E = q_E(t)$ the position of the planet with mass $1-\mu$ at time t and $q_M = q_M(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- After a symplectic change to polar coordinates $(r, \alpha, P_r, P_\alpha)$, the symplectic form becomes $\omega = dr \wedge dP_r + d\alpha \wedge dP_\alpha$, and the Hamiltonian is expressed

$$H(r, \alpha, P_r, P_\alpha) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - P_\alpha + U(r, \alpha).$$

- Introduce **McGehee coordinates** $(x, \alpha, P_r =: y, P_\alpha =: G)$, where $r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+$, can be then extended to infinity $(x = 0)$.
- The symplectic structure becomes a singular object

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

Symplectic and contact geometry of these systems

(b^m-symplectic)

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

or (m-folded)

$$\omega = \mathbf{x_1^m dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Contact Geometry

The restriction to $H = ct$ induces a contact structure whenever there exists a *Liouville vector field* is transverse to it. This contact structure may admit singularities.

How are these singularities?

Restricted planar circular 3-body problem

- Time-dependent potential: $U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
- Time-dependent Hamiltonian:
  $H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in \mathbb{R}^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$
- Rotating coordinates $\rightsquigarrow$ Time independent Hamiltonian
  $H(q,p) = \frac{p^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

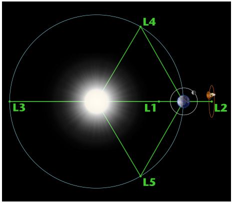


Figure: Lagrange points (Source: NASA/WMAP Science Team)

- H has 5 critical points: L_i Lagrange points $(H(L_1) \leq \cdots \leq H(L_5))$
- Periodic orbits of X_H? Perturbative methods (dynamical systems) or.... contact geometry!

Contact geometry of the restricted 3-body problem

Weinstein conjecture 

The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

- Proved in dimension 3 by Taubes.
- Every Reeb vector field has at least two periodic orbits (Cristofaro-Gardiner and Hutchings 2016).
- Examples in celestial mechanics show that Reeb vector fields generically tend to have infinitely many periodic orbits.
- Every nondegenerate Reeb vector field has either two or infinitely many periodic orbits (Colin, Dehornoy, Rechtman).

Theorem (Albers-Frauenfelder-Van Koert-Paternain)

For any value $c < H(L_1)$, the regularized planar circular restricted three-body problem has a closed orbit with energy c.

Periodic orbits in the singular set-up

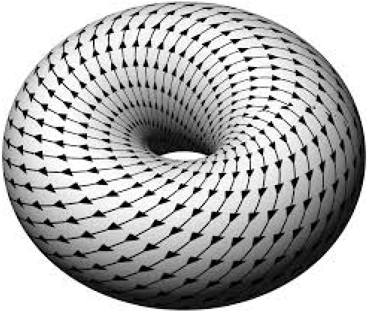
- **What if we consider the b^3-symplectic model? Do we have an associated singular contact structure?**
- **Can we still prove the existence of periodic orbits? Can we localize these periodic orbits with respect to the line at infinity?**
- Can we prove a "two-or-infinite" dichotomy in this case? What about other contact structures?

Zooming in and out on Symplectic/Contact Geometry

Singular forms

A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



b-forms

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

Key point: A b-form of degree k decomposes as:

$$\omega = \alpha \wedge \frac{dz}{z} + \beta, \quad \alpha \in \Omega^{k-1}(M),\ \beta \in \Omega^k(M) \quad d\omega := d\alpha \wedge \frac{dz}{z} + d\beta.$$

- This defines the b-cohomology groups. Mazzeo-Melrose

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

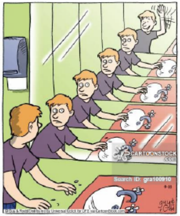
- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.

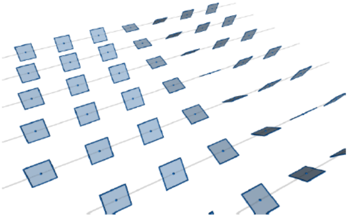
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path methods.
- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$ (so a b-volume form).
- The associated *Reeb vector field* is the unique b-vector field R such that

$$\begin{cases} \iota_R d\alpha = 0 \\ \iota_R \alpha = 1. \end{cases}$$

- The b-tangent bundle can be replaced by other algebroids (E-symplectic) known to Nest and Tsygan.
- A classical generalization is that of the b^m-tangent bundle whose sections are given by vector fields which are tangent to the hypersurface Z to order m.

The Symplectic/Contact mirror "reloaded"





b-Symplectic	**b-Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

Definition of escape orbit

Definition

Let (M, Z, α) be a b-contact manifold. A *singular periodic orbit* is an integral curve $\gamma : \mathbb{R} \to M \setminus Z$ of the b-Reeb vector field such that $\lim_{t\to\pm\infty} \gamma(t) = p_\pm \in Z$. An *escape orbit* is an integral curve such that at least one of the semiorbits has a stationary limit point on Z.

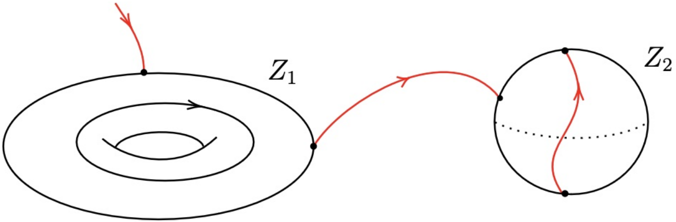


Figure: Examples of an escape orbit (on the left, tending to a point on the critical torus) and two singular periodic orbits. The critical set is a disjoint union of a torus Z_1 and a sphere Z_2, and a b-Reeb orbit on the critical torus is depicted in black.

Attacking the b^m-Weinstein's conjecture

Theorem

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Key point

For $\alpha = u\frac{dz}{z^m} + \beta$ The restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian.

Theorem (The restricted three-body problem)

After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field everywhere transverse to the energy level-sets Σ_c for $c > 0$ and $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are b^3-contact manifolds.

1. *The critical set is a cylinder.*
2. *The Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

Away from Z?

There are compact b^m-contact manifolds (M, Z) of any dimension for all $m \in \mathbb{N}$ without periodic Reeb orbits on $M \setminus Z$.

Example

- $S^3 \subset (\mathbb{R}^4, \omega)$ Construct a b-contact form $\alpha = \iota_X \omega$ on $(\mathbb{S}^3, \mathbb{S}^2)$ by contracting a Liouville vector field X transverse to $\mathbb{S}^3$ given by $X = \frac{1}{2}(x_1\partial_{x_1} + 2y_1\partial_{y_1} + x_2\partial_{x_2} + y_2\partial_{y_2})$.
  The Reeb vector field is $R = \frac{2}{1+y_1^2}(-x_1y_1\partial_{x_1} + x_1^2\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2})$. On $Z = S^2$: rotation, away from Z, no periodic orbits.

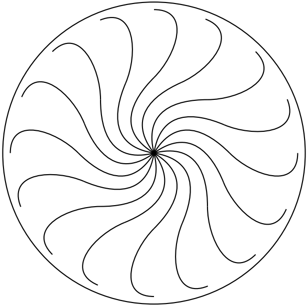
Example

- $(\mathbb{T}^3, \sin\phi\frac{dx}{\sin x} + \cos\phi dy)$. $R_\alpha = \sin\phi\sin x\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}$.
- $Z =$ two disjoint copies of the 2-torus $\mathbb{T}^2$ and the Reeb flow restricted to it is given by $\cos\phi\frac{\partial}{\partial y}$.

Periodic orbits away from Z?

Definition

$(M^3, \xi = \ker \alpha)$ is *overtwisted* if there exists D^2 s.t. $TD \cap \xi$ defines a 1-dimensional foliation given by



A contact manifold that is not overtwisted is called *tight*.

Overtwisted manifolds and periodic orbits

Theorem (Hofer '93)

Let (M^3, α) a closed OT contact manifold. Then there exists a periodic orbit.

Not true for open OT manifolds!

A b^m-contact manifold is overtwisted if there exists an overtwisted disk away from the critical hypersurface Z.

Definition

A b^m-contact form α is $\mathbb{R}^+$-invariant around the critical set if $\alpha = udz + \beta$, where $u \in C^\infty(Z)$ and $\beta \in \Omega^1(Z)$

Singular Hofer

Theorem

Let (M^3, α) be a closed b^m-contact manifold with critical set Z. Assume there exists an overtwisted disk in $M \setminus Z$ and assume that α is $\mathbb{R}^+$-invariant in a tubular neighbourhood around Z. Then there exists

1. *a periodic Reeb orbit in $M \setminus Z$ or/and*
2. *a family of periodic Reeb orbits approaching the critical set Z.*

Furthermore, the periodic orbits are contractible loops in the symplectization.

The proof is an adaptation of Hofer's techniques.

An image is worth more than a thousand words...

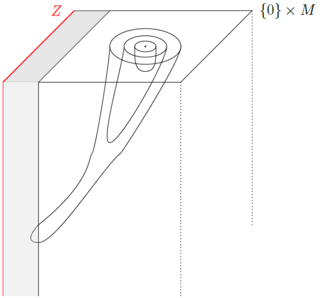
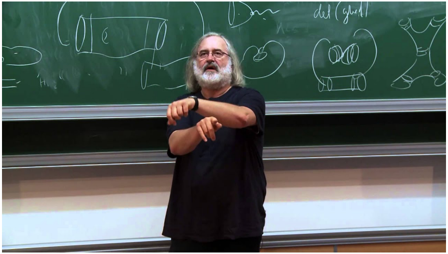


Figure 5.3: Bishop family blowing-up in the $\mathbb{R}^+$-invariant part



More generally

Theorem

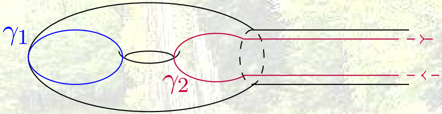
Let (M^3, α) be a $\mathbb{R}^+$-invariant contact manifold that is OT away from the $\mathbb{R}^+$-invariant part. Then there exists a 1-parametric family of periodic Reeb orbits in the $\mathbb{R}^+$-invariant part of M or a periodic Reeb orbit away from the $\mathbb{R}^+$-invariant part.

Mantra

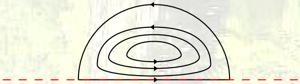
Non-compactness $+$ $\mathbb{R}^+$-invariance $=$ compactness

The singular Weinstein conjecture re-loaded

A true singular Weinstein structures should also admit singular orbits as below:



Or,



Singular Weinstein conjecture

Let (M, α) be a compact b-contact manifold with critical hypersurface Z. Then there exists always a Reeb orbit $\gamma : \mathbb{R} \to M \setminus Z$ such that $\lim_{t\to\pm\infty} \gamma(t) = p_\pm \in Z$ and $R_\alpha(p_\pm) = 0$ (singular periodic orbit).

From $2N$ to infinitely many escape orbits

Theorem

Let α be a b-contact form on a 3-dimensional manifold (M, Z) without boundary, with Z a closed embedded surface in M. Then there exists a b-contact form C^∞-close to α, such that the associated b-Reeb vector field has either

1. *infinitely many escape orbits if $b_1(Z) > 0$, or*
2. *at least $2N$ escape orbits if $b_1(Z) = 0$, where N is the number of connected components of Z.*

Moreover, the set of b-contact forms exhibiting these properties is open in the C^∞-topology.

Proof:

- Write $\alpha = f\frac{dz}{z} + \beta$, where $f \in C^\infty(\mathcal{N}(Z))$ and $\beta \in \Omega^1(\mathcal{N}(Z))$.

We can assume that f restricts to a Morse function on Z

Choose a b-contact form that is C^∞-close to α as

$$\widetilde{\alpha} := (f + \epsilon h)\frac{dz}{z} + \beta, \tag{1}$$

where h is a C^∞-small function such that $(f + \epsilon h)|_Z$ is a Morse function.

- The Reeb vector field $R = gz\frac{\partial}{\partial z} + Y$, where $g \in C^\infty(\mathcal{N}(Z))$ and $Y \in \mathfrak{X}(\mathcal{N}(Z))$ such that $\iota_Y(dz) = 0$.
- The restriction of the smooth 2-form $\omega := fd\beta + \beta \wedge df$ to Z is symplectic $\rightsquigarrow$ at a critical point $p \in Z$ of $f|_Z$, $f(p) \neq 0$.
- Recall $R|_Z$ is a Hamiltonian vector field with respect to $\omega|_Z$, and the exceptional Hamiltonian is given by $H := -f|_Z$. We denote this Hamiltonian vector field by $\overline{R} := R|_Z = Y|_Z$.

Proof:

- At a critical point $p \in Z$ of H, $R_p = \overline{R}_p = 0$. R thus admits at least 2 zeroes (corresponding to the maximum and minimum values of H).
- At a critical point $p \in Z$, the Reeb condition $\alpha(R) = 1$:
  $1 = \alpha(R)|_p = f(p)g(p) + \beta_p(Y_p) = f(p)g(p)$, ⇝g(p)$\neq 0$.
- At a critical point p, the differential of R at p is given by

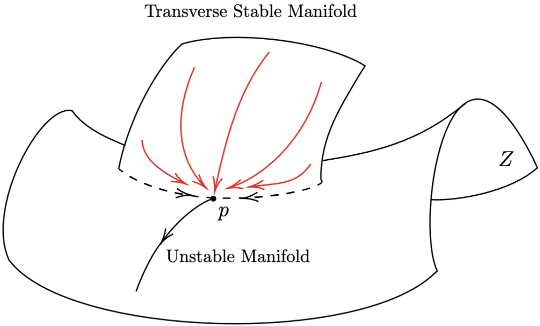
$$dR(p) = \left.\begin{pmatrix} d_p\overline{R} & * \\ 0 & g(p) \end{pmatrix}\right|_p .$$

- Choose a Darboux chart around the critical point in Z such that $\omega = dy \wedge dx$ so that

$$dR(p) = \left.\begin{pmatrix} H_{xy} & H_{yy} & * \\ -H_{xx} & -H_{xy} & * \\ 0 & 0 & g(p) \end{pmatrix}\right|_p .$$

Proof:

- The eigenvalues are λ_+, λ_- and λ_z, where λ_+ and λ_- are eigenvalues of the first 2×2 minor, $\lambda_\pm = \pm\sqrt{-\operatorname{Hess} H(p)}$ (so $\neq 0$ as H is Morse), and $\lambda_z = g(p) \neq 0$.
-
  - $\operatorname{Hess} H(p) < 0$: The critical point of R is hyperbolic and there is a two-dimensional (un)stable manifold at p that is transverse to Z.
  - $\operatorname{Hess} H(p) > 0$: The critical point of R is non-hyperbolic and there is a one-dimensional (un)stable manifold at p that is transverse to Z.



Proof:

- When the transverse invariant manifold is of dimension two, there are infinite many periodic orbits.
- A transverse invariant manifold of dimension one guarantees just two escape orbits (one on each side of Z) with limit point p.
- Let C_k be the number of critical points of H of index k on Z and b_k the k-th Betti number. We use the Morse inequality $C_k \geq b_k(Z)$
  - Case $b_1(Z) > 0$. There is at least one critical point of H of index one (in fact, at least two, because the first Betti number is even), so there is a saddle point and therefore infinitely many escape orbits.
  - Case $b_1(Z) = 0$. Each connected component of the critical set is diffeomorphic to $\mathbb{S}^2$. There are least two escape orbits for each critical point (one on each side of Z) $\rightsquigarrow$ there must be at least $2N$ escape orbits. Note that it can still be that the exceptional Hamiltonian has a saddle point on Z, in which case there would be infinitely many escape orbits.

What about the singular Weinstein conjecture?

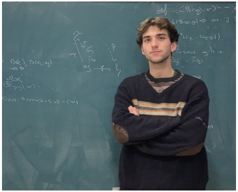
Can we always find singular periodic orbits?

Filling the space with singular bubbles....

We can customize the number of singular periodic orbits.

Theorem (Fontana-McNally, M., Peralta-Salas)

Given any 3-dimensional manifold M for any positive $0 \leq k \leq N$, there exists a b-contact form on M whose critical set Z is composed of N 2-spheres and such that the number of singular periodic orbits is exactly k. Furthermore, each of these spheres has an infinite number of generalized escape orbits converging to them (orbits whose α- or ω-limit sets are on Z).



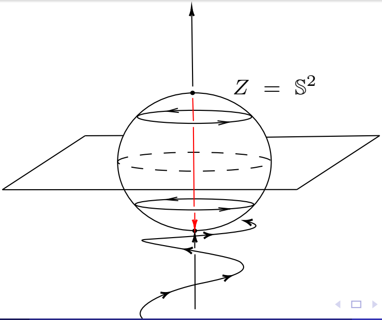
Building blocks

Fontana-M.- Peralta-Salas

The b-manifold $(\mathbb{R}^3, \mathbb{S}^2 = \{\mathrm{r=1}\})$ with $r^2 = x^2 + y^2 + z^2$ with the 1-form

$$\alpha = z(3 + r^2)\frac{rdr}{r^2 - 1} - \frac{r^2 + 1}{2}dz + xdy - ydx$$

is a b-contact form and its Reeb field has exactly one singular periodic orbit.



Key point in the proof

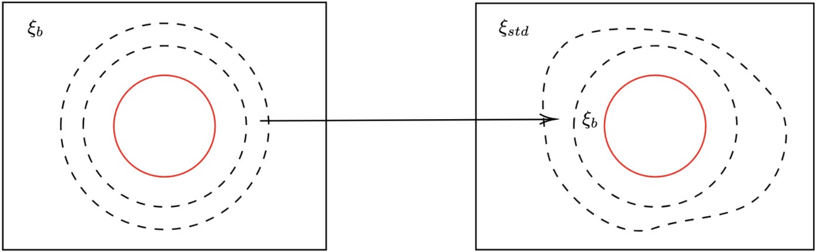
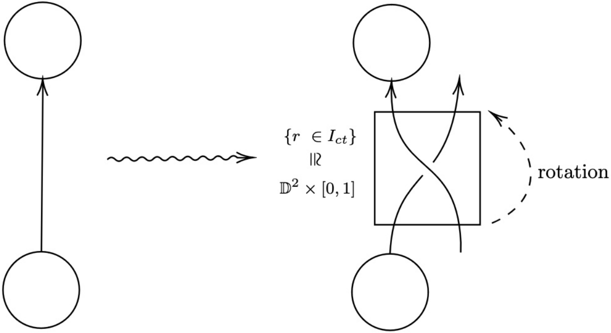


FIGURE 2. Gluing the singular bubble with b-contact structure ξ_b in the standard contact structure of $\mathbb{R}^3$ along an open shell (the area within the dashed annulus). The critical surface is colored red.

Breaking singular periodic orbits

We can perturb our contact structure to eliminate singular periodic orbits



If a 3-dimensional manifold M with b-contact form α has finitely many escape orbits and N singular periodic orbits, then for every $k \leq N$ there is another b-contact form $\tilde{\alpha}$ arbitrarily close to α with exactly k singular periodic orbits.

The b-Hopf vector field

- Consider the standard b-symplectic Euclidean space $(\mathbb{R}^4, \omega)$, with

$$\omega = \frac{dx_1}{x_1} \wedge dy_1 + dx_2 \wedge dy_2.$$

- Construct a b-contact form $\alpha = \iota_X \omega$ on $(\mathbb{S}^3, \mathbb{S}^2)$ by contracting a Liouville vector field X transverse to $\mathbb{S}^3$ given by $X = \frac{1}{2}(x_1 \partial_{x_1} + 2y_1 \partial_{y_1} + x_2 \partial_{x_2} + y_2 \partial_{y_2})$.

$$\alpha = \frac{1}{2}(dy_1 - 2y_1 \frac{dx_1}{x_1} + x_2 dy_2 - y_2 dx_2).$$

- The Reeb vector field is

$$R = \frac{2}{1+y_1^2}(-x_1 y_1 \partial_{x_1} + x_1^2 \partial_{y_1} - y_2 \partial_{x_2} + x_2 \partial_{y_2}). \qquad (2)$$

Since y_1 is bounded on $\mathbb{S}^3$, the factor $\frac{2}{1+y_1^2}$ does not affect the dynamics at all.

The orbits of the b-Hopf vector field on $\mathbb{S}^3$ are either

a) a singular periodic orbit, if passing along the z axis;
b) stationary, if on the north or south pole of the critical $\mathbb{S}^2$;
c) periodic if on a parallel of the critical $\mathbb{S}^2$;
d) a generalized escape orbit otherwise.

In particular, there are only two singular periodic orbits and no other escape orbits.

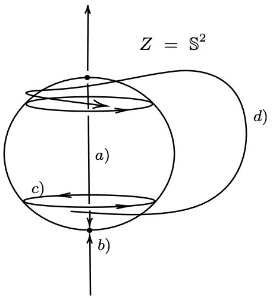


Figure: The four types of orbits depicted. The orbits exiting from the north pole and arriving at the south pole outside of the $\mathbb{S}^2$ are part of the same singular periodic orbit on $\mathbb{S}^3$.

A counterexample to the singular Weinstein conjecture

- The b-Hopf vector field was already shown to have no periodic orbits away from the critical set $Z = \mathbb{S}^2$. It has finitely many escape orbits. The b-Reeb field described initially shares the same dynamical properties. We then break all singular periodic orbits by perturbation without creating new ones.

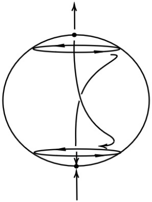


FIGURE 7. Orbits of the perturbed b-Hopf vector field. Note that the singular periodic orbit passing through the z axis becomes a generalized escape orbit after the perturbation.

Fontana-McNally-M.-Peralta-Salas

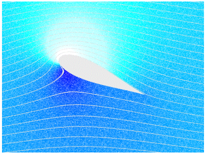
This gives a b-contact form on $(\mathbb{S}^3, \mathbb{S}^2)$ which has no singular periodic orbits and no periodic orbits away from the critical set.


Another mirror…
Reflecting the stars on the sea…
Arnold's dream of establishing a connection between the dynamical complexity of celestial mechanics and of stationary solutions of hydrodynamics:
"Car les écoulements avec curl v = λ v admettent, probablement, des lignes de courant avec une topologie aussi compliquée que celle des orbites en mécanique céleste."

Euler Flows and contact structures

- Euler equations model the dynamics of an **inviscid and incompressible fluid flow**.

$$\begin{cases} \frac{\partial X}{\partial t} + (X \cdot \nabla)X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$

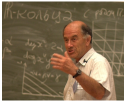
- Beltrami fields: curl$X = fX \quad \operatorname{div} X = 0.$ $\leftrightsquigarrow$ stationary solutions with constant Bernoulli function $B = P + \frac{1}{2}g(X, X)$.
- In terms of $\alpha = \iota_X g$, μ volume form **stationary Euler equations** read

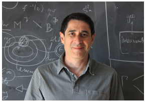
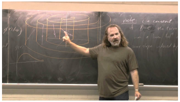
$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$

The contact-Beltrami correspondence

Main trick: If X is non-vanishing rotational Beltrami then $\alpha = \iota_X g$ is contact.

- The Beltrami equation $\leftrightsquigarrow$ $d\alpha = f\iota_X\mu$. Since $f > 0$ and X does not vanish $\rightsquigarrow$ $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.
- X satisfies $\iota_X(d\alpha) = \iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$ $\leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.




- Etnyre-Ghrist: Beltrami fields $\leftrightsquigarrow$ contact structures.
- With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.

A magic mirror

- Weinstein conjecture for Reeb vector fields $\rightsquigarrow$ **periodic orbits for Beltrami vector fields**
- Escape orbits/singular orbits for singular Reeb vector fields $\rightsquigarrow$ **flows of Beltrami fields as escape orbits**

The Beltrami case

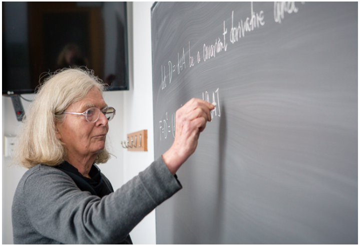
Exact b-metric $\leftrightsquigarrow$ Melrose b-contact forms:

$$g = \frac{dz^2}{z^2} + \pi^* h \qquad (3)$$

with h Riemannian metric on Z.

Theorem (M-Oms-Peralta, "lockdown theorem")

There exists at least $2 + b_1(Z)$ escape orbits for Reeb vector fields of generic Melrose b-contact forms on (M, Z).



Proof: The Beltrami equation $\rightsquigarrow$ the Hamiltonian function associated to (R, Z) is an eigenfunction of the induced Laplacian on Z $\rightsquigarrow$ (Uhlenbeck) generically Morse.

2N or infinity periodic orbits...

Indeed, we can do better (as we do not need to use the mirror)

Theorem (Fontana-M-Oms-Peralta, "after lockdown theorem")

Let (M, Z) be a 3-dimensional b-manifold without boundary, and Z a closed hypersurface in M. There exists a residual set $\widehat{\mathcal{G}_b^k} \subset \mathcal{G}_b^k$ in the space of asymptotically exact b-metrics such that any b-Beltrami vector field on (M, Z, g) with $g \in \widehat{\mathcal{G}_b^k}$, which is not identically zero on Z, has either

1. *infinitely many escape orbits if $b_1(Z) > 0$, or*
2. *at least $2N$ escape orbits if $b_1(Z) = 0$, where N is the number of connected components of Z.*

... coming back to the *2 or infinity* periodic orbits dichotomy...

Outside the Beltrami box