The Toda Lattice and Symplectic Balls

Yaron Ostrover

Tel-Aviv University

Based on a joint work with Vinicius Ramos and Daniele Sepe.

Symplectic Dynamics at INdAM

Rome, May 2023

The Plan of the Talk

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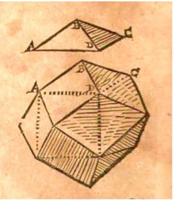

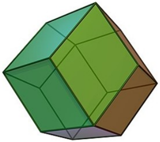

Rhombic Dodecahedron

Epitome Astronomiae Copernicanae [Kepler, 1618]

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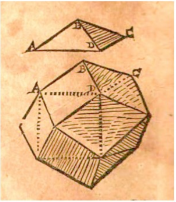

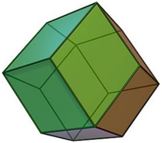

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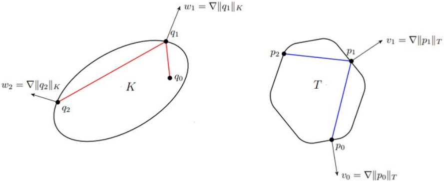

Minkowski Billiard Dynamics

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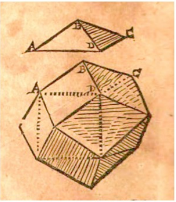

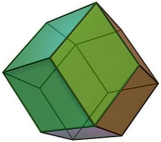

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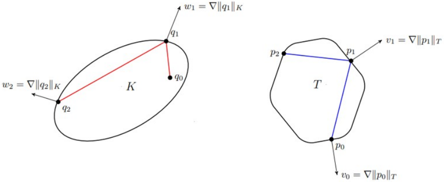

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The Toda Lattice Model (M. Toda, 1967)

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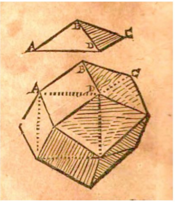

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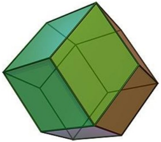

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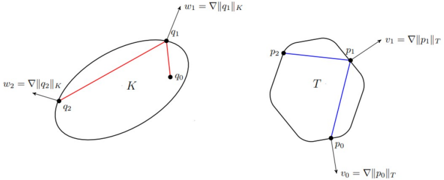

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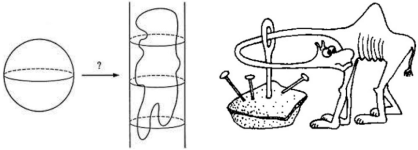

Symplectic Balls

Symplectic Balls and Lagrangian Products

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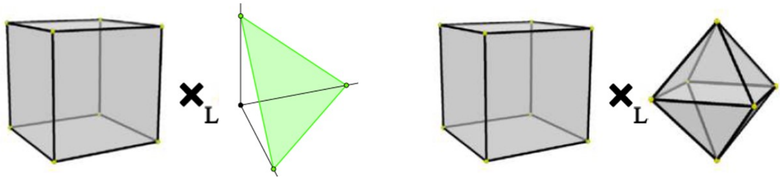

A Symplectic Ball in Disguise

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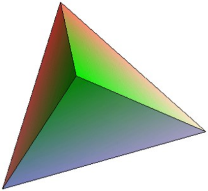
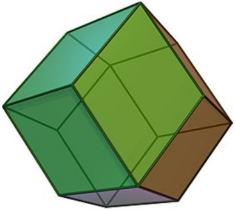
Theorem (O-Ramos-Sepe, 2023):

 $\times_L$  $\cong B$

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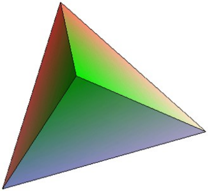

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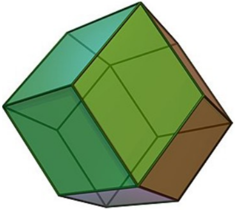


$\cong B$

More precisely:

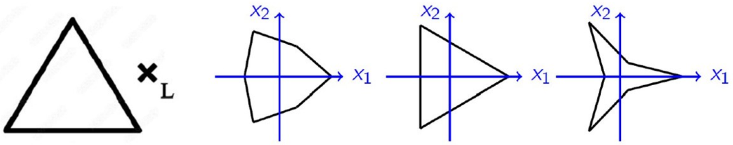
$$\mathcal{S}^n = \left\{ \mathbf{q} \in \mathbb{R}^{n+1} \mid \sum_i q_i = 0,\ q_i - q_{i+1} < 1 \text{ for all i} \right\},$$

$$\mathcal{R}^n = \left\{ \mathbf{p} \in \mathbb{R}^{n+1} \mid \sum_i p_i = 0,\ \max_i p_i - \min_i p_i < 1 \right\}.$$

Theorem (O-Ramos-Sepe, 2023): $\mathcal{S}^n \times \mathcal{R}^n$ *is symplectomorphic to a ball.*

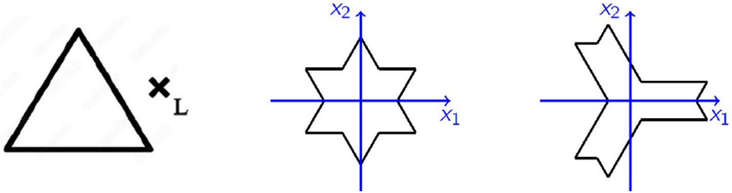
Other Examples

The Ellipsoid E(a,b):



$a < b < 2a$ $b = 2a$ $b > 2a$

The Polydiscs P(1,1) and P(1,3):



Lagrangian Products and Minkowski Billiards

Assume $K \subset \mathbb{R}^n_q$ and $T \subset \mathbb{R}^n_p$ are convex sets.

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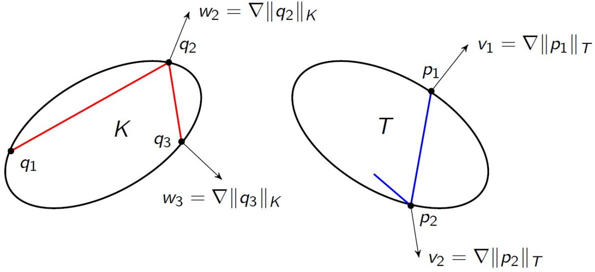

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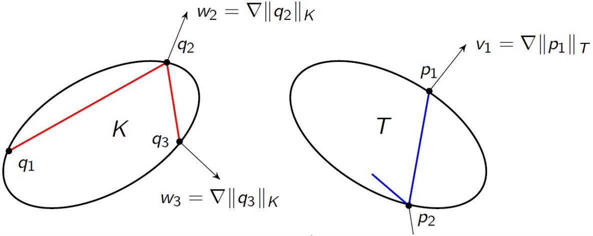

The Reeb flow of the Lagrangian pair (K,T) corresponds to billiard dynamics in K w.r.t the “geometry induced by T”.

A Special Billiard Dynamics in the Simplex

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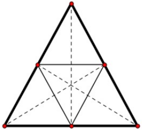
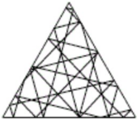


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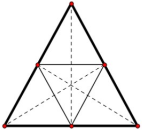
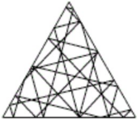
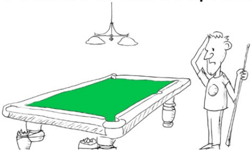
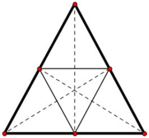




Answer (in dim 2): the hexagonal geometry.



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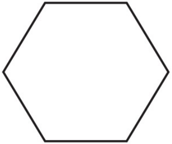

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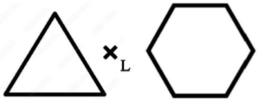

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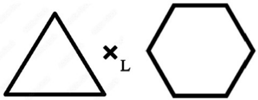

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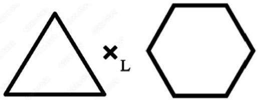

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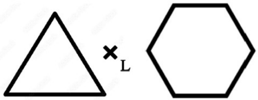

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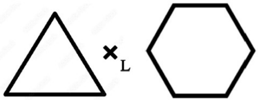

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Serendipity: this can be done via the Toda lattice model.

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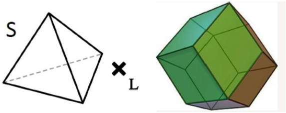

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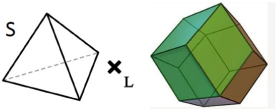

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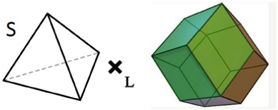

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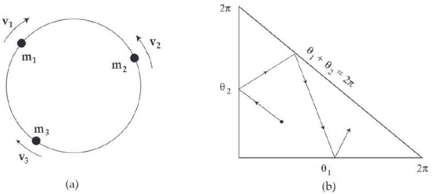
Idea #2: Billiard in the Simplex ⟷ elastic collisions of particles on a finite (frictionless) ring.



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Idea #2: Billiard in the Simplex $\longleftrightarrow$ elastic collisions of particles on a finite (frictionless) ring.

Idea #3: such systems of collisions of particles are related to the (completely integrable) Toda lattice systems:

$$H(p,q) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{q_j - q_{j+1}}$$

A Deformation of the Toda Lattice

$$H_c(\mathbf{q},\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 + e^{-c}\sum_{i=1}^{n+1} e^{c(q_i - q_{i+1})}.$$

As $c \to \infty$, the potential converges to

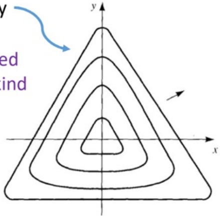
$$\begin{cases} 0, \text{ if } q_i - q_{i+1} < 1, \text{ for all } i = 1, \ldots, n, \\ \infty, \text{ if } q_i - q_{i+1} > 1, \text{ for some } i = 1, \ldots, n. \end{cases}$$

The flow of X_{H_c} converges to the billiard flow in

$$\{\mathbf{q} \in \mathbb{R}^n \mid q_i - q_{i+1} < 1, \text{ for all } i = 1, \ldots, n\}.$$

The level lines of the potential energy

Billiard in an equilateral triangle can be obtained from a three particles Toda lattice with some kind of limiting procedure



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$$H(L) = \frac{1}{2}\mathrm{Trace}(L^2)$$

$$\dot{L} = BL - LB := [B, L]$$

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & \cdots \\ b_1 & a_2 & b_2 & & \cdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix} \quad B = \begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots \\ -b_1 & 0 & b_2 & & \cdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & -b_{n-1} & 0 \end{pmatrix}$$

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Note: the Spectrum of L is invariant under the flow.

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$$\dot{L} = BL - LB := [B, L]$$

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & \cdots \\ b_1 & a_2 & b_2 & & \cdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix} \quad B = \begin{pmatrix} 0 & b_1 & 0 & \cdots & \cdots \\ -b_1 & 0 & b_2 & & \cdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & -b_{n-1} & 0 \end{pmatrix}$$

Lax pair.

Note: the Spectrum of L is invariant under the flow.

Theorem [Toda, Flaschka–McLaughlin, Moser.....]: The system is completely integrable, and there are global action angle coordinates.

The Action-Angle Coordinates

There is a difference equation related to L:

$$a_{k-1}y_{k-1}(\lambda) + b_k y_k(\lambda) + a_k y_{k+1}(\lambda) = \lambda y_k(\lambda).$$

We can associate to it a discriminant $\Delta(\lambda)$.

Theorem (Flaschka–McLaughlin, van Moerbeke, Moser)

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2n+2}$ be the roots of $\Delta(\lambda)^2 - 4$.
Then the action coordinates $\phi = (I_1, \ldots, I_n)$ are given by

$$I_i = (n+1)\int_{\lambda_{2i}}^{\lambda_{2i+1}} \cosh^{-1}\left|\frac{\Delta(\lambda)}{2}\right| d\lambda,$$

and they induce a symplectomorphism

$$\Phi : \left\{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n+1} \mid \sum_i q_i = \sum_i p_i = 0\right\} \longrightarrow \mathbb{R}^{2n}.$$

The Main Result

Let $\rho : \mathbb{R}^{n+1} \to [0,\infty)^n$ defined by

$$\rho(p_1,\dots,p_{n+1}) = (p_{\sigma(1)} - p_{\sigma(2)},\dots,p_{\sigma(n)} - p_{\sigma(n+1)}),$$

where $\sigma \in S_{n+1}$ such that $p_{\sigma(1)} \geq p_{\sigma(2)} \geq \cdots \geq p_{\sigma(n+1)}$.

Theorem (O-Ramos-Sepe, 2023):

- *If A is symmetric, then for every $\varepsilon > 0$,*

$$(1-\epsilon)\Delta^n \times_L A \hookrightarrow \mathbb{X}_{(n+1)\rho(A)} \hookrightarrow (1+\epsilon)\Delta^n \times_L A.$$

- *If A is balanced, then $\Delta^n \times_L A$ is symplectomorphic to $\mathbb{X}_{(n+1)\rho(A)}$.*

Open Questions

For which polytopes P is the product $\Delta^n \times P$ symplectomorphic to a ball?

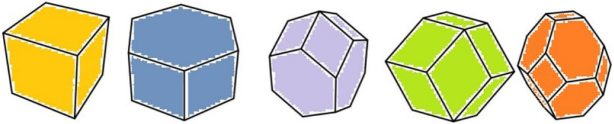


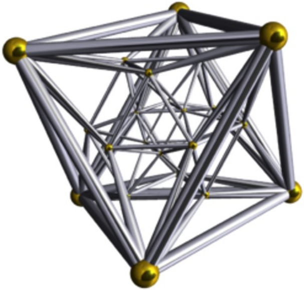
Figure: The Fedorov polyhedra

What about other root-systems?

Symplectic Packing Problems?

Is this a Symplectic Ball?

The Octacube (24-cell).



Equality in Viterbo's Conjecture and "combinatorially Zoll" (Chaidez-Hutchings)

Thank you for your attention!

Any questions?

The Rhombic Dodecahedron as a Torus

Configuration spaces of hard spheres

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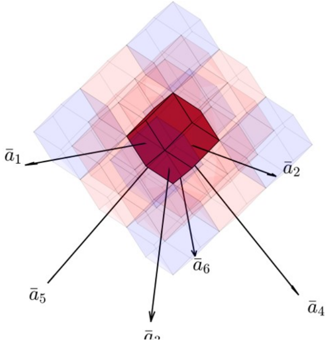


FIG. 2: A 3-torus $\mathbb{T}^3$ is obtained by identifying opposite faces of a rhombic dodecahedron. The vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are respectively the unit directions in the x and y axes. The remaining four $\mathbf{a}_i$ pass through the centers of the lower faces.

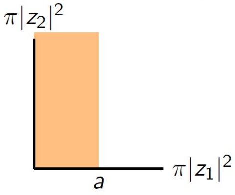
Toric domains

Definition

A toric domain $\mathbb{X}_\Omega \subset \mathbb{C}^n$ is a set of the form $\mathbb{X}_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset [0,\infty)^n$ is an open set and

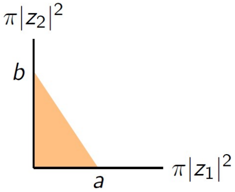
$$\mu : \mathbb{C}^n \to [0,\infty)^n \quad \mu(z_1,\ldots,z_n) = (\pi|z_1|^2,\ldots,\pi|z_n|^2)$$

Example (Cylinder)



$$Z(a) := \{(z_1,z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \le a\}$$

Example (Ellipsoid)



$$E(a,b) := \left\{(z_1,z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1\right\}$$

The Arnold-Liouville Theorem

The Arnold-Liouville theorem

Fix (M^{2n}, ω) and let $F = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ such that $\{H_i, H_j\} = 0$ for all i, j.

- If $c \in \mathbb{R}^n$ is a regular value of F and $F^{-1}(c)$ is compact and connected, then $F^{-1}(c) \cong \mathbb{T}^n$.
- Let U be an open set such that $F(U)$ is simply-connected and does not contain critical values. Then there exists a diffeomorphism $\phi : F(U) \to \Omega$ and a symplectomorphism $\Phi : U \to \mathbb{X}_\Omega$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathbb{X}_\Omega \\ \Big\downarrow F & & \Big\downarrow \mu \\ F(U) & \xrightarrow{\phi} & \Omega \end{array}$$

- The map ϕ can be obtained by action coordinates:

$$\phi(c) = \left(\oint_{\gamma_1^c} \lambda, \ldots, \oint_{\gamma_n^c} \lambda \right).$$