Bézout's theorem and topological persistence

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with Lev Buhovsky, Jordan Payette, Iosif Polterovich, Egor Shelukhin and Vukašin Stojisavljević

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Tool - persistence modules and barcodes: convenient algebraic/combinatorial tool for book-keeping information on oscillation and topology of (sub)level sets of functions on manifolds.

Barcodes

Edelsbrunner, Harer, Carlsson,... Last decade in the context of topological data analysis.

Barcode $\mathcal{B} = \{I_j, m_j\}$-finite collection of intervals I_j with multiplicities m_j, $I_j = (a_j, b_j]$, $a_j < b_j \leq +\infty$.

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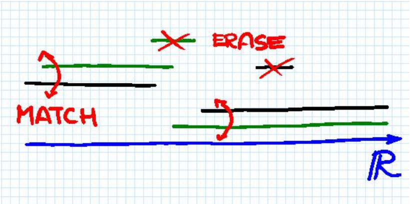
Bottleneck distance between barcodes: $\mathcal{B}, \mathcal{C}$ are δ-matched, $\delta > 0$ if after erasing some intervals in $\mathcal{B}$ and $\mathcal{C}$ of length $< 2\delta$ we can match the rest in 1-to-1 manner with error at most δ at each end-point.

$$d_{bot}(\mathcal{B}, \mathcal{C}) = \inf \delta \ .$$

Figure: Matching



Persistence modules

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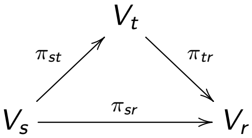
Persistence module: a pair (V, π), where V_t, $t \in \mathbb{R}$ are $\mathcal{F}$-vector spaces, $\dim V_t < \infty$, $V_s = 0$ for all $s \ll 0$.

$\pi_{st} : V_s \to V_t$, $s < t$ linear maps: $\forall s < t < r$



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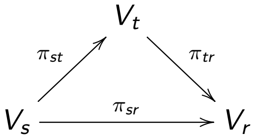
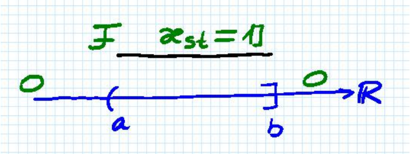


Regularity: For all but finite number of jump points $t \in \mathbb{R}$, there exists a neighborhood U of t such that π_{sr} is an isomorphism for all $s, r \in U$. Extra assumption ("semicontinuity") at jump points.

Structure theorem

Interval module $(\mathcal{F}(a,b],\kappa)$, $a\in\mathbb{R}$, $b\in\mathbb{R}\cup+\infty$:
$\mathcal{F}(a,b]_t=\mathcal{F}$ for $t\in(a,b]$ and $\mathcal{F}(a,b]_t=0$ otherwise;
$\kappa_{st}=1\!\!1$ for $s,t\in(a,b]$ and $\kappa_{st}=0$ otherwise.

Figure: Interval module



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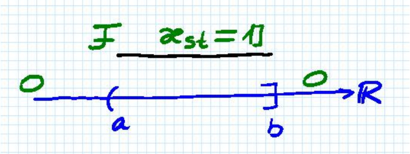

Structure theorem: For every persistence module (V,π) there exists unique barcode $\mathcal{B}(V) = \{(I_j, m_j)\}$ such that $V = \oplus \mathcal{F}(I_j)^{m_j}$.

Persistence in Morse theory

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H_*-homology with coefficients in a field.
Persistence morphisms are induced by the inclusions of sublevels
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Stability Theorem (Cohen-Steiner,Edelsbrunner,Harer, 2007)
$||f|| := \max |f|$-uniform norm
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"Long" bars: $N_\delta(f)$ - number of bars in $\mathcal{B}(f)$ of length $> \delta$.
Cohen-Steiner-Edelsbrunner-Mileyko (2010)

Long bars vs. Sobolev norms

Theorem[BP^3S^2], 2022

$N_\delta(|f|) \le C_1 \delta^{-n/k} ||f||_{k,p}^{n/k} + C_2, \ \forall \delta > 0$

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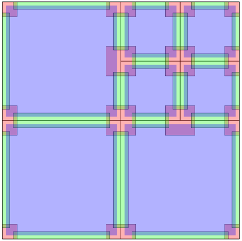
Symplectic digression: Floer persistence modules P.-Shelukhin (2014). N_δ-count related to entropy Cineli-Ginzburg-Gurel (2021), G-G-Mazzucchelli (2022)

Step 1. Approximate by polynomials on small cubes, use Milnor's bound (1964) $\sharp(\text{critical points}) \leq \deg^{\dim}$, and Morse theory. Cf. Yomdin, innovation: multiscale/stopping time.



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Cf. Amplitudes (Giunti, Nolan, Otter, Waas, 2021) - mind 2δ.

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Example: On sphere S^n with round metric,
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Random setting: Expectation $\approx \lambda^{n/2}$ (Gichev, 2009). Similar bound on certain homogeneous Riemannian manifolds (Akhiezer-Kazarnovskii, 2017).

Persistent intersection count: $Z_f := \{f = 0\}$
$z_0(f, \delta) = \dim \operatorname{Im}(H_0(Z_f) \to H_0(\{|f| < \delta\}))$

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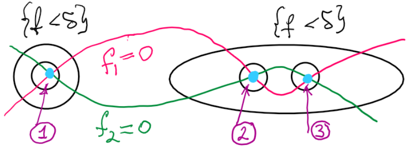

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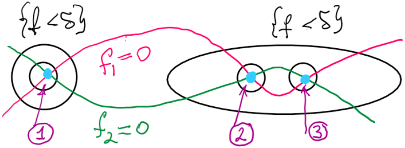

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Theorem $[BP^3S^2]$, 2022 Let $k > n/2$ be an integer, $\delta > 0$.

$$z_0(f, \delta) \leq \frac{C_1}{\delta^{n/k}}(\lambda + 1)^{\frac{n}{2}} + C_2,$$

where C_1, C_2 depend on n, k and metric.

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Resolution: replace the notion of the degree of a polynomial.

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- Let $\zeta(f,r)$ be the number of zeros of an entire function $f : \mathbb{C} \to \mathbb{C}$ inside a ball B_r, $f(0) \neq 0$. Then, for $a > 1$, $\zeta(f,r) \leq C \log \mu(f, ar)$ $\forall r > 0$, where C- positive constant depending on a and $f(0)$.

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In Example above ζ and $\log \mu$ grow linearly in r.

Cornalba-Shiffman Example (1972)

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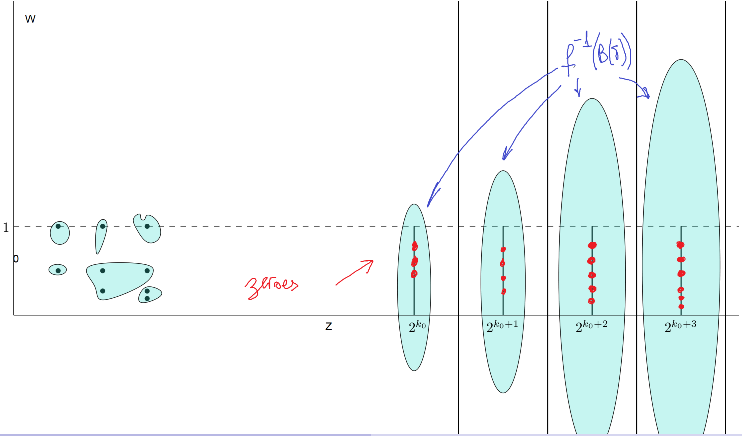
Griffiths: *"This is the first instance known to this author when the analogue of a general result in algebraic geometry fails to hold in analytic geometry."*



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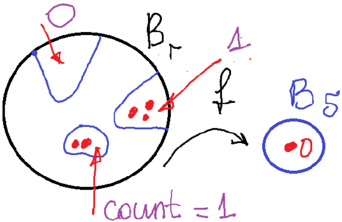
Coarse zero count:

$$\zeta(f, r, \delta) = \dim Im(H_0(\{f = 0\} \cap B_r) \to H_0(\{|f| < \delta\} \cap B_r)$$

This is the number of connected components of the set $f^{-1}(B_\delta) \cap B_r$ which contain zeros of f.



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Question: Is the power $2n - 1$ at log in Theorem sharp?

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Look at the cancellations. Q.E.D.

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Conclude (lemma):
$N_{2\delta}(V) \leq N_0(V') \leq N_0(U') + N_0(W') = N_\delta(U) + N_\delta(W)$. QED.

The End

THANK YOU!