# Symplectic almost squeezings of $B^4$

joint with **Emmanuel Opshtein**

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**Gromov** : $B^4(a) \overset{s}{\hookrightarrow} Z^4(1) = D^2(1) \times \mathbb{R}^2$ only for $a \leq 1$

**Question** : For $a > 1$, what is the "smallest" set $S(a) \subset B^4(a)$ such that

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**Theorem** $\geq 2$ (**Sackel–Song–Varolgunes–Zhu 2021**)
The Minkowski-dimension of $S(a)$ must be $\geq 2$.

**Theorem** $\leq 2$ : $\dim S(a) = 2$ works for $a < 3$ :
**Sackel–Song–Varolgunes–Zhu 2021** for $a \leq 2$ :

$$B^4(2) \setminus \{(x_1, x_2) = 0\} \overset{s}{=} E(1,4) \setminus \{(x_1, y_1) = 0\}$$

**Brendel 2022** : for $a < 3$

**Question** : $a_{\text{crit}} = 3$ or $a_{\text{crit}} = \infty$ ?

**Main theorem** : $a_{\text{crit}} = \infty$ : *Take $d$ even. Then*

$$\boxed{B^4(d/2) \setminus \Delta_d \overset{s}{\hookrightarrow} Z^4(1)}$$

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**Biran 2001** : take the finite group

$$G_d = \left\{ \begin{pmatrix} \xi^k & 0 \\ 0 & \xi^\ell \end{pmatrix} \mid \xi = e^{2\pi i/d} \right\}$$

of order $d^2$.

$$\boxed{\Delta_d := G_d \cdot \mathbb{R}^2(x_1, x_2)}$$

# Corollaries

## 1. Capacity killing

For all normalized symplectic capacities $c$ :

$$c_G \leq c \leq c^Z$$

Biran showed

$$c_G(B^4(1) \setminus \Delta_d) \leq 1/d$$

but the main theorem shows

$$c(B^4(1) \setminus \Delta_d) \leq c^Z(B^4(1) \setminus \Delta_d) \leq 2/d.$$

## 2. Lagrangian intersection with $\Delta_d$

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**3. Reeb chords to $\Lambda \cup \Delta_d$**

Let $U \subset B^4(1)$ be starshaped, $\Lambda \subset \partial U \setminus \Delta_d$ a Legendrian knot.

Then there exists a Reeb chord from $\Lambda$ to $\Lambda \cup \Delta_d$ of length $T \leq \frac{2}{d}$.

# Variations

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**2. Main Theorem'** *For every $\varepsilon > 0$ there exists a symplectic embedding*

$$B^4(1) \setminus \Delta_d \overset{s}{\hookrightarrow} (1+\varepsilon)\, E\,(1/d, d)$$

*for $d$ sufficiently large.*

**Corollary** *Let $(M^4, \omega)$ be a closed symplectic manifold of volume $1/2$. Then for every $\varepsilon > 0$ there exists $d$ such that*

$$B^4(1-\varepsilon) \setminus \Delta_d \overset{s}{\hookrightarrow} (M, \omega).$$

**Sketch of proof** of

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$(M^4, \omega)$ closed

Polarization : $\Sigma \subset M$ symplectic surface such that

$$[\Sigma] = \mu \, \mathrm{PD}[\omega], \quad \mu > 0$$

Then $\omega = d\lambda$ on $M \setminus \Sigma$

Liouville vector field $X_\lambda$ : $\quad \iota_{X_\lambda} d\lambda := \lambda$

Get **two** decompositions of $M$ :

1. $M = \Sigma \cup M \setminus \Sigma$

   hypersurface + Liouville domain

2. $M = \mathrm{SDB}(M, \Sigma, \lambda) \cup \Delta$

   **disc bundle over $\Sigma$** + **isotropic skeleton**

   **basin of attraction** + **unstable manifold**

**Examples** on $(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}})$

**1.** $\Sigma = \mathbb{C}\mathrm{P}^1$

$$\text{with} \quad \lambda = \sum_j x_j dy_j - y_j dx_j = \sum_j R_j \, d\theta_j \quad (R_j := r_j^2)$$

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**2.** $\Sigma_d = \left\{ z_0^d + z_1^d + z_2^d = 0 \right\}, \quad [\Sigma_d] = d \, \mathrm{PD}[\omega_{\mathrm{FS}}]$

$$\text{with} \quad \lambda_d = -d^c \log \frac{\left| z_0^d + z_1^d + z_2^d \right|}{(|z_0|^2 + |z_1|^2 + |z_2|^2)^2}$$

have $\boxed{\Delta = \overline{\Delta}_d = G_d \cdot \mathbb{R}\mathrm{P}^2}$

Since $X_\lambda|_{\mathbb{C}\mathrm{P}^1}$ is tangent to $\mathbb{C}\mathrm{P}^1$ : $\quad B^4 \setminus \Delta_d \overset{s}{=} \mathrm{SDB}(\mathring{\Sigma}_d)$

Polarize **also** $S^2 \times S^2$

For this : Use **singular** polarization (**Opshtein**) :

$$\Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_k, \qquad \Sigma_i \cap \Sigma_j \quad \omega\text{-orthogonal}$$

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In this setting, Opshtein showed :

**One can choose the Liouville form $\lambda$ on $M \setminus \bigcup_j \Sigma_j$ such that $X_\lambda$ has a normal form near $\Sigma_j$, determined by $\mu_j$, pointing inwards**

## **Examples** on $S^2(1) \times S^2(b)$

**1.** $\Sigma_1 = (S^2(1), \mu_1 = b), \quad \Sigma_2 = (S^2(b), \mu_2 = 1)$

$$\text{with} \quad \lambda = (R_1 - 1)\, d\theta_1 + (R_2 - b)\, d\theta_2$$

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**2.** $\Sigma_0 = \Sigma_{m,n}$ : smoothing of $m$ $A$-spheres and $n$ $B$-spheres

$$\begin{aligned} \mathrm{PD}[\omega_{1,b}] &= bA + B \\ &= \mu_0 \, (\underbrace{mA + nB}_{[\Sigma_{m,n}]}) + (b - \mu_0 m)\, A + (1 - \mu_0 n)\, B \end{aligned}$$

So : **have** singular polarization of $S^2(1) \times S^2(b)$ if

$$\mu_0 m < b \quad \text{and} \quad \mu_0 n \leq 1$$

**Want** : $$B^4(d/2) \setminus \Delta_d \overset{s}{=} \mathrm{SDB}\left(\mathring{\Sigma}_d, \mu = \tfrac{1}{2}\right) \overset{s}{\hookrightarrow} \mathrm{SDB}\left(\Sigma_{m,n}, \mu_0 = \tfrac{1}{2}\right)$$

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$$\begin{aligned} \text{genus}(\Sigma_d) &\leq \text{genus}(\Sigma_{m,n}) \\ \text{area}(\Sigma_d) &< \text{area}(\Sigma_{m,n}) \\ \tfrac{1}{2}\, m &< b \\ \tfrac{1}{2}\, n &\leq 1. \end{aligned}$$

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With $n = 2$, this becomes :

$$\begin{aligned} \tfrac{1}{2}(d-1)(d-2) &< m-1 \\ \tfrac{1}{2}d^2 &< m+2b \\ \tfrac{1}{2}\,m &< b. \end{aligned}$$

OK for $m = b$ large