BARCODE ENTROPY OF GEODESIC FLOWS

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Abstract. We introduce and study the barcode entropy for geodesic flows of closed Riemannian manifolds, which measures the exponential growth rate of the number of not-too-short bars in the Morse-theoretic barcode of the energy functional. We prove that the barcode entropy bounds from below the topological entropy of the geodesic flow and, conversely, bounds from above the topological entropy of any hyperbolic compact invariant set. As a consequence, for Riemannian metrics on surfaces, the barcode entropy is equal to the topological entropy. A key to the proofs and of independent interest is a crossing energy theorem for gradient flow lines of the energy functional.

Contents

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1. Introduction

We introduce and study the *barcode entropy* of geodesic flows of closed Riemannian manifolds, an entropy-type invariant based on the Morse theory of the energy functional on the free loop space. Intuitively, the barcode entropy measures the exponential growth rate of the "noise" in the Morse-theoretic persistence module associated to the energy functional. Our main results show that barcode entropy is closely related to topological entropy and that, in dimension two, the two invariants of the metric are actually equal. This work can be viewed as an analogue for geodesic flows of the constructions and results for compactly supported Hamiltonian diffeomorphisms from [ÇGG21] by Çineli and the first two authors. Nevertheless, our arguments are technically very different, and the two papers are independent of each other. In fact, barcode entropy is a very general concept which can be adopted in a variety of frameworks, and the setting of geodesic flows is at least as natural and perhaps even more natural than that of Hamiltonian diffeomorphisms.

Let us now give an overview of our results and discuss the broader context. Consider the energy functional $\mathcal{E}\colon \Lambda \to [0,\infty)$ on the free loop space Λ of a closed Riemannian manifold (M,g), whose non-trivial critical points are the closed geodesics. The homology of the sublevel sets $\{\mathcal{E} < c^2\} \subset \Lambda$ with coefficients in some field $\mathbb{F}$ naturally forms a persistence module, which has an associated barcode. We denote by $b_{\epsilon,c}$ the number of bars of size at least $\epsilon > 0$ and intersecting the interval $[0,c]$. The ϵ-barcode entropy $\hbar_\epsilon$ is the exponential growth rate of the function $c \mapsto b_{\epsilon,c}$, and the barcode entropy $\hbar = \hbar(g;\mathbb{F})$ is obtained by taking the limit as $\epsilon \to 0^+$; see Definition 2.1.

Two other notions of entropy associated with the geodesic flow $\phi_t\colon SM \to SM$ of (M,g) are its topological entropy $h_{\mathrm{top}} = h_{\mathrm{top}}(g)$, i.e., the topological entropy of the time-one map ϕ_1, and the volume-growth entropy $h_{\mathrm{vol}} = h_{\mathrm{vol}}(g)$ defined here as the exponential growth rate of the function $t \mapsto \mathrm{vol}(\mathrm{graph}(\phi_t))$. (Note that this is different from other more standard notions of volume entropy.) The celebrated Yomdin theorem, [Yom87], implies that $h_{\mathrm{vol}} \le h_{\mathrm{top}}$. Our first main result, Theorem A, asserts that

$$\hbar \le h_{\mathrm{vol}}.$$

In particular, since the topological entropy of any (smooth) geodesic flow is always finite, we infer that the barcode entropy $\hbar$ is finite as well.

The second main result of the paper, Theorem B, guarantees that the barcode entropy is a non-trivial invariant. Namely, $\hbar \ge h_{\mathrm{top}}(I)$ for any hyperbolic compact invariant subset $I \subset SM$, where $h_{\mathrm{top}}(I)$ denotes the topological entropy of the restricted geodesic flow $\phi_t|_I$. As a consequence, when $\dim(M) = 2$,

$$\hbar = h_{\mathrm{vol}} = h_{\mathrm{top}}.$$

This is Corollary C which follows from Theorems A and B and the results from [Kat80, LY12, LS19] asserting that the topological entropy of any flow on a 3-dimensional closed manifold can be approximated by the topological entropy of a suitable hyperbolic compact invariant subset. Surprisingly, the identity $h_{\mathrm{vol}} = h_{\mathrm{top}}$ for Riemannian closed surfaces appears to be new, although it is reminiscent of an identity due to Mañé, [Mn97].

The central new feature of our approach distinguishing it from many other results on topological entropy of geodesic flows via closed geodesics (see, e.g., [Din71, Kat82, Pat99] and references therein) is that the topology of M plays no role here. The barcode entropy can be positive regardless of the growth of $\pi_1(M)$ or $H_*(\Lambda)$, e.g., even when M is the sphere or a torus. For instance, by Theorem B, this is the case when ϕ_t has a localized hyperbolic set with positive entropy. However, exponential growth of the fundamental group and, in many instances, of the homology implies exponential growth of the function $c \mapsto b_{\epsilon,c}$ (for any $\epsilon > 0$) and thus Theorem A generalizes some of these results. Overall, without Theorems A and B, it is *a priori* unclear if the Morse complex of $\mathcal{E}$ carries enough information, and if it does how to extract it, to detect positive topological entropy in the absence of homological or the fundamental group growth.

Likewise, our approach is different from modern generalizations in, e.g., [AASS21, Alv16, Alv19, ACH19, AM19, ADMM22, AP20, MS11], of these classical results, relating topological entropy of Reeb flows to their Floer theoretic invariants (e.g., symplectic or contact homology). Again, the contact topology of the underlying manifold – SM in our case – is central to those results but essentially immaterial to Theorems A and B. There is however some overlap between methods used, e.g., in [MS11, Mei18] and in the present paper.

One should also keep in mind that in the definition of barcode entropy one cannot replace the bar count $b_{\epsilon,c}$ by the number of closed geodesics with energy less than c^2, i.e., the number of critical circles of the energy functional $\mathcal{E}$ in the sublevel set $\{\mathcal{E} < c^2\}$. Indeed, the number of closed geodesics of energy at most c^2 may grow arbitrarily fast in c, possibly super-exponentially, even for bumpy metrics (see [BP96], and also [Asa17, Kal99] for results on a similar phenomenon in the Hamiltonian setting), whereas the barcode entropy is always finite. On the technical level, the key point is that $b_{\epsilon,c}$ is stable under small perturbations of the energy functional $\mathcal{E}$, while the count of critical circles is not.

The proof of Theorem A employs an analogue for geodesic flows of Lagrangian tomographs from [ÇGG21, ÇGG22b], a tool from integral geometry reminiscent of the constructions in [APF98, GS94, GS05]. We prove Theorem B by an argument of different nature, based on a result of independent interest. This is Theorem D which, in particular, implies that every closed geodesic in a locally maximal hyperbolic compact invariant set $I \subset SM$ gives rise to a bar of length at least δ, for some constant $\delta > 0$ depending only on I. In turn, the proof of Theorem D ultimately relies on a uniform crossing energy bound, Proposition 6.6, which is a Morse-theoretic counterpart of the crossing energy bound for Hamiltonian diffeomorphisms and Floer cylinders from [GG14, GG18]; see also [All22] where the lower bound is proved using generating functions. However, Proposition 6.6 is not a consequence of these results; its proof is technically and conceptually quite different and hinges on finite-dimensional approximations.

It is worth emphasizing that in spite of obvious parallels, Theorems A and B do not follow from their counterparts for compactly supported Hamiltonian diffeomorphisms, [ÇGG21, Thms. A and B]. While there are ways to study geodesic flows via Floer theory by encapsulating the geodesic flow ϕ_t into an appropriate Hamiltonian diffeomorphism of T^*M (see [AS06, SW06, Vit99]), this approach does not seem suitable for establishing the lower bound on the barcode entropy as in Theorem B. The essential reason is that a hyperbolic compact invariant subset of the geodesic flow corresponds to an invariant subset of any associated Hamiltonian diffeomorphism that is only partially hyperbolic.

One may expect the results from this paper to extend to more general Reeb flows when a suitable analogue of Morse theory (e.g., symplectic homology or cylindrical contact homology) is available. Then two problems arise. The first one is defining barcode entropy and establishing a variant of Theorem A; after our paper has appeared, this has been done for Reeb flows on the boundary of certain Liouville domains [FLS23]. The second problem is proving an analogue of Theorem B for general Reeb flows. The difficulty here is that it is absolutely unclear at the moment how to prove a version of the crossing energy bound in such a general setting.

Finally, while the exponential growth rate is relevant to the relation between barcode entropy and topological entropy, in other contexts the noise or chaos in the system might be captured by the function $c \mapsto b_{c,\epsilon}$ in different ways. For instance, one may consider polynomial growth rate as in slow-entropy type invariants; see, e.g., [ÇGG22a, Sect. 2.3] for applications to the γ-norm bounds and [BPP$^+$22, CSEHM10] for applications to nodal count.

2. Main results

2.1. **Morse-theoretic barcodes for closed geodesics.** Let (M, g) be a closed Riemannian manifold of dimension at least two (throughout this paper, all Riemannian metrics are tacitly assumed to be smooth, that is, C^∞). We denote by SM the associated unit tangent bundle, i.e., $SM = \big\{v \in TM \ \big|\ \|v\|_g = 1\big\}$. The geodesic flow is defined by $\phi_t \colon SM \to SM$, $\phi_t(\dot\gamma(0)) = \dot\gamma(t)$, where $\gamma\colon \mathbb{R} \to M$ is a geodesic parametrized with unit speed $\|\dot\gamma\|_g = 1$. The closed geodesics γ of length c correspond to the closed orbits $\dot\gamma$ with minimal period c of the geodesic flow. An alternative way of characterizing closed geodesics is by means of their classical variational principle which we now recall.

We consider the free loop space $\Lambda := W^{1,2}(S^1, M)$, where $S^1 = \mathbb{R}/\mathbb{Z}$, and the energy functional $\mathcal{E}\colon \Lambda \to [0,\infty)$ given by

$$\mathcal{E}(\gamma) = \int_0^1 \|\dot\gamma(t)\|_g^2 \, dt. \tag{2.1}$$

The circle S^1 acts on Λ by time translation as $t \cdot \gamma = \gamma(t + \cdot) \in \Lambda$, where $t \in S^1$ and $\gamma \in \Lambda$, and the energy functional $\mathcal{E}$ is invariant under this action. Besides the constant curves $\Lambda^0 := \mathcal{E}^{-1}(0)$, the critical point set

$$\mathrm{crit}^+(\mathcal{E}) := \mathrm{crit}(\mathcal{E}) \cap \mathcal{E}^{-1}(0,\infty)$$

consists of all critical circles $S^1 \cdot \gamma$, where γ is any 1-periodic closed geodesic. Since closed geodesics are parametrized proportionally to arc length, any critical value $\mathcal{E}(\gamma)$ is the squared length of the closed geodesic γ, where γ is viewed as a path $\gamma\colon [0,1] \to M$, i.e.

$$\sqrt{\mathcal{E}(\gamma)} = \int_0^1 \|\dot\gamma\|_g \, dt.$$

In this paper we shall work with the critical values of the functional $\sqrt{\mathcal{E}}$. For this reason, for each subset $\mathcal{U} \subseteq \Lambda$ and $b > 0$, we shall denote

$$\mathcal{U}^{<b} := \mathcal{U} \cap \mathcal{E}^{-1}[0, b^2).$$

We emphasize that the variational principle of $\mathcal{E}$ allows us to detect all the closed geodesics of (M, g), since any closed geodesic can be reparametrized to become 1-periodic. Moreover, any closed geodesic γ with minimal period 1 gives rise to infinitely many critical circles $S^1 \cdot \gamma^m \subset \mathrm{crit}^+(\mathcal{E})$, where $m \in \mathbb{N}$ and $\gamma^m := \gamma(m\,\cdot)$ is the m-th iterate of γ.

We shall employ the language of persistence modules and barcodes, and refer the reader to [PRSZ20] for a comprehensive introduction to the subject. We consider the persistence module $(H_b,\, i_{b,a})_{b>a>0}$, where $H_b := H_*(\Lambda^{<b}, \Lambda^0; \mathbb{F})$ and the maps $i_{b,a}\colon H_a \to H_b$ are the homomorphisms induced by the inclusion. The value $b = \infty$ is allowed, in which case we have $H_\infty = H_*(\Lambda, \Lambda^0; \mathbb{F})$. Hereafter, all singular homology groups will be taken with coefficients in an arbitrary field $\mathbb{F}$, usually suppressed in the notation.

For $c \in (0, \infty]$ and $h \in H_c$, we define the birth and death values

$$\alpha(h) := \inf \big\{ a < c \bigm| h \in \mathrm{im}(i_{c,a}) \big\} \in [0, c),$$
$$\beta(h) := \inf \big\{ b > c \bigm| h \in \ker(i_{b,c}) \big\} \in [c, \infty].$$

Here, we adopt the usual convention that $\inf \varnothing = \infty$. If $h \neq 0$, we have $\alpha(h) > 0$. The *closed geodesics barcode* $\mathcal{B} = \mathcal{B}(g; \mathbb{F})$ is the collection of all pairs $([a, b), n)$, where $[a, b) \subset (0, \infty)$ is an interval (of finite or infinite length) and $n = n_{[a,b)}$ is a non-negative integer, defined as follows. For any interval $[a, b)$ and any $c \in (a, b]$, consider the vector spaces

$$V := \big\{ h \in H_c \bigm| \alpha(h) \leq a,\ \beta(h) \leq b \big\},$$
$$W := \big\{ h \in H_c \bigm| \alpha(h) < a,\ \beta(h) \leq b \big\},$$
$$Z := \big\{ h \in H_c \bigm| \alpha(h) \leq a,\ \beta(h) < b \big\},$$

and define $n_{[a,b)} = \dim V/(W + Z)$. The vector space V is finite-dimensional, and the quotient dimension $n_{[a,b)}$ is independent of the choice of the value $c \in (a, b]$; see Lemma 3.5.

The barcode should be seen as a collection of real intervals $[a, b)$, called *bars*, with possible repetitions: an element $([a, b), n) \in \mathcal{B}$ corresponds to n copies of the bar $[a, b)$; if $n = 0$, the bar $[a, b)$ is not contained in the barcode. The size of a bar $[a, b)$ is its length $b - a \in (0, \infty]$. For instance, we will say that $c > 0$ is a boundary point of at least m bars with size at least δ if there exist distinct elements $([a_1, c), n_1), \ldots, ([a_k, c), n_k), ([c, b_1), m_1), \ldots, ([c, b_h), m_h) \in \mathcal{B}$ such that $n_1 + \ldots + n_k + m_1 + \ldots + m_h \geq m$, and all bars $[a_i, c)$ and $[c, b_i)$ have size $\geq \delta$.

2.2. **Barcode entropy.** For $\epsilon > 0$ and $c > 0$, we denote by $\mathcal{B}_{\epsilon,c} \subset \mathcal{B} = \mathcal{B}(g; \mathbb{F})$ the subcollection of those bars of size at least ϵ that intersect the interval $(0, c]$ non-trivially, i.e.,

$$\mathcal{B}_{\epsilon,c} := \big\{ ([a, b), n) \in \mathcal{B} \bigm| a \leq c,\ b - a \geq \epsilon \big\}.$$

We set $b_{\epsilon,c}$ to be the number of bars in $\mathcal{B}_{\epsilon,c}$, namely

$$b_{\epsilon,c} := \sum_{([a,b),n) \in \mathcal{B}_{\epsilon,c}} n. \tag{2.2}$$

Even though the barcode $\mathcal{B}$ can contain infinitely many bars intersecting the bounded interval $(0, c]$, the value $b_{\epsilon,c}$ is always finite for any $\epsilon > 0$; see Lemma 3.7.

Definition 2.1. The *barcode entropy* $\hbar = \hbar(g; \mathbb{F})$ is the limit

$$\hbar := \lim_{\epsilon \to 0^+} \hbar_\epsilon,$$

where

$$\hbar_\epsilon := \limsup_{c\to\infty} \frac{\log^+(b_{\epsilon,c})}{c}.$$

Here $\log^+ := \log(\max\{1,\cdot\})$ with the logarithm taken base 2.

We shall compare the barcode entropy with two other classical notions of entropy for a closed Riemannian manifold (M,g), and refer the reader to, e.g., Paternain's monograph [Pat99] for the background on the subject. The first one is the topological entropy $h_{\rm top} = h_{\rm top}(g)$ of the geodesic flow $\phi_t : SM \to SM$, which is the same as the topological entropy of its time-1 map ϕ_1. Since ϕ_t is smooth, $h_{\rm top}$ is always finite.

The second notion of entropy, which we call *volume-growth entropy*[1] and denote by $h_{\rm vol} = h_{\rm vol}(g)$, is defined by

$$h_{\rm vol} := \limsup_{t\to\infty} \frac{\log(V(t))}{t},$$

where $V(t)$ is the volume of the graph of $\phi_t : SM \to SM$ measured with respect to an arbitrary Riemannian metric on $SM \times SM$. The value of $h_{\rm vol}$ is independent of the choice of this Riemannian metric, but clearly depends on the Riemannian metric g defining the geodesic flow ϕ_t. The celebrated Yomdin theorem [Yom87] implies

$$h_{\rm vol} \le h_{\rm top}.$$

Indeed, the flow $\psi_t = (\mathrm{id}, \phi_t) : SM\times SM \to SM\times SM$ also has topological entropy $h_{\rm top}$, and the graph of ϕ_t is precisely $\psi_t(\Delta)$, where $\Delta = \{(v,v) \mid v \in SM\}$ is the diagonal submanifold.

Our first main result is the following. Together with Yomdin theorem and the finiteness of topological entropy, it implies in particular that the barcode entropy is always finite.

Theorem A. *On any closed Riemannian manifold and for any coefficient field, we have $\hbar \le h_{\rm vol}$.*

The proof of this theorem is based on an inequality reminiscent of the classical Crofton formula applied to certain Lagrangian tomographs. For general Reeb flows of closed contact-type hypersurfaces of symplectic manifolds, a different argument due to Meiwes [Mei18, Prop. 10.9] and still involving Lagrangian tomographs allows to bound $h_{\rm vol}$ from below by the exponential growth-rate of certain leafwise intersections.

Knowing that the barcode entropy is always finite, it remains to establish whether it is a non-trivial invariant, that is, whether it does not always vanish. Our second main result implies that the barcode entropy is positive when the geodesic flow admits a suspended horseshoe. For each compact subset $I \subset SM$ invariant under the geodesic flow (i.e., $\phi_t(I) = I$ for all $t \in \mathbb{R}$), we denote by $h_{\rm top}(I) = h_{\rm top}(I; M, g)$

[1] The volume-growth entropy should not be confused with another classical notion of entropy involving volumes: the exponential growth rate of the volume of Riemannian balls in the universal cover of the Riemannian manifold. The inequality of Theorem A is reminiscent of Manning's inequality, [Man79], involving this latter notion of volume-growth entropy, but is not directly related to it.

the topological entropy of the restricted geodesic flow $\phi_t|_I$. The precise statement is the following.

Theorem B. *On any closed Riemannian manifold and for any coefficient field, if I is a hyperbolic compact invariant subset of the geodesic flow, then $\hbar \geq h_{\mathrm{top}}(I)$.*

For flows on 3-dimensional closed manifolds, the topological entropy can be approximated by the topological entropy of suitable hyperbolic compact invariant subsets; see [Kat80, LY12, LS19]. Combining this fact with Theorems A and B, we obtain the following corollary for Riemannian closed surfaces.

Corollary C. *On any closed Riemannian surface and for any coefficient field, we have $\hbar = h_{\mathrm{vol}} = h_{\mathrm{top}}$.*

Remark 2.2. A priori, the barcode entropy depends on the choice of the coefficient field $\mathbb{F}$ employed in the homology group of the persistence module, although we do not know examples where this actually happens. Corollary C implies in particular that, at least for Riemannian surfaces, the barcode entropy is independent of the coefficient field.

Remark 2.3. Surprisingly, the identity $h_{\mathrm{vol}} = h_{\mathrm{top}}$ on all closed Riemannian surfaces provided by Corollary C is new. We do not know whether the identity holds for higher dimensional closed Riemannian manifolds as well. Nevertheless, the following similar identity due to Mañé, [Mn97], does hold in every dimension:

$$h_{\mathrm{top}} = \lim_{t\to\infty} \frac{1}{t} \log \Bigg(\underbrace{\int_M \mathrm{vol}(\phi_t(S_xM))\, dx}_{(*)} \Bigg).$$

Here dx is the Riemannian volume form, and the volume of $\phi_t(S_xM)$ is measured with respect to the Riemannian metric on SM induced by the one on M. Despite the similarities, the quantity $(*)$ seems different from the volume $V(t)$ entering in the definition of h_{vol}, and Mañé's identity does not seem to be equivalent to ours. Both our identity and Mañé's one rely on Yomdin's inequality to bound h_{top} from below, but Mañé's identity further relies on the Przytycki inequality, [Prz80], to bound h_{top} from above, whereas we employ a completely different argument based on a crossing energy bound, as we shall explain in Section 2.3.

Remark 2.4. There are other alternatives to the definition of barcode entropy adopted here. First, we could have instead worked with a variant of *sequential entropy* for geodesic flows defined similarly to sequential barcode entropy for compactly supported Hamiltonian diffeomorphisms; see [ÇGG22b]. Theorems A and B would hold for sequential barcode entropy, and while we do not know if in general the two types of barcode entropy are equal, this would be the case when M is a surface. Secondly, we could have used the S^1-equivariant Morse theory rather than the ordinary Morse theory. By the Gysin sequence and [BPP$^+$22, Thm. 3.1], the equivariant barcode entropy is greater than or equal to the barcode entropy, and hence Theorem B would still hold for it. However, we do not know whether Theorem A remain true in the equivariant setting. Finally, the definition of barcode entropy and Theorem A has also a relative analogue along the lines of relative barcode entropy from [ÇGG21]; we will touch upon it in Section 4.4.

2.3. **Invariant subsets and a lower bound on the bar size.** The main ingredient of the proof of Theorem B is a uniform lower bound on the size of the bars associated with a locally maximal, hyperbolic, compact invariant subset of the geodesic flow. This lower bound actually requires a slightly weaker assumption on the invariant subset than hyperbolicity: expansivity. Before stating the result, Theorem D, let us introduce some notation and terminology.

Let $I \subset SM$ be a compact invariant subset for the geodesic flow ϕ_t, i.e., $\phi_t(I) = I$ for all $t \in \mathbb{R}$. We denote by $\mathcal{P}(I) \subset \mathrm{crit}^+(\mathcal{E})$ the space of 1-periodic closed geodesics tangent to I, i.e.

$$\mathcal{P}(I) := \big\{ \gamma \in \mathrm{crit}^+(\mathcal{E}) \ \big|\ \tfrac{\dot\gamma(t)}{\|\dot\gamma(t)\|_g} \in I,\ \forall t \in S^1 \big\}. \tag{2.3}$$

For each $c > 0$, set $\mathcal{P}^c(I) := \mathcal{P}(I) \cap \mathcal{E}^{-1}(c^2)$. The *length spectrum* of I is defined as

$$\sigma(I) := \{ c > 0 \mid \mathcal{P}^c(I) \neq \varnothing \}.$$

In other words, $\sigma(I)$ consists of the energy of all (possibly iterated) 1-periodic geodesics tangent to I. We denote the total local homology of the set formed by closed geodesics tangent to I with energy c^2 by

$$C_*(\mathcal{P}^c(I)) := H_*(\Lambda^{<c} \cup \mathcal{P}^c(I), \Lambda^{<c}).$$

We need to impose two conditions on compact invariant sets $I \subset SM$. The first one is that I is *locally maximal*: the set I admits an open neighborhood $U \subset SM$, called an *isolating neighborhood*, such that

$$I = \bigcap_{t \in \mathbb{R}} \phi_t(U).$$

In other words, I is the largest invariant subset contained in U. The second condition is that I is *expansive*: for every $\epsilon > 0$, there exists $\delta > 0$ such that for any $z_1, z_2 \in I$ and continuous function $s\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$s(0) = 0, \qquad \sup_{t \in \mathbb{R}} \tilde d \left(\phi_t(z_1), \phi_{s(t)}(z_2) \right) < \delta,$$

we have $z_2 = \phi_t(z_1)$ for some $t \in [-\epsilon, \epsilon]$. Here $\tilde d\colon SM \times SM \to [0,\infty)$ is the distance on the unit tangent bundle induced by the Riemannian metric g. This condition was first introduced by Bowen and Walters in [BW72].

We are now in a position to state our third main result.

Theorem D. *Let (M,g) be a closed Riemannian manifold, let $I \subset SM$ be a locally maximal, expansive, compact invariant subset of its geodesic flow, and let $\mathbb{F}$ be any coefficient field. There exists $\delta > 0$ such that every $c \in \sigma(I)$ is a boundary point of at least $\dim C_*(\mathcal{P}^c(I))$ bars of size at least δ in the closed geodesics barcode $\mathcal{B}(g;\mathbb{F})$.*

Next, let $\gamma \in \mathrm{crit}^+(\mathcal{E})$ be a closed geodesic of energy $\mathcal{E}(\gamma) = c^2$. The corresponding c-periodic orbit of the geodesic flow is given by $t \mapsto \phi_t(v)$, where $v = \dot\gamma(0)/\|\dot\gamma(0)\|_g$. The closed geodesic γ is called *non-degenerate* when $d\phi_c(v)$ does not have 1 among its eigenvalues; equivalently, the critical circle $S^1 \cdot \gamma \subset \mathrm{crit}^+(\mathcal{E})$ is non-degenerate in the sense of Morse–Bott theory. An invariant subset $I \subset SM$ for the geodesic flow ϕ_t is called non-degenerate when every closed geodesic tangent to I is non-degenerate. Note that this definition requires all iterates of closed geodesics tangent to I to be non-degenerate.

Furthermore, we say that a critical circle $S^1 \cdot \gamma \subset \mathrm{crit}^+(\mathcal{E})$ is *prime* when γ is not iterated, i.e., when $\gamma \neq \zeta^m$ for any integer $m \geq 2$ and $\zeta \in \mathrm{crit}^+(\mathcal{E})$. For each $c \in \sigma(I)$, we denote by $n_c(I)$ the number of non-degenerate, prime, critical circles in $\mathcal{P}^c(I)$. Theorem D has the following corollary.

Corollary E. *Let (M, g) be a closed Riemannian manifold, let $I \subset SM$ be a locally maximal, expansive, compact invariant subset of the geodesic flow, and let $\mathbb{F}$ be any coefficient field. There exists $\delta > 0$ such that every $c \in \sigma(I)$ is a boundary point of at least $2n_c(I)$ bars of size at least δ in the closed geodesics barcode $\mathcal{B}(g;\mathbb{F})$.*

The proof of Theorem D is based on a uniform crossing energy bound for locally maximal, expansive, compact invariant subsets of the geodesic flow. This is Proposition 6.6 which can be viewed as a Morse theoretic analogue for closed geodesics of the crossing energy theorem from [GG14, GG18] for Hamiltonian diffeomorphisms. In the context of generating functions of Hamiltonian diffeomorphisms, an analogous statement was proved by Allais, [All22, Sect. 7]. Similarly to the Hamiltonian version that lies at the heart of the arguments in [ÇGG21, GG14, GG18], we expect Proposition 6.6 to have further applications to the study of closed geodesics.

To illustrate the crossing energy theorem for geodesic flows, consider the particular case where the locally maximal, expansive compact invariant set is a periodic orbit γ, which corresponds to a critical circle $S^1 \cdot \gamma \subset \mathrm{crit}^+(\mathcal{E})$. The local maximality and expansiveness imply that there exists an open neighborhood $U \subset M$ of the support of γ such that no other closed geodesic has support entirely contained in U. This implies that there exists $\delta_U(\gamma) > 0$ with the following property. Consider an energy gradient flow line starting close to γ, i.e., a solution $u\colon \mathbb{R} \to \Lambda U$ of the ordinary differential equation $\dot{u} = \nabla\mathcal{E}(u)$ such that $u(0)$ is sufficiently C^0-close to γ, where ΛU the loop space of U. If $u(0) \neq \gamma$, there exists a large positive or negative time s such that the support of the loop $u(s) \in \Lambda$ is not entirely contained in the open set U, and we have a uniform energy drop $|\mathcal{E}(u(0)) - \mathcal{E}(u(s))| \geq \delta_U(\gamma)$. If we repeat the same argument replacing γ with its m-th iterate γ^m, an analogous energy gradient flow line would have energy drop $\delta_U(\gamma^m)$, and *a priori* $\delta_U(\gamma^m)$ may depend on the order of iteration m and shrink as m grows. The uniform crossing energy bound gives a positive lower bound for $\delta_U(\gamma^m)$ independent of m. Indeed, we shall prove that $\delta_U(\gamma^m) \to \infty$, but with a caveat: we are not able to derive such a statement in the infinite dimensional setting Λ, and instead we shall establish it using finite dimensional approximations of Λ, as in Milnor's [Mil63]. This will not prevent us from obtaining Theorem D in the infinite dimensional setting Λ.

2.4. **Organization of the paper.** After recalling some preliminary elementary facts on the closed geodesics barcode in Section 3, we prove Theorem A in Section 4. Assuming Corollary E, in Section 5 we establish Theorem B and Corollary C. In Section 6 we introduce a finite dimensional setting for the energy action functional and prove the uniform crossing energy bound, Proposition 6.6. Finally, in Section 7, we prove Theorem D and Corollary E.

3. Closed geodesics barcode

Let (M, g) be a closed Riemannian manifold of dimension at least two, and let $\mathcal{E}\colon \Lambda \to [0, \infty)$ be the associated energy functional defined by (2.1). The length spectrum of the Riemannian manifold is the set

$$\sigma = \sigma(g) := \big\{ \sqrt{\mathcal{E}(\gamma)} \;\big|\; \gamma \in \mathrm{crit}^+(\mathcal{E}) \big\}.$$

Since $\mathcal{E}$ satisfies the Palais–Smale condition, [Pal63], σ is closed. By Sard's theorem, σ has measure zero, and hence it is nowhere dense. Consider the persistence module

$$H_a \xrightarrow{i_{b,a}} H_b, \quad 0 < a < b \leq \infty,$$

where $H_a := H_*(\Lambda^{<a}, \Lambda^0)$ and the maps $i_{b,a}$ are the homomorphisms induced by the inclusion. It is well known that

$$H_\epsilon = \{0\}, \qquad \forall \epsilon \in (0, 2\,\mathrm{injrad}(g)).$$

The vector space H_∞ is always infinite-dimensional. If the Riemannian metric g is bumpy (i.e., all closed geodesics, including the iterated ones, are non-degenerate), H_c is finite-dimensional for any $c \in (0,\infty)$ since the critical set $\mathrm{crit}(\mathcal{E}) \cap \mathcal{E}^{-1}(0, c^2)$ consists of finitely many non-degenerate critical circles. On the other hand, without the bumpy assumption on the Riemannian metric g, the vector space H_c may be infinite-dimensional even for some finite c, but only if $c \in \sigma$.

Lemma 3.1. *For any $b \in (0,\infty) \setminus \sigma$, the relative homology group H_b is finite-dimensional.*

Proof. Since the length spectrum σ is closed, for each $b \in (0,\infty) \setminus \sigma$ there exists $a \in (0, b)$ such that $[a, b] \cap \sigma = \varnothing$. Namely, the interval $[a, b]$ consists of regular values of the energy $\mathcal{E}$. The usual gradient flow deformation result from Morse theory guarantees that the inclusion $\Lambda^{<a} \hookrightarrow \Lambda^{<b}$ is a homotopy equivalence and hence $i_{b,a} : H_a \to H_b$ is an isomorphism.

By the bumpy metric theorem from [Ano82], for any $\epsilon > 0$ there exists a bumpy Riemannian metric h on M such that

$$(1+\epsilon)^{-1}\|v\|_g \leq \|v\|_h \leq (1+\epsilon)\|v\|_g, \qquad \forall v \in TM. \tag{3.1}$$

Let $\mathcal{F}\colon \Lambda \to [0,\infty)$ be the energy functional associated with h, i.e.,

$$\mathcal{F}(\gamma) = \int_0^1 \|\dot\gamma\|_h^2 \, dt.$$

The inequalities (3.1) imply that $(1+\epsilon)^{-2}\mathcal{E}(\gamma) \leq \mathcal{F}(\gamma) \leq (1+\epsilon)^2\mathcal{E}(\gamma)$ for all $\gamma \in \Lambda$. Then, choosing $\epsilon > 0$ small enough so that $(1+\epsilon)^2 a < b$ and setting $c := (1+\epsilon)a$, we have the inclusion of sublevel sets

$$\Lambda^{<a} = \mathcal{E}^{-1}[0, a^2) \;\subseteq\; \mathcal{F}^{-1}[0, c^2) \;\subseteq\; \mathcal{E}^{-1}[0, b^2) = \Lambda^{<b}.$$

Therefore, the isomorphism $i_{b,a}\colon H_a \to H_b$ factors through the finite-dimensional vector space $H_*(\mathcal{F}^{-1}[0, c^2), \mathcal{F}^{-1}(0))$, and we conclude that H_b is finite-dimensional. □

Lemma 3.2. *For all a, b with $0 < a < b \leq \infty$, the image $\mathrm{im}(i_{b,a})$ is finite-dimensional.*

Proof. Fix a point $c \in (a, b) \setminus \sigma$. By Lemma 3.1, H_c is finite-dimensional, and so must be $\mathrm{im}(i_{b,c})$. Since $i_{b,a} = i_{b,c} \circ i_{c,a}$, we have $\mathrm{im}(i_{b,a}) \subseteq \mathrm{im}(i_{b,c})$ and hence $\mathrm{im}(i_{b,a})$ is finite-dimensional. □

In the literature, persistence modules satisfying the assertion of Lemma 3.2 are sometimes called *q-tame*, [CdSGO16]. To keep this paper self-contained, in the rest of this section we briefly present some foundational results from the theory of abstract persistence modules relevant for us and apply them in our setting.

Given any $c \in (0,\infty]$ and homology class $h \in H_c$, recall from Section 2.1 the birth and death values

$$\begin{aligned}\alpha(h) &:= \inf\big\{a \le c \;\big|\; h \in \mathrm{im}(i_{c,a})\big\} \in [0,c),\\ \beta(h) &:= \inf\big\{b > c \;\big|\; h \in \ker(i_{b,c})\big\} \in [c,\infty],\end{aligned}$$

where as usual $\inf \varnothing = \infty$. If $h \neq 0$, since $H_\epsilon = \{0\}$ for a sufficiently small $\epsilon > 0$, we have $\alpha(h) \ge \epsilon$. For all $0 < a < b \le \infty$, we define the vector subspace

$$I_{b,a} := \big\{h \in H_b \;\big|\; \alpha(h) \le a\big\} = \bigcap_{c\in(a,b]} \mathrm{im}\,(i_{b,c})\,.$$

Lemma 3.2 implies that $I_{b,a}$ is finite-dimensional. Its dimension $\dim I_{b,a}$ is the number of bars containing $[a,b)$.

Lemma 3.3. *For all a,b with $0 < a < b \le \infty$ and for all sufficiently small $\epsilon \in (0,b-a)$, we have $I_{b,a} = \mathrm{im}(i_{b,a+\epsilon})$.*

Proof. For any $\epsilon \in (0,b-a)$, we have $I_{b,a} \subseteq I_{b,a+\epsilon}$. This, together with the fact that these vector spaces are finite-dimensional, implies that $I_{b,a} = I_{b,a+\epsilon}$, provided that ϵ is small enough. Therefore, $I_{b,a} = \mathrm{im}(i_{b,a+\epsilon})$ for any sufficiently small $\epsilon \in (0,b-a)$. □

Let $0 < a \le b \le \infty$ and consider the subspace of H_a given by

$$K_{b,a} := \big\{h \in H_a \;\big|\; \beta(h) \le b\big\} = \begin{cases} \displaystyle\bigcap_{c>b} \ker\,(i_{c,a}) & \text{if } b < \infty,\\ H_a & \text{if } b = \infty.\end{cases}$$

The dimension $\dim K_{b,a}$ is the number of bars of the form $[a',b')$, where $a' < a \le b' \le b$. Notice that, unlike $I_{b,a}$, the vector space $K_{b,a}$ can be infinite-dimensional (even when $b < \infty$). Next, for $c \in (a,b]$, we define

$$V_{[a,b),c} := \big\{h \in H_c \;\big|\; \alpha(h) \le a,\ \beta(h) \le b\big\} = K_{b,c} \cap I_{c,a}.$$

By Lemma 3.2, $V_{[a,b),c}$ is a finite-dimensional vector subspace of H_c. The dimension $\dim V_{[a,b),c}$ is the number of bars of the form $[a',b')$ with $a' \le a < c \le b' \le b$.

Lemma 3.4. *If $b < \infty$, for any sufficiently small $\epsilon \in (0,b-a)$ we have*

$$V_{[a,b),c} = \ker(i_{b+\epsilon,c}) \cap \mathrm{im}(i_{c,a+\epsilon}).$$

Proof. For all $\epsilon > 0$, we have $V_{[a,b),c} \subseteq V_{[a,b+\epsilon),c}$. Since these vector spaces are finite-dimensional, $V_{[a,b),c} = V_{[a,b+\epsilon),c}$ whenever $\epsilon > 0$ is sufficiently small. Hence $V_{[a,b),c} = \ker(i_{b+\epsilon,c}) \cap I_{c,a}$. Finally, applying Lemma 3.3, we obtain the desired equality. □

We define the vector subspaces

$$\begin{aligned} W_{[a,b),c} &:= \big\{ h \in H_c \;\big|\; \alpha(h) < a,\ \beta(h) \leq b \big\} = K_{b,c} \cap \mathrm{im}(i_{c,a}) \quad \text{and} \\ Z_{[a,b),c} &:= \big\{ h \in H_c \;\big|\; \alpha(h) \leq a,\ \beta(h) < b \big\} = \ker(i_{b,c}) \cap I_{c,a} \end{aligned}$$

of $V_{[a,b),c}$. In view of Lemma 3.4, for $\epsilon > 0$ small enough we have

$$Z_{[a,b),c} = \ker(i_{b,c}) \cap \mathrm{im}(i_{c,a+\epsilon})$$

and, if $b < \infty$,

$$W_{[a,b),c} = \ker(i_{b+\epsilon,c}) \cap \mathrm{im}(i_{c,a}).$$

Recall from Section 2.1 that the closed geodesics barcode $\mathcal{B} = \mathcal{B}(g; \mathbb{F})$ is the collection of all pairs $([a,b), n)$, where $[a,b) \subset (0,\infty)$ and

$$n = \dim\left(\frac{V_{[a,b),c}}{W_{[a,b),c} + Z_{[a,b),c}} \right) < \infty, \qquad \forall c \in (a,b].$$

Lemma 3.5. *The dimension* $\dim(V_{[a,b),c}/(W_{[a,b),c} + Z_{[a,b),c}))$ *is independent of the value* $c \in (a,b]$.

Proof. Let $a < b < \infty$. In order to simplify the notation, let us suppress $[a,b)$ in the notation and simply write $V_c = V_{[a,b),c}$, $W_c = W_{[a,b),c}$, and $Z_c = Z_{[a,b),c}$ for all $c \in (a,b]$. We fix c and d so that $a < c < d \leq b$, and choose $\epsilon > 0$ small enough so that the assertion of Lemma 3.4 holds. The homomorphism $i_{d,c}$ restricts to surjective homomorphisms

$$\begin{aligned} V_c = \ker(i_{b+\epsilon,c}) \cap \mathrm{im}(i_{c,a+\epsilon}) &\xrightarrow{i_{d,c}} \ker(i_{b+\epsilon,d}) \cap \mathrm{im}(i_{d,a+\epsilon}) = V_d, \quad \text{and} \\ W_c = \ker(i_{b+\epsilon,c}) \cap \mathrm{im}(i_{c,a}) &\xrightarrow{i_{d,c}} \ker(i_{b+\epsilon,d}) \cap \mathrm{im}(i_{d,a}) = W_d. \end{aligned}$$

Consider $h \in V_c$ such that $i_{d,c}(h) = w' + z' \in W_d + Z_d$, with $w' \in W_d$ and $z' \in Z_d$. Since $i_{d,c}(W_c) = W_d$, there exists $w \in W_c$ such that $i_{d,c}(w) = w'$. The vector $z := h - w$ satisfies $i_{d,c}(z) = z'$, and since $i_{b,c}(z) = i_{b,d}(z') = 0$, we infer that $z \in Z_c$, and $h \in W_c + Z_c$. This shows that the kernel of the induced homomorphism

$$V_c \xrightarrow{i_{d,c}} \frac{V_d}{W_d + Z_d}$$

is $W_c + Z_c$, and hence $i_{d,c}$ induces an isomorphism

$$\frac{V_c}{W_c + Z_c} \xrightarrow[\cong]{i_{d,c}} \frac{V_d}{W_d + Z_d}.$$

The case of $b = \infty$ is analogous. □

Proposition 3.6. *For any* $c \in (0,\infty) \setminus \sigma$ *and* $([a,b), n) \in \mathcal{B}$ *such that* $c \in (a,b]$, *there exists an* n*-dimensional vector subspace* $B_{[a,b),c} \subset H_c$ *such that* $\alpha(h) = a$ *and* $\beta(h) = b$ *for all* $h \in B_{[a,b),c} \setminus \{0\}$. *These vector subspaces can be chosen so that* H_c *decomposes as a direct sum*

$$H_c = \bigoplus_{[a,b) \ni c} B_{[a,b),c}.$$

Proof. Note that H_c is finite-dimensional as $c \in (0,\infty)\setminus\sigma$. Since $I_{c,a} \subseteq I_{c,a'}$ and $K_{b,c} \subseteq K_{b',c}$ for all $a < a'$ and $b < b'$, there exist finitely many values a_i and b_j with

$$0 =: a_0 < a_1 < \ldots < a_h < c < b_1 < \ldots < b_k \leq \infty$$

such that

- $\mathrm{im}(i_{c,a_1}) = \{0\}$ and $I_{c,a_h} = H_c$,
- $\mathrm{im}(i_{c,a}) = I_{c,a_i} \subsetneq I_{c,a_{i+1}}$ for all $i \in \{1,\ldots,h-1\}$ and $a \in (a_i, a_{i+1}]$,
- $\ker(i_{b_1,c}) = H_c$ and $K_{b_k,c} = \{0\}$,
- $\ker(i_{b,c}) = K_{b_j,c} \subsetneq K_{b_{j+1},c}$ for all $j \in \{1,\ldots,k-1\}$ and $b \in (b_j, b_{j+1}]$.

Namely, all bars containing c are of the form $[a_i, b_j)$ for some $i > 0$ and j. Let

$$V_{j,i} := V_{[a_i,b_j),c} = K_{b_j,c} \cap I_{c,a_i},$$

and notice that

$$V_{j_1,i_1} \cap V_{j_2,i_2} = V_{j_3,i_3}, \qquad \text{where } i_3 = \min\{i_1,i_2\},\ j_3 = \min\{j_1,j_2\}.$$

Now, set $\mathcal{Y}_{0,0} = \mathcal{Y}_{1,0} = \mathcal{Y}_{0,1} = \varnothing$ and choose a basis $\mathcal{Y}_{1,1}$ of $V_{1,1}$. We next proceed inductively for increasing values of the integer $m = 2,\ldots,h+k$: assume that we have already chosen the bases $\mathcal{Y}_{j,i}$ for $V_{j,i}$ for all j,i such that $j+i < m$; for all j,i such that $j+i = m$, we choose a basis $\mathcal{Y}_{j,i}$ of $V_{j,i}$ by completing the set $\mathcal{Y}_{j-1,i} \cup \mathcal{Y}_{j,i-1}$. At the end of the process, we obtain bases that satisfy

$$\mathcal{Y}_{j_1,i_1} \cap \mathcal{Y}_{j_2,i_2} = \mathcal{Y}_{j_3,i_3}, \qquad \text{where } i_3 = \min\{i_1,i_2\},\ j_3 = \min\{j_1,j_2\}.$$

We define the vector subspaces

$$B_{j,i} := \mathrm{span}(\mathcal{Y}_{j,i} \setminus (\mathcal{Y}_{j-1,i} \cup \mathcal{Y}_{j,i-1})) \subset V_{j,i}.$$

Notice that $\alpha(h) = a_i$ and $\beta(h) = b_j$ for all $h \in B_{j,i}\setminus\{0\}$. The union $\mathcal{Y}_{j-1,i} \cup \mathcal{Y}_{j,i-1}$ is a basis for the vector space

$$V_{j,i-1} + V_{j-1,i} = W_{[a_i,b_j),c} + Z_{[a_i,b_j),c}.$$

Thus the quotient maps $V_{j,i} \to V_{j,i}/(V_{j,i-1} + V_{j-1,i})$ restrict to isomorphisms

$$B_{j,i} \xrightarrow{\cong} \frac{V_{j,i}}{V_{j,i-1} + V_{j-1,i}}. \tag{3.2}$$

This, together with the definition of the barcode, implies that $n_{j,i} := \dim B_{j,i}$ is the multiplicity of the bar $[a_i, b_j)$; namely $([a_i,b_j), n_{j,i}) \in \mathcal{B}$. The basis $\mathcal{Y}_{k,h}$ of $V_{k,h} = H_c$ can be decomposed as a disjoint union

$$\mathcal{Y}_{k,h} = \bigcup_{\substack{i=1,\ldots,h\\ j=1,\ldots,k}} \mathcal{Y}_{j,i} \setminus (\mathcal{Y}_{j-1,i} \cup \mathcal{Y}_{j,i-1}),$$

where $\mathcal{Y}_{j,i} \setminus (\mathcal{Y}_{j-1,i} \cup \mathcal{Y}_{j,i-1})$ is the chosen basis of $B_{j,i}$. Therefore, we have the direct sum decomposition

$$H_c = \bigoplus_{\substack{i=1,\ldots,h\\ j=1,\ldots,k}} B_{j,i}. \qquad \square$$

Recall from Section 2.2 that, for any $\epsilon > 0$ and $c > 0$, the set $\mathcal{B}_{\epsilon,c} \subset \mathcal{B}$ is the subcollection of those bars of size at least ϵ, and $b_{\epsilon,c}$ is its cardinality counted with multiplicities, i.e.,

$$b_{\epsilon,c} = \sum_{\substack{a \leq c \\ b-a \geq \epsilon}} \dim B_{[a,b),c}.$$

Lemma 3.7. *For all real values $\epsilon > 0$ and $c > 0$, we have $b_{\epsilon,c} < \infty$.*

Proof. Fix non-spectral values $0 =: c_0, c_1, \ldots, c_n \in (0,\infty) \setminus \sigma$ such that $c_n > c$ and $c_i < c_{i+1} < c_i + \epsilon$ for all $i \in \{0, \ldots, n\}$. Notice that H_{c_i} is finite-dimensional for each $i \in \{1, \ldots, n\}$ by Lemma 3.1. Moreover, any bar $[a, b)$ with $a \leq c$ and $b-a \geq \epsilon$ must contain some value c_i. Along with Proposition 3.6, this implies that

$$b_{\epsilon,c} \leq \sum_{i=1}^{n} \dim H_{c_i} < \infty. \qquad \square$$

4. Upper bound on the barcode entropy

In this section we prove Theorem A, and briefly discuss a minor generalization to a relative version of barcode entropy. The proof roughly follows the same path as the proof of [ÇGG21, Theorem A], based on an argument in the spirit of integral geometry involving Lagrangian tomographs. In the setting of geodesic flows, we need to work with specific Lagrangian tomographs.

4.1. **Tomographs.** Let (M, g) be a closed Riemannian manifold. If $\dim(M) \leq 1$, the barcode entropy $\hbar = \hbar(g; \mathbb{F})$ vanishes, and Theorem A is trivially satisfied. Therefore, from now on we shall assume that $\dim(M) \geq 2$. For some integer $k > 0$, consider an open neighborhood $Z \subset \mathbb{R}^k$ of the origin, and a smooth map

$$\psi \colon Z \times M \to M,$$

which we treat as a family of maps $\psi_z = \psi(z, \cdot) \colon M \to M$ parameterized by $z \in Z$, satisfying the following three conditions:

(i) $\psi_0 = \mathrm{id}$,

(ii) $\psi_z \colon M \to M$ is a diffeomorphism for each $z \in Z$,

(iii) the associated map $\Psi \colon Z \times TM \to TM \times TM$, given by

$$\Psi(z, v) = (d\psi_z(x)^* v, v), \quad \forall v \in T_{\psi_z(x)} M,$$

has surjective differential $d\Psi(z, v)$ at all points $(z, v) \in Z \times TM \setminus 0$-section.

In point (iii), we denoted by $d\psi_z(x)^* \colon T_{\psi_z(x)} M \to T_x M$ the adjoint of $d\psi_z(x)$ with respect to the Riemannian metric g, which is defined by

$$g(d\psi_z(x)^* v, w) = g(v, d\psi_z(x) w).$$

In integral geometry, maps such as Ψ are sometimes referred to as *tomographs*. With a slight abuse of terminology, we shall instead call ψ a tomograph.

Lemma 4.1. *For some integer $k > 0$ and some open neighborhood Z of the origin, there exists a smooth map $\psi \colon Z \times M \to M$ satisfying conditions (i), (ii), (iii).*

Proof. Let $\chi\colon \mathbb{R}^n \to [0,1]$ be a smooth bump function such that $\chi|_{B^n(1/2)} \equiv 1$ and $\mathrm{supp}(\chi) \subset B^n(1)$, where $B^n(r) \subset \mathbb{R}^n$ denotes the open ball of radius r centered at the origin. We define the smooth map

$$\sigma\colon \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma(A,b,x) = \sigma_{A,b}(x) = x + \chi(x)\big(Ax+b\big),$$

where $A \in \mathbb{R}^{n\times n}$ is treated as an $n\times n$ matrix. Notice that, for all (A,b) in a sufficiently small neighborhood $U \subset \mathbb{R}^{n\times n}\times\mathbb{R}^n$ of the origin, the following three conditions hold:

- $\sigma_{A,b}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism,
- $\|\sigma_{A,b}^{-1}(0)\| < 1/2$,
- $\sigma_{A,b}(B^n(1)) = B^n(1)$.

For each non-zero vector $v \in \mathbb{R}^n \setminus \{0\}$, the map $U \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$(A,b) \mapsto (d\sigma_{A,b}(\sigma_{A,b}^{-1}(0))^T v, \sigma_{A,b}^{-1}(0)) = ((I+A)v, -(I+A)^{-1}b) \tag{4.1}$$

has surjective differential at the origin $0 \in U$. Here, the superscript ^{T} denotes the transpose matrix, i.e., the adjoint with respect to the Euclidean metric.

Let (M,g) be a closed Riemannian manifold of dimension $n \geq 2$. For each point $x_0 \in M$, take a chart $\phi_{x_0}\colon W_{x_0} \to \mathbb{R}^n$ that provides geodesic normal coordinates centered at x_0. Namely, $W_{x_0} \subset M$ is an open neighborhood of x_0, $\phi_{x_0}(x_0) = 0$, and, if we denote by $(x_1,\dots,x_n) = \phi_{x_0}(x)$ the coordinates associated with the chart, the Riemannian metric g can be written as

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx_i \otimes dx_j,$$

and its coefficients satisfy $g_{ij}(0) = \epsilon_{ij}$ and $dg_{ij}(0) = 0$. Up to modifying this chart outside a neighborhood of x_0, we can assume that the image $\phi_{x_0}(W_{x_0})$ contains the unit ball $B^n(1)$. Consider the smooth map $\theta_{x_0}\colon U \times M \to M$ given by

$$\theta_{x_0}(A,b,x) = \theta_{x_0,A,b}(x) := \begin{cases} x & \text{if } x \in M \setminus W_{x_0}, \\ \phi_{x_0}^{-1} \circ \sigma_{A,b} \circ \phi_{x_0}(x) & \text{if } x \in W_{x_0}. \end{cases}$$

Notice that each $\theta_{x_0,A,b}\colon M \to M$ is a diffeomorphism. We define the lift

$$\Theta_{x_0}\colon U \times TM \to TM \times TM, \qquad \Theta_{x_0}(A,b,v) = (d\theta_{x_0,A,b}(\theta_{x_0,A,b}^{-1}(\pi(v)))^* v, v),$$

where $\pi\colon TM \to M$ is the base projection of the tangent bundle. By the surjectivity of the differential of the above map (4.1) at the origin, we readily see that, for all unit tangent vectors $v \in S_{x_0}M$, the differential $d\Theta_{x_0}(0,0,v)$ is surjective. Since the surjectivity of the differential is an open condition, there exists an open neighborhood $V_{x_0} \subset M$ of x_0 such that, for each $x \in V_{x_0}$ and $v \in S_x V_{x_0}$, the differential $d\Theta_{x_0}(0,0,v)$ is surjective.

By the compactness of M, there exist finitely many points $x_1,\dots,x_h \in M$ such that $V_{x_1} \cup \ldots \cup V_{x_h} = M$. We define the map $\psi\colon U^{\times h} \times M \to M$ by

$$\psi(\underbrace{A_1,b_1,\dots,A_h,b_h}_{z},x) = \psi_z(x) = \theta_{x_1,A_1,b_1} \circ \ldots \circ \theta_{x_h,A_h,b_h}(x).$$

By construction, we have $\psi_0(x) = x$, and each $\psi_z\colon M \to M$ is a diffeomorphism. The associated map $\Psi\colon U^{\times h} \times TM \to TM \times TM$ given by

$$\Psi(z,v) = \Psi_z(v) = (d\psi_z(\psi_z^{-1}(x))^* v, v), \qquad \forall z \in U^{\times h},\ v \in T_xM$$

has surjective differential at every point of the form $(0,v)\in U^{\times h}\times SM$. Using one last time the fact that the surjectivity of the differential is an open condition, we find a sufficiently small open neighborhood $Z\subset U^{\times h}$ of the origin such that $d\Psi(z,v)$ is surjective for all $(z,v)\in Z\times SM$. Finally, since $v\mapsto\Psi(z,v)$ is linear along the fibers of TM, we conclude that $d\Psi(z,v)$ is surjective for all $(z,v)\in Z\times TM\setminus 0$-section. $\square$

4.2. **Variational principles associated with a tomograph.** Fix a constant

$$\epsilon\in(0,\tfrac12\,\mathrm{injrad}(g))>0.$$

Using the notation from Section 4.1, up to possibly shrinking the open neighborhood $Z\subset\mathbb{R}^k$ around the origin, we can assume that the tomograph ψ satisfies

$$\sup\big\{d(x,\psi_z(x))\ \big|\ (z,x)\in Z\times M\big\}<\epsilon, \tag{4.2}$$

where $d\colon M\times M\to[0,\infty)$ denotes the Riemannian distance.

We now employ ψ in connection with a twisted version of the variational principle for closed geodesics, introduced by Grove [Gro73]. For each $z\in Z$, consider the path space

$$\Omega_z:=\big\{\gamma\in W^{1,2}([0,1],M)\ \big|\ \gamma(1)=\psi_z(\gamma(0))\big\},$$

and denote the energy functional over it by

$$\mathcal{E}_z\colon\Omega_z\to[0,\infty),\qquad \mathcal{E}_z(\gamma)=\int_0^1\|\dot\gamma\|_g^2\,dt.$$

Notice that $\Omega_0=\Lambda$ is the usual free loop space, and $\mathcal{E}_0=\mathcal{E}$ is the usual energy functional of 1-periodic curves. The critical set $\mathrm{crit}^+(\mathcal{E}_z):=\mathrm{crit}(\mathcal{E}_z)\cap\mathcal{E}_z^{-1}(0,\infty)$ consists of those geodesic segments $\gamma\colon[0,1]\to M$ such that $\gamma(1)=\psi_z(\gamma(0))$ and $d\psi_z(\gamma(0))^*\dot\gamma(1)=\dot\gamma(0)$. As usual, we consider the critical values of the square root of the energy $\sqrt{\mathcal{E}_z}$. For each subset $\mathcal{U}\subseteq\Omega_z$ and real number $c>0$, we employ the usual notation $\mathcal{U}^{<c}:=\mathcal{U}\cap\mathcal{E}_z^{-1}[0,c^2)$. For each interval $[a,b]\subset(0,\infty)$, we set

$$\mathrm{crit}(\mathcal{E}_z)^{[a,b]}:=\mathrm{crit}(\mathcal{E}_z)\cap\mathcal{E}_z^{-1}[a^2,b^2].$$

Notice that, by (4.2) and since $\epsilon<\frac12\,\mathrm{injrad}(g)$, the interval $[\epsilon,2\,\mathrm{injrad}(g)-\epsilon]$ does not contain critical values of $\sqrt{\mathcal{E}_z}$.

Conditions (i) and (ii) imply the following statement.

Lemma 4.2. *There exists continuous maps $\sigma\colon\Lambda\to\Omega_z$ and $\nu\colon\Omega_z\to\Lambda$ such that*

$$\begin{aligned}\sqrt{\mathcal{E}_z(\sigma(\gamma))}&<\sqrt{\mathcal{E}(\gamma)}+\epsilon,\qquad\forall\gamma\in\Lambda,\\ \sqrt{\mathcal{E}(\nu(\gamma))}&<\sqrt{\mathcal{E}_z(\gamma)}+\epsilon,\qquad\forall\gamma\in\Omega_z.\end{aligned}\tag{4.3}$$

Moreover, there exists a continuous homotopy $h_s\colon\Lambda\to\Lambda$ such that $h_0=\mathrm{id}$, $h_1=\nu\circ\sigma$, and

$$\sqrt{\mathcal{E}(h_s(\gamma))}<\sqrt{\mathcal{E}(\gamma)}+2\epsilon,\qquad\forall\gamma\in\Lambda,\ s\in[0,1].$$

Similarly, there exists a continuous homotopy $k_s\colon\Omega_z\to\Omega_z$ such that $k_0=\mathrm{id}$, $k_1=\sigma\circ\nu$, and

$$\sqrt{\mathcal{E}_z(k_s(\gamma))}<\sqrt{\mathcal{E}_z(\gamma)}+2\epsilon,\qquad\forall\gamma\in\Omega_z,\ s\in[0,1].$$

Proof. For each $x, y \in M$ with $d(x,y) < \mathrm{injrad}(g)$, we denote by $\alpha_{x,y}$ the unit-speed geodesic segment joining x and y. If $\gamma_1\colon [0,\tau_1] \to M$ and $\gamma_2\colon [0,\tau_2] \to M$ are continuous paths such that $\gamma_1(\tau_1) = \gamma_2(0)$, we denote their concatenation by

$$\gamma_1 * \gamma_2\colon [0,\tau_1+\tau_2] \to M, \qquad \gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [0,\tau_1], \\ \gamma_1(t-\tau_1), & \text{if } t \in [\tau_1, \tau_1+\tau_2]. \end{cases}$$

For each $W^{1,2}$-path $\gamma\colon [0,\tau] \to M$, let $\overline{\gamma}\colon [0,1] \to M$ be the same path reparametrized with constant speed on the unit interval, so that

$$\int_0^t \|\dot{\overline{\gamma}}(r)\|_g \, dr = t \int_0^\tau \|\dot\gamma(r)\|_g \, dr, \qquad \forall t \in [0,1].$$

Notice that $\sqrt{\mathcal{E}(\overline{\gamma})}$ is the length of the original γ. Moreover, if $\gamma\colon [0,1] \to M$ is already defined on the unit interval, the Hölder inequality implies that $\mathcal{E}(\overline{\gamma}) \le \mathcal{E}(\gamma)$. The continuous map[2] $u\colon \Lambda \to \Lambda$, $u(\gamma) = \overline{\gamma}$ is homotopic to the identity via the continuous homotopy $u_s\colon \Lambda \to \Lambda$ defined as follows: for each $\gamma \in \Lambda$ and $s \in [0,1]$, the curve $\gamma_s := u_s(\gamma)$ satisfies $\gamma_s|_{[s,1]} = \gamma|_{[s,1]}$, whereas $\gamma_s|_{[0,s]}$ is the constant speed reparametrization of $\gamma|_{[0,s]}$. This homotopy preserves the sublevel sets of $\mathcal{E}$, for

$$\begin{aligned} \mathcal{E}(\gamma_s) &= \int_0^s \|\dot\gamma_s\|_g^2 \, dt + \int_s^1 \|\dot\gamma_s\|_g^2 \, dt = \frac{1}{s}\left(\int_0^s \|\dot\gamma_s\|_g \, dt\right)^2 + \int_s^1 \|\dot\gamma_s\|_g^2 \, dt \\ &= \frac{1}{s}\left(\int_0^s \|\dot\gamma\|_g \, dt\right)^2 + \int_s^1 \|\dot\gamma\|_g^2 \, dt \le \int_0^s \|\dot\gamma\|_g^2 \, dt + \int_s^1 \|\dot\gamma\|_g^2 \, dt = \mathcal{E}(\gamma). \end{aligned}$$

Let $z \in Z$, and define the maps $\sigma\colon \Lambda \to \Omega_z$ and $\nu\colon \Omega_z \to \Lambda$ by

$$\sigma(\gamma) = \overline{\gamma * \alpha_{\gamma(1),\psi_z(\gamma(1))}}, \qquad \nu(\gamma) = \overline{\gamma * \alpha_{\gamma(1),\psi_z^{-1}(\gamma(1))}}.$$

Clearly, these maps are continuous and satisfy the energy bounds (4.3). Observe that for the continuous homotopy $\zeta_s\colon M \to M$ from $\zeta_0 = \mathrm{id}$ to $\zeta_1 = \psi_z$ given by

$$\zeta_s(x) = \alpha_{x,\psi_z(x)}(s\, d(x,\psi_z(x))) \quad \text{where} \quad 0 \le s \le 1,$$

$d(x,\zeta_s(x)) = s\, d(x,\psi_z(x)) < \epsilon$ for all $x \in M$ and $s \in [0,1]$. Next, employing ζ_s, we define another continuous homotopy $v_s\colon \Lambda \to \Lambda$ by

$$v_s(\gamma) = \overline{\gamma * \alpha_{\gamma(1),\zeta_s(\gamma(1))} * \alpha_{\zeta_s(\gamma(1)),\gamma(1)}}.$$

This homotopy satisfies the properties $v_0(\gamma) = \overline{\gamma}$, $v_1(\gamma) = \nu \circ \sigma(\gamma)$ and

$$\sqrt{\mathcal{E}(v_s(\gamma))} = \sqrt{\mathcal{E}(\overline{\gamma})} + 2s\, d(\gamma(1),\psi_z(\gamma(1))) < \sqrt{\mathcal{E}(\gamma)} + 2\epsilon.$$

Finally, we construct our desired continuous homotopy $h_s\colon \Lambda \to \Lambda$ by juxtaposition of the homotopies u_s and v_s, i.e.,

$$h_s := \begin{cases} u_{2s}, & \text{if } s \in [0,1/2], \\ v_{2s-1} & \text{if } s \in [1/2,1]. \end{cases}$$

The construction of the other continuous homotopy $k_s\colon \Omega_z \to \Omega_z$ is analogous. □

As in Section 2.2, we denote by $b_{\epsilon,c}$ the number of bars of size at least ϵ and intersecting the interval $(0,c]$ in the closed geodesics barcode $\mathcal{B} = \mathcal{B}(g;\mathbb{F})$, for a fixed coefficient field $\mathbb{F}$ which we suppress in the notation.

[2]The continuity of the map u, as well as the continuity of the homotopy u_s, is intuitive but perhaps not obvious. A full proof was provided in [Ano80, Theorem 2] by Anosov himself!

Lemma 4.3. *For each $z \in Z$ such that all the critical points in $\mathrm{crit}^+(\mathcal{E}_z)$ are non-degenerate, we have*

$$\#\mathrm{crit}(\mathcal{E}_z)^{[\epsilon,c+\epsilon]} \geq b_{2\epsilon,c}, \qquad \forall c > \epsilon.$$

Proof. Recall from Section 2.1 that $\mathcal{B}$ is the barcode associated with the persistence module $(H_b := H_*(\Lambda^{<b}, \Lambda^0), i_{b,a})_{b>a>0}$, where the maps $i_{b,a} \colon H_a \to H_b$ are the homomorphisms induced by the inclusion. Pick $z \in Z$ such that all the critical points in $\mathrm{crit}^+(\mathcal{E}_z)$ are non-degenerate. Consider the analogous persistence module $(K_b := H_*(\Omega_z^{<b}, \Omega_z^{<\epsilon}), j_{b,a})_{b>a>\epsilon}$, where $j_{b,a} \colon K_a \to K_b$ are again the homomorphisms induced by the inclusion and let $\mathcal{C}$ be the barcode associated with $(K_b, j_{b,a})_{b>a>\epsilon}$.

Since we chose $\epsilon < \frac{1}{2}\,\mathrm{injrad}(g)$, the functional $\sqrt{\mathcal{E}}$ does not have critical values in $(0, 2\epsilon]$ and $\sqrt{\mathcal{E}_z}$ does not have critical values in $[\epsilon, 3\epsilon]$. Therefore, the inclusions $\Lambda^0 \hookrightarrow \Lambda^{<2\epsilon}$ and $\Omega_z^{<\epsilon} \hookrightarrow \Omega_z^{<3\epsilon}$ are homotopy equivalences and induce isomorphisms

$$H_c = H_*(\Lambda^{<c}, \Lambda^0) \xrightarrow{\cong} H_*(\Lambda^{<c}, \Lambda^{<2\epsilon}), \tag{4.4}$$

$$K_c = H_*(\Omega_z^{<c}, \Omega_z^{<\epsilon}) \xrightarrow{\cong} H_*(\Omega_z^{<c}, \Omega_z^{<3\epsilon}). \tag{4.5}$$

The maps $\sigma \colon \Lambda \to \Omega_z$ and $\nu \colon \Omega_z \to \Lambda$ from Lemma 4.2 induce homomorphisms

$$\sigma_* \colon H_c \to K_{c+\epsilon}, \qquad \nu_* \colon K_c \to H_*(\Lambda^{<c}, \Lambda^{<2\epsilon})$$

by the first and, respectively, the second energy bound in (4.3). Composing the latter homomorphisms with the inverse of the isomorphisms (4.4), we obtain homomorphisms

$$\nu_* \colon K_c \to H_{c+\epsilon},$$

which we still denote by ν_* with a slight abuse of notation. The homotopies provided by Lemma 4.2, together with the isomorphisms (4.4) and (4.5) induced by the inclusion, imply that we have commutative diagrams

$$\begin{tikzcd}
H_c \arrow[rr, "i_{c+2\epsilon,c}"] \arrow[dr, "\sigma_*"] & & H_{c+2\epsilon} \arrow[dr, "\sigma_*"] & \\
& K_{c+\epsilon} \arrow[ur, "\nu_*"] \arrow[rr, "j_{c+3\epsilon,c+\epsilon}"] & & K_{c+3\epsilon}
\end{tikzcd}$$

In the language of persistence modules, these commutative diagrams mean precisely that $(H_b, i_{b,a})_{b>a>0}$ and $(K_b, j_{b,a})_{b>a>\epsilon}$ are ϵ*-interleaved*. This property allows us to invoke the isometry theorem from the theory of persistence modules [CdSGO16, Theorem 5.14], which provides a so called ϵ*-matching* between the barcodes $\mathcal{B}$ and $\mathcal{C}$. Namely, consider each barcode as a collection of intervals (bars) in which every interval is allowed to be repeated according to its multiplicity. For each $\delta \geq 0$, we denote by $\mathcal{B}_\delta \subset \mathcal{B}$ and $\mathcal{C}_\delta \subset \mathcal{C}$ the respective subcollections of bars $[a, b)$ of size $b - a \geq \delta$. The ϵ-matching is a bijection $f \colon \mathcal{B}' \to \mathcal{C}'$ from subcollections $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{B}_{2\epsilon} \subset \mathcal{B}'$, $\mathcal{C}_{2\epsilon} \subset \mathcal{C}'$, and if $f([a, b)) = [c, d)$ then $[a, b) \subset [c - \epsilon, d + \epsilon)$ and $[c, d) \subset [a - \epsilon, b + \epsilon)$. For each $\delta \geq 0$ and $c > \epsilon$, we define $\mathcal{B}_{\delta,c} \subset \mathcal{B}_\delta$ and $\mathcal{C}_{\delta,c} \subset \mathcal{C}_\delta$ as the respective subcollections of bars $[a, b)$ such that $a \leq c$. Notice that, with the notation recalled just before the statement of the lemma, $b_{\delta,c}$ is the cardinality of $\mathcal{B}_{\delta,c}$.

Since all critical points in $\mathrm{crit}^+(\mathcal{E}_z)$ are non-degenerate, for each compact interval $[a,b] \subset (0,\infty)$ the set of critical points $\mathrm{crit}(\mathcal{E}_z)^{[a,b]}$ is finite. Let $c > 0$ be a critical value of $\sqrt{\mathcal{E}_z}$ and fix $\delta > 0$ so small that $\sqrt{\mathcal{E}_z}$ has no critical values in the interval $(c, c+\delta] = \varnothing$. Morse theory implies that

$$\dim H_*(\Omega_z^{<c+\delta}, \Omega_z^{<c}) = \#\big(\mathrm{crit}(\mathcal{E}_z) \cap \mathcal{E}_z^{-1}(c^2)\big).$$

Consider the homomorphism $j_{c+\delta,c}\colon K_c \to K_{c+\delta}$. Notice that $\dim\ker(j_{c+\delta,c})$ is the number of bars in $\mathcal{C}$ of the form $[a,c)$ for some $a < c$, whereas $\dim\mathrm{coker}(j_{c+\delta,c})$ is the number of bars in $\mathcal{C}$ of the form $[c,b)$ for some $b > c$. The homology exact triangle associated with the inclusion $K_c \subset K_{c+\delta}$

$$\begin{tikzcd} K_c \arrow[r, "j_{c+\delta,c}"] & K_{c+\delta} \arrow[d] \\ & H_*(\Omega_z^{<c+\delta}, \Omega_z^{<c}) \arrow[ul, "\partial_*"] \end{tikzcd}$$

implies that

$$H_*(\Omega_z^{<c+\delta}, \Omega_z^{<c}) \cong \ker(j_{c+\delta,c}) \oplus \mathrm{coker}(j_{c+\delta,c}),$$

and, therefore, every critical point in $\mathrm{crit}(\mathcal{E}_z) \cap \mathcal{E}_z^{-1}(c^2)$ is the endpoint of exactly one bar in $\mathcal{C}$.

By the conclusion of the previous paragraph, for each $c > \epsilon$ the cardinality $\mathcal{C}_{0,c}$ is bounded from above by the cardinality of $\mathrm{crit}(\mathcal{E}_z)^{[\epsilon,c]}$. The ϵ-matching f readily implies that $f(\mathcal{B}_{2\epsilon,c}) \subset \mathcal{C}_{0,c+\epsilon}$. Thus we conclude that

$$b_{2\epsilon,c} = \#\mathcal{B}_{2\epsilon,c} = \#f(\mathcal{B}_{2\epsilon,c}) \leq \#\mathcal{C}_{0,c+\epsilon} \leq \#\mathrm{crit}(\mathcal{E}_z)^{[\epsilon,c+\epsilon]}. \qquad \square$$

4.3. **Lagrangian intersections.** We momentarily consider the geodesic flow on the whole tangent bundle, $\phi_t\colon TM \to TM$, $\phi_t(\dot\gamma(0)) = \dot\gamma(t)$, where $\gamma\colon \mathbb{R} \to M$ is a geodesic or a constant curve, and we lift it to an embedding

$$\Phi_t\colon TM \hookrightarrow TM \times TM, \qquad \Phi_t(v) = (v, \phi_t(v)). \tag{4.6}$$

Let as above $\Psi\colon Z \times TM \to TM \times TM$ be the associated tomograph map, which we treat as a family of maps $\Psi_z = \Psi(z,\cdot)\colon TM \to TM \times TM$ parametrized by $z \in Z$. There is a one-to-one correspondence

$$\begin{aligned} \mathrm{crit}(\mathcal{E}_z) &\overset{1:1}{\longleftrightarrow} \Psi_z(TM) \cap \Phi_1(TM) \\ \gamma &\longleftrightarrow (\dot\gamma(0), \dot\gamma(1)). \end{aligned} \tag{4.7}$$

A straightforward calculation shows that a critical point $\gamma \in \mathrm{crit}(\mathcal{E}_z)$ is non-degenerate if and only if $(\dot\gamma(0), \dot\gamma(1))$ is a transverse intersection point of $\Psi_z(TM)$ and $\Phi_1(TM)$. Notice that, by condition (iii) and the parametric transversality theorem, $\Psi_z(TM \setminus 0\text{-section}) \pitchfork \Phi_1(TM)$ for almost all $z \in Z$. Therefore, for almost all $z \in Z$, all critical points of the energy functional $\mathcal{E}_z$ with positive critical values are non-degenerate.

Since $\phi_t(cv) = c\,\phi_{ct}(v)$ for each $c > 0$, the one-to-one correspondence (4.7) can be rewritten as the following family of one-to-one correspondences for each $c > 0$:

$$\begin{aligned} \mathrm{crit}(\mathcal{E}_z) \cap \mathcal{E}_z^{-1}(c^2) &\overset{1:1}{\longleftrightarrow} \Psi_z(SM) \cap \Phi_c(SM) \\ \gamma &\longleftrightarrow \tfrac{1}{c}(\dot\gamma(0), \dot\gamma(1)). \end{aligned}$$

Let us equip the tangent bundle TM with the Sasaki Riemannian metric $\tilde{g}$ induced by the Riemannian metric g on M. This metric, in turn, gives rise to the product Riemannian metric on $TM \times TM$ and on all its submanifolds and, in particular, on $\Phi_t(SM)$. We denote by $\mathrm{vol}(\Phi_t(SM))$ the volume obtained by integrating the Riemannian density. Fix an open neighborhood $B \subset \mathbb{R}^k$ of the origin such that $\overline{B} \subset Z$. Next we need the following "Crofton inequality" (cf. [ÇGG21, Lemma 5.3], which is reminiscent of the classical Crofton formula.

Lemma 4.4. *There exists a constant $C > 0$ such that, for each compact interval $[c_0, c_1] \subset (0, \infty)$ with non-empty interior,*

$$\int_B \#\mathrm{crit}(\mathcal{E}_z)^{[c_0,c_1]}\, dz \leq C \int_{c_0}^{c_1} \mathrm{vol}(\Phi_t(SM))\, dt.$$

Proof. Throughout the proof we employ the formalism of densities in order to avoid orientability issues. If S is a submanifold of a Riemannian manifold M, we denote by $|dV_S|$ the Riemannian density of S induced by the restricted Riemannian metric and by $\mathrm{vol}(S)$ the volume of S with respect to this density, i.e.,

$$\mathrm{vol}(S) = \int_S |dV_S|.$$

For each $t \in \mathbb{R}$, set $L_t := \Phi_t(SM) \subset SM \times SM$. Fix a compact interval $[c_0, c_1] \subset (0, \infty)$ with non-empty interior and let

$$L := \bigcup_{t \in [c_0, c_1]} L_t \times \{t\}. \tag{4.8}$$

This is a submanifold with boundary of $SM \times SM \times [c_0, c_1]$. Equip $[c_0, c_1]$ with the Euclidean Riemannian metric and $SM \times SM \times [c_0, c_1]$ with the product metric. Let $\tau\colon L \to [c_0, c_1]$ be the projection onto the last factor, i.e., $\tau(v, \phi_t(v), t) = t$. By the smooth coarea formula (see, e.g., [BZ88, Sect. 13.4.3]), we have

$$\mathrm{vol}(L) = \int_{c_0}^{c_1} \left(\int_{L_t} \frac{1}{\|d\tau(y)\|_{\tilde{g}}} |dV_{L_t}(y)| \right) dt. \tag{4.9}$$

Let $X = \frac{d}{dt}\big|_{t=0} \phi_t$ be the geodesic vector field on SM. We recall that $\|X(v)\|_{\tilde{g}} = \|v\|_g = 1$ for all $v \in SM$. Therefore,

$$\|d\tau(v, \phi_t(v), t)\|_{\tilde{g}} = \frac{d\tau(v, \phi_t(v), t)(0, X(\phi_t(v)), 1)}{\sqrt{\|X(\phi_t(v))\|_{\tilde{g}}^2 + 1}} = \frac{1}{\sqrt{2}}.$$

Plugging this into the coarea formula (4.9) yields

$$\mathrm{vol}(L) = \frac{1}{\sqrt{2}} \int_{c_0}^{c_1} \mathrm{vol}(L_t) dt. \tag{4.10}$$

Next, let us lift Ψ to a map

$$\tilde{\Psi}\colon Z \times TM \times \mathbb{R} \to TM \times TM \times \mathbb{R}, \quad \tilde{\Psi}(z, v, t) = (\Psi_z(v), t).$$

By condition (iii), $\tilde{\Psi}$ is a submersion outside $B \times 0\text{-section} \times \mathbb{R}$. Hence the preimage $\tilde{\Psi}^{-1}(L) \subset Z \times SM \times [c_0, c_1]$ is a manifold with boundary of dimension $\dim B$. We set

$$P := \tilde{\Psi}^{-1}(L) \cap \big(B \times TM \times \mathbb{R}\big) \subset B \times SM \times [c_0, c_1]$$

and let $\pi\colon P \to B$ be the projection onto the first factor, i.e., $\pi(z,v,t)=z$. Then

$$N(z) := \#\mathrm{crit}(\mathcal{E}_z)^{[c_0,c_1]} = \#\big(\pi^{-1}(z)\big)$$

is finite for almost all $z \in B$. Therefore,

$$\int_B N(z)\,dz = \int_P |\pi^* dz|. \tag{4.11}$$

Here dz is the Euclidean volume form on B and $|\pi^* dz|$ is the density associated with the pullback $\pi^* dz$. Let us equip $E := B \times SM \times \mathbb{R}$ with the product Riemannian metric which is Euclidean on the factors B and $\mathbb{R}$, and $P \subset E$ with the induced Riemannian metric. Notice that the differential $d\pi(y)$ is a contraction for all $y \in P$; namely, the Riemannian norm of any tangent vector $w \in T_y P$ is larger than or equal to the Euclidean norm of its image $d\pi(y)w$. This readily implies that $|\pi^* dz| = f|dV_P|$ for some smooth function $f\colon P \to [0,1]$ and hence

$$\int_P |\pi^* dz| \le \int_P |dV_P| = \mathrm{vol}(P). \tag{4.12}$$

Since $\tilde\Psi|_P\colon P \to L$ is a submersion, the volume of P can be computed by means of the smooth coarea formula as

$$\mathrm{vol}(P) = \int_L \left(\int_{\tilde\Psi^{-1}(y)} \frac{1}{J(w)}\,|dV_{\tilde\Psi^{-1}(y)}(w)| \right) |dV_L(y)|, \tag{4.13}$$

where J denotes the Riemannian Jacobian of $\tilde\Psi|_P$ normal to the fibers, i.e.,

$$J(w) = \sqrt{\det\big(d\tilde\Psi|_P(w)\, d\tilde\Psi|_P(w)^*\big)}.$$

We set

$$j(w) := \min_V \sqrt{\det\big(d\tilde\Psi(w)|_V\, d\tilde\Psi(w)|_V^*\big)} > 0,$$

where V ranges over all vector subspaces $V \subset T_w\overline{E}$ of dimension $\dim L$ orthogonal to the fiber $\tilde\Psi^{-1}(\tilde\Psi(w))$. The function j is defined on the non-compact space $\overline{E} = \overline{B} \times SM \times \mathbb{R}$. Nevertheless, since $\tilde\Psi$ is the direct sum of Ψ with the identity on $\mathbb{R}$, we readily see that $j(z,v,t)$ is independent of $t \in \mathbb{R}$ and hence it has a positive lower bound

$$j_0 := \min_{\overline{E}} j > 0.$$

For any $w \in P$, if $N_w \subset T_w P$ is the orthogonal complement of $\ker(d\tilde\Psi(w))$, we have

$$J(w) = \sqrt{\det\big(d\tilde\Psi(w)|_{N_w}\, d\tilde\Psi(w)|_{N_w}^*\big)} \ge j(w) \ge j_0.$$

Together with the coarea formula (4.13), this implies that

$$\mathrm{vol}(P) \le \underbrace{\frac{1}{j_0} \max_{y \in SM \times SM} \big\{\mathrm{vol}(\Psi^{-1}(y))\big\}}_{C} \mathrm{vol}(L).$$

Notice that the finite quantity C is independent of the value c_1. Finally, from this inequality combined with (4.10), (4.11) and (4.12), we obtain the Crofton inequality

$$\int_B N(z)\,dz \le \frac{C}{\sqrt{2}} \int_{c_0}^{c_1} \mathrm{vol}(L_t)\,dt. \qquad \square$$

Having established Lemmas 4.3, and 4.4 we are now in a position to prove Theorem A. We recall that the barcode entropy $\hbar = \hbar(g;\mathbb{F})$ and the volume-growth entropy $h_{\rm vol} = h_{\rm vol}(g)$ are defined by

$$\hbar := \lim_{\epsilon\to 0^+} \limsup_{c\to\infty} \frac{\log^+(b_{2\epsilon,c})}{c}, \qquad h_{\rm vol} := \limsup_{t\to\infty} \frac{\log(V(t))}{t},$$

where $V(t) = \mathrm{vol}(\Phi_t(SM))$.

Theorem A. *On any closed Riemannian manifold and for any coefficient field, we have $\hbar \leq h_{\rm vol}$.*

Proof. For each $r > h_{\rm vol}$ there exists $k > 0$ such that $V(t) < k\, r\, e^{rt}$ for all $t > 0$ large enough. Therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\int_0^t V(s)\,ds\right) < \limsup_{t\to\infty} \frac{\log(k) + rt}{t} = r.$$

In a similar vein, if $h_{\rm vol} > 0$, for any $r \in (0, h_{\rm vol})$ there exists $k > 0$ such that $V(t) > k\, r\, e^{rt}$ for all $t > 0$ large enough, and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\int_0^t V(s)\,ds\right) > \limsup_{t\to\infty} \frac{\log(k) + rt}{t} = r.$$

We conclude

$$h_{\rm vol} = \limsup_{t\to\infty} \frac{1}{t} \log\left(\int_0^t V(s)\,ds\right) = \limsup_{c\to\infty} \frac{1}{c} \log\left(\int_\epsilon^c V(s)\,ds\right) \tag{4.14}$$

for any $\epsilon > 0$. Let us fix $\epsilon \in (0, \frac{1}{2}\,\mathrm{injrad}(g))$, and consider a tomograph satisfying the displacement bound (4.2). Lemma 4.4 and (4.14) imply that

$$h_{\rm vol} \geq \limsup_{c\to\infty} \frac{1}{c} \log^+\left(\int_B N_c(z)\,dz\right), \tag{4.15}$$

where $N_c(z) := \#\mathrm{crit}(\mathcal{E}_z)^{[\epsilon,c]}$. For almost every $z \in B$, all critical points in $\mathrm{crit}^+(\mathcal{E}_z)$ are non-degenerate. By Lemma 4.3, for all $c > \epsilon$ we have

$$\int_B N_c(z)\,dz \geq \mathrm{vol}(B)\, b_{2\epsilon,c-\epsilon}.$$

This inequality, together with the lower bound (4.15), implies that

$$h_{\rm vol} \geq \limsup_{c\to\infty} \frac{\log^+(\mathrm{vol}(B)\, b_{2\epsilon,c-\epsilon})}{c} = \limsup_{c\to\infty} \frac{\log^+(b_{2\epsilon,c})}{c}.$$

Finally, taking the limit for $\epsilon \to 0^+$, we conclude that $h_{\rm vol} \geq \hbar$. ☐

Remark 4.5 (Floer-theoretic approach)**.** The inequality $\hbar \leq h_{\rm top}$ can be proved by means of Floer-theoretic methods, although the argument is quite indirect. Here we briefly outline it.

Consider a closed Riemannian manifold (M, g). With a common abuse of notation, we still denote by g the Riemannian metric on covectors, and we consider the function $r : T^*M \to [0,\infty)$, $r(p) := \|p\|_g$. As in Section 3, we denote by $\sigma(g)$ the length spectrum of our Riemannian manifold. For $\epsilon > 0$ small enough and $a \notin \sigma(g)$, let $G = G_{a,\epsilon} : T^*M \to \mathbb{R}$ be the piecewise smooth Hamiltonian such that $G \equiv 0$ on $\{r \leq 1 + \epsilon\}$, and $G = ar - a(1+\epsilon)$ on $\{1 + \epsilon \leq r\}$. Let $F = F_{a,\epsilon} : T^*M \to \mathbb{R}$ be a smooth, fiberwise monotone increasing and convex Hamiltonian approximating G, and equal to G outside the shell $S = S_\epsilon = \{1 \le r \le 1+\epsilon\}$. Notice that all the periodic orbits of the Hamiltonian flow ϕ_F^t with integer period lie in the compact set $\{r \le 1+\epsilon\}$. Hamiltonians of this type are routinely employed in the construction of the symplectic homology. The topological entropy of the restricted Hamiltonian flow $\phi_F^t|_S$ coincides with the topological entropy of the reparametrized geodesic flow $\psi^t(v) = \phi^{at}(v)$, which is $a\, h_{\mathrm{top}}(g)$.

For $k \in \mathbb{N}$ and $c > 0$, consider the filtered Floer complex $CF^{(-\infty,c)}(kF)$, which is generated by the k-periodic orbits of the Hamiltonian flow ϕ_F^t with action in $(-\infty, c)$. The associated filtered Floer homology $HF^{(-\infty,c)}(kF)$ forms a persistence module, which defines the barcode entropy $\hbar(F)$, even though the Hamiltonian F is not compactly supported and does not exactly fit into the framework of [ÇGG21]. After suitable modifications, the proof of [ÇGG21, Thm. A] carries over, and therefore $\hbar(F)$ is bounded from above by the topological entropy of $\phi_F^t|_S$. In other words,

$$\hbar(F) \le a\, h_{\mathrm{top}}(g).$$

With the notation from Section 2.1, let $H_a := H_*(\Lambda^{<a}, \Lambda^0)$ be the homology in the persistence module associated with the energy functional. The results from [AS06, SW06] and [Web06] or alternatively directly from [Vit99] assert that $H_a \cong HF^{(-\infty,b)}(F_{a,\epsilon})$ provided $|a-b|$ and $\epsilon > 0$ are small enough. As a consequence, after a suitable manipulation of limits, it is not hard to see that

$$\hbar(g) = \lim_{\epsilon \to 0^+} \lim_{a \to 1^+} \hbar(F_{a,\epsilon}),$$

and hence $\hbar(g) \le h_{\mathrm{top}}(g)$.

After our work has appeared, the inequality $\hbar \le h_{\mathrm{top}}$, for a suitable notion of barcode entropy defined via symplectic homology, has been proved by Fender, Lee, and Sohn [FLS23] for the Reeb flows on the boundary of certain Liouville domains.

4.4. **Relative barcode entropy.** The notion of barcode entropy has a relative analogue similar to the one in [ÇGG21, Def. 2.1]. Namely, let (M, g) be a closed Riemannian manifold and let $Q \subset M \times M$ be a closed submanifold of the product. Equip $M \times M$ with the product Riemannian metric $\frac{1}{2} g \oplus g$. We denote by $TQ^\perp$ the normal bundle of Q with respect of this metric and by $SQ^\perp := TQ^\perp \cap S(M \times M)$ the unit normal bundle of Q. Let Ω be the space of $W^{1,2}$-paths $\gamma\colon [0,1] \to M$ such that $(\gamma(0), \gamma(1)) \in Q$. The energy functional $\mathcal{E}\colon \Omega \to [0,\infty)$ is again defined by the expression (2.1). The critical points of $\mathcal{E}$ are geodesics or constant curves $\gamma \in \Omega$ such that $(-\dot\gamma(0), \dot\gamma(1)) \in TQ^\perp$. Consider the geodesic flow $\phi_t : SM \to SM$ and its lift

$$\Phi_t : SM \to SM \times SM, \qquad \Phi_t(v) = (-v, \phi_t(v)).$$

This map is the same as the one defined in (4.6), except for the minus sign in the first factor; clearly, the exponential growth rate of the volume of $\Phi_t(SM)$, measured with respect to the Riemannian volume form, is still the volume-growth entropy $h_{\mathrm{vol}} = h_{\mathrm{vol}}(g)$. The filtration $\mathcal{E}^{-1}[0, a^2) \subset \Omega$ forms a persistence module, which allows us to define the *relative barcode entropy* $\hbar(Q) = \hbar(Q; g)$, arguing as in Definition 2.1. For each $c > 0$, we have a one-to-one correspondence

$$\begin{aligned} \mathrm{crit}(\mathcal{E}_z) \cap \mathcal{E}_z^{-1}(c^2) &\overset{1:1}{\longleftrightarrow} SQ^\perp \cap \Phi_c(SM) \\ \gamma &\longleftrightarrow \tfrac{1}{c}(\dot\gamma(0), \dot\gamma(1)). \end{aligned}$$

Generalizing the arguments in the proof of Theorem A, one can prove that

$$\hbar(Q) \leq h_{\mathrm{vol}}. \tag{4.16}$$

In the special case where $Q = \Delta = \{(x, x) \mid x \in M\}$, we have $\Omega = \Lambda$, and the relative barcode entropy $\hbar(\Delta)$ reduces to the ordinary barcode entropy $\hbar$. Therefore, the inequality of Theorem A is a special case of (4.16). On the other hand, it is not clear to us how to extend Theorem B to relative barcode entropy.

Another interesting particular case of (4.16) is when $Q = N_0 \times N_1$ where N_0 and N_1 are closed submanifolds of M. Then Ω is the space of paths in M connecting N_0 to N_1 and the critical points of $\mathcal{E}$ are the geodesics connecting these submanifolds and orthogonal to both of them. This version of relative barcode entropy is a literal Morse-theoretic counterpart for geodesic flows of [ÇGG21, Def. 2.1] and (4.16) turns then into an analogue of [ÇGG21, Thm. 5.1].

5. Lower bound on the barcode entropy

Postponing the proof of Theorem D and Corollary E to the end of Section 7, let us now derive Theorem B and Corollaries C from Corollary E. The arguments are similar to the ones from [ÇGG21, Thm. B and C], and require some familiarity with ergodic theory. Let $\phi_t \colon SM \to SM$ be the geodesic flow on the unit tangent bundle SM of a closed Riemannian manifold (M, g). Given a ϕ_t-invariant probability measure μ on SM, we denote by h_μ the measure-theoretic entropy of the geodesic flow ϕ_t with respect to μ; see, e.g., [FH19, Def. 4.1.1]. We briefly refer to h_μ as to the entropy of μ.

Theorem B. *On any closed Riemannian manifold and for any coefficient field, if I is a hyperbolic compact invariant subset of the geodesic flow, then $\hbar \geq h_{\mathrm{top}}(I)$.*

Proof. Let $I \subset SM$ be a hyperbolic compact invariant subset of the geodesic flow $\phi_t \colon SM \to SM$ with positive topological entropy $h_{\mathrm{top}}(I) > 0$. By the variational principle for entropy, [FH19, Cor. 4.3.9], for each $\rho > 0$ there exists an invariant probability measure μ supported in I with entropy $h_\mu \geq h_{\mathrm{top}}(I) - \rho$. Since h_μ is the average of $\nu \mapsto h_\nu$ over the ergodic components of μ, [Wal82, Th. 8.4], there exists an ergodic invariant probability measure ν whose support is contained in the support of μ such that $h_\nu \geq h_\mu - \rho$. Since I is a hyperbolic compact invariant subset, the restricted geodesic flow $\phi_t|_I$ has only one zero Lyapunov exponent at every point: the one corresponding to the geodesic vector field. In particular, ν has only one zero Lyapunov exponent. Suppose that $\rho > 0$ is so small that $h_\nu > 0$. The measure ν satisfies the assumptions of a theorem of Lian and Young [LY12, Th. D'], which extends to flows a result of Katok and Mendoza for surface diffeomorphisms [KH95, Th. S.5.9(1)], and implies the existence of a locally maximal hyperbolic compact invariant subset $I_\rho \subset SM$ such that $h_{\mathrm{top}}(I_\rho) \geq h_\nu - \rho \geq h_{\mathrm{top}}(I) - 2\rho$.

Let $p(c)$ be the number of closed orbits of minimal period at most c of the restricted geodesic flow $\phi_t|_{I_\rho}$. We recall that a critical circle $S^1 \cdot \gamma \subset \mathrm{crit}^+(\mathcal{E})$ is called prime when the closed geodesic γ is not iterated. By definition, $p(c)$ is equal to the number of prime critical circles of $\mathcal{E}$ in $\mathcal{P}^{\leq c}(I_\rho) := \mathcal{P}(I_\rho) \cap \mathcal{E}^{-1}(0, c^2]$. Since the compact invariant set I_ρ is hyperbolic, we have

$$h_{\mathrm{top}}(I_\rho) = \limsup_{c \to \infty} \frac{\log^+(p(c))}{c}, \tag{5.1}$$

see, e.g., [FH19, Thm. 4.2.24 and Rem. 4.2.25].

Let $\mathcal{B} = \mathcal{B}(g;\mathbb{F})$ be the closed geodesics barcode. As in Section 2.2, we denote by $b_{\epsilon,c}$ the number of bars in $\mathcal{B}$ having size at least ϵ and intersecting the interval $(0,c]$. Since I_ρ is hyperbolic, it is expansive (see, e.g., [FH19, Thm. 5.4.22]) and all closed geodesics tangent to I_ρ are non-degenerate. Therefore, Corollary E provides a constant $\delta > 0$ such that $b_{\delta,c} \geq \frac{1}{2}p(c)$ for all $c > 0$. Since $b_{\epsilon,c} \geq b_{\delta,c}$ for all $\epsilon \in (0,\delta)$, we have

$$\hbar = \lim_{\epsilon\to 0^+} \limsup_{c\to\infty} \frac{\log^+(b_{\epsilon,c})}{c} \geq \limsup_{c\to\infty} \frac{\log^+(p(c))}{c} = h_{\rm top}(I_\rho) \geq h_{\rm top}(I) - 2\rho.$$

This inequality holds for an arbitrarily small $\rho > 0$, and hence $\hbar \geq h_{\rm top}(I)$. □

Corollary C. *On any closed Riemannian surface and for any coefficient field, we have* $\hbar = h_{\rm vol} = h_{\rm top}$.

Proof. Theorem A and Yomdin theorem provide the inequalities

$$\hbar \leq h_{\rm vol} \leq h_{\rm top}. \tag{5.2}$$

Assume now that our closed Riemannian manifold M has dimension 2, and its geodesic flow has positive topological entropy $h_{\rm top} > 0$. Applying the variational principle for entropy and the ergodic decomposition as in the proof of Theorem B, for each sufficiently small $\epsilon > 0$ we obtain an ergodic probability measure μ on SM that is invariant under the geodesic flow and has entropy bounded from below as $h_\mu \geq h_{\rm top} - \epsilon > 0$. By the Ruelle inequality [KH95, Thm. S.2.13], the measure μ has one positive Lyapunov exponent and one negative Lyapunov exponent. Since SM has dimension 3, μ has three Lyapunov exponents, and therefore only one zero Lyapunov exponent: the one corresponding to the geodesic vector field. This allows us to apply [LY12, Thm. D′] as in the proof of Theorem B, and obtain a locally maximal hyperbolic compact invariant subset $I_\epsilon \subset SM$ such that $h_{\rm top}(I_\epsilon) \geq h_\mu - \epsilon \geq h_{\rm top} - 2\epsilon$. This, combined with Theorem B, implies that $\hbar \geq h_{\rm top} - 2\epsilon$ for any arbitrarily small $\epsilon > 0$, and therefore $\hbar \geq h_{\rm top}$. This, together with (5.2), implies the desired identity $\hbar = h_{\rm vol} = h_{\rm top}$. □

6. Crossing energy bound

The rest of the paper is devoted to the proof of Theorem D, which provides a uniform lower bound for the size of the bars associated with a locally maximal, expansive, compact invariant subset of the geodesic flow. As we explained in Section 2.3, the proof of Theorem D hinges on a uniform crossing energy bound, which we will establish at the end of this section in Proposition 6.6, after necessary preliminaries.

6.1. Functional setting in period τ.

Let (M,g) be a closed Riemannian manifold. In order to study its closed geodesics, it suffices to focus on the 1-periodic ones. Nevertheless, to establish a uniform crossing energy bound, it will be useful for us to work with closed geodesics of any period. Let us introduce the setting.

Let $\Pi := W^{1,2}_{\rm loc}(\mathbb{R},M)$ be the free path space, endowed with the $W^{1,2}_{\rm loc}$-topology. For each $\tau > 0$, we denote the subspace of τ-periodic loops by

$$\Lambda_\tau = \big\{\gamma \in \Pi \ \big|\ \gamma(t) = \gamma(t+\tau) \ \ \forall t \in \mathbb{R}\big\}.$$

Then we have the energy functional

$$\mathcal{E}_\tau\colon \Lambda_\tau \to [0,\infty), \qquad \mathcal{E}_\tau(\gamma) = \int_0^\tau \|\dot\gamma(t)\|_g^2\, dt,$$

whose positive critical set $\mathrm{crit}^+(\mathcal{E}_\tau) := \mathrm{crit}(\mathcal{E}_\tau) \cap \mathcal{E}_\tau^{-1}(0,\infty)$ consists of the τ-periodic closed geodesics. For $\tau = 1$, this functional setting reduces to the one from Section 2.1.

6.2. **Finite-dimensional approximations of the path space.** Recall that the Riemannian distance of (M,g) is

$$d\colon M \times M \to [0,\infty), \qquad d(x,y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\|_g\, dt,$$

where the infimum is taken over all absolutely continuous curves $\gamma\colon [0,1] \to M$ with endpoints $\gamma(0) = x$ and $\gamma(1) = y$. Let

$$\rho := \mathrm{injrad}(g) > 0$$

be the injectivity radius of (M,g). Consider the space of sequences

$$P = \big\{ \boldsymbol{x} = (x_i)_{i\in\mathbb{Z}} \in M^{\mathbb{Z}} \ \big|\ d(x_i, x_{i+1}) \le \rho/2 \ \ \forall i \in \mathbb{Z} \big\}.$$

For every $\sigma > 0$, we can view P as a subspace of the free path space Π via the embedding

$$\iota_\sigma\colon P \hookrightarrow \Pi, \qquad \boldsymbol{x} \mapsto \iota_\sigma(\boldsymbol{x}) = \gamma_{\boldsymbol{x},\sigma}. \tag{6.1}$$

Here $\gamma_{\boldsymbol{x},\sigma} \in \Pi$ is the unique curve such that, for each $i \in \mathbb{Z}$, the restriction $\gamma_{\boldsymbol{x},\sigma}|_{[i\sigma,(i+1)\sigma]}$ is a smooth geodesic segment and $\gamma_{\boldsymbol{x},\sigma}(i\sigma) = x_i$. Notice that the smaller the parameter σ, the better P "approximates" the path space Π.

For each integer $k \ge 2$, we define the space of k-periodic sequences

$$L_k = \big\{ \boldsymbol{x} \in P \ \big|\ x_i = x_{i+k} \ \ \forall i \in \mathbb{Z} \big\}.$$

This is a finite-dimensional compact manifold with corners, of dimension

$$\dim L_k = k \dim M.$$

Then we can identify L_k with a submanifold of the free loop space $\Lambda_{k\sigma}$ via the restriction $\iota_\sigma\colon L_k \hookrightarrow \Lambda_{k\sigma}$ of the embedding (6.1). We denote the restriction of the energy functional to L_k by

$$E_{k,\sigma} := \mathcal{E}_{k\sigma} \circ \iota_\sigma, \qquad E_{k,\sigma}(\boldsymbol{x}) = \frac{1}{\sigma} \sum_{i=0}^{k-1} d(x_i, x_{i+1})^2.$$

The positive critical set $\mathrm{crit}^+(E_{k,\sigma}) := \mathrm{crit}(E_{k,\sigma}) \cap E_{k,\sigma}^{-1}(0,\infty)$ consists of $\boldsymbol{x} \in L_k$ such that the associated curve $\gamma_{\boldsymbol{x},\sigma} = \iota_\sigma(\boldsymbol{x})$ is a $k\sigma$-periodic closed geodesic.

Let us equip the space L_k with the Riemannian metric

$$g_k(\boldsymbol{v}, \boldsymbol{w}) := \sum_{i=0}^{k-1} g(v_i, w_i),$$

which induces the Riemannian distance $d_k\colon L_k \times L_k \to [0,\infty)$. Notice that d_k and d are related by

$$d_k(\boldsymbol{x}, \boldsymbol{y})^2 \ge \sum_{i=0}^{k-1} d(x_i, y_i)^2,$$

and actually the equality holds at least when $\boldsymbol{x}$ and $\boldsymbol{y}$ are contained in the interior of L_k and are sufficiently close therein. One may also introduce a Riemannian distance on the infinite-dimensional free loop space Λ. Instead, we will only need to consider the C^0 distance on the free path space Π, which is given by

$$d_{C^0}\colon \Pi\times\Pi\to[0,\infty),\qquad d_{C^0}(\gamma_1,\gamma_2)=\sup_{t\in\mathbb{R}} d(\gamma_1(t),\gamma_2(t)).$$

When restricted to L_k via the embedding $\iota_{k,\sigma}$, the distances d_{C^0} and d_k are equivalent. Here we just prove the following weaker statement, which is sufficient for our purposes.

Lemma 6.1. *For every $R>0$, there exists $r>0$ with the following property. For each $\sigma>0$ and $\boldsymbol{x},\boldsymbol{y}\in P$ such that $d_{C^0}(\gamma_{\boldsymbol{x},\sigma},\gamma_{\boldsymbol{y},\sigma})\geq R$, there exists $i\in\mathbb{Z}$ such that $d(x_i,y_i)\geq r$.*

Proof. It is sufficient to prove the lemma for the space of pairs

$$K:=\big\{\boldsymbol{x}=(x_0,x_1)\in M\times M\ \big|\ d(x_0,x_1)\leq\rho/2\big\}.$$

For each $\boldsymbol{x}\in K$, let $\gamma_{\boldsymbol{x}}\colon[0,1]\to M$ be the unique shortest geodesic segment joining $x_0=\gamma_{\boldsymbol{x}}(0)$ and $x_1=\gamma_{\boldsymbol{x}}(1)$, i.e., $\gamma_{\boldsymbol{x}}(t)=\exp_{x_0}(t\exp_{x_0}^{-1}(x_1))$. We denote the C^0-distance function on K by

$$D\colon K\times K\to[0,\infty),\qquad D(\boldsymbol{x},\boldsymbol{y})=\max_{t\in[0,1]} d(\gamma_{\boldsymbol{x}}(t),\gamma_{\boldsymbol{y}}(t)).$$

The continuous function $F\colon[0,\infty)\to[0,\infty)$ defined by

$$F(r)=\max\big\{D(\boldsymbol{x},\boldsymbol{y})\ \big|\ d(x_0,y_0)^2+d(x_1,y_1)^2\leq 4r^2\big\}$$

is everywhere finite due to the compactness of K. Notice that $F(0)=0$, $F(r)>0$ for each $r>0$, and F is monotone increasing (perhaps not strictly monotone increasing). Given $R>0$, we set

$$r:=\inf\big\{r'>0\ \big|\ F(r')\geq R\big\}\in(0,\infty],$$

where as usual $\inf\varnothing=\infty$. If $D(\boldsymbol{x},\boldsymbol{y})\geq R$ for $\boldsymbol{x},\boldsymbol{y}\in K$, then $d(x_0,y_0)^2+d(x_1,y_1)^2\geq 4r^2$, and the inequality $d(x_i,y_i)\geq r$ must hold for at least one value of $i\in\{0,1\}$. □

6.3. **Abstract crossing energy bound.** The gradient of the energy $E_{k,\sigma}$ with respect to the Riemannian metric g_k is given by $\nabla E_{k,\sigma}(\boldsymbol{x})=\boldsymbol{w}$, where

$$w_i=2\big(\dot\gamma_{\boldsymbol{x},\sigma}(\sigma i^-)-\dot\gamma_{\boldsymbol{x},\sigma}(\sigma i^+)\big),\qquad \forall i\in\mathbb{Z}.$$

The following statement is a refinement of the classical Palais–Smale compactness condition for the energy functional $E_{k,\sigma}$.

Lemma 6.2. *For all sequences*

$$k_n\in\mathbb{N}\cap[2,\infty),\qquad \boldsymbol{x}_n=(x_{n,i})_{i\in\mathbb{Z}}\in L_{k_n},\qquad \sigma_n\in[\sigma_0,\sigma_1]\subset(0,\infty),$$

satisfying

$$\lim_{n\to\infty}\|\nabla E_{k_n,\sigma_n}(\boldsymbol{x}_n)\|_{g_{k_n}}=0,\tag{6.2}$$

the corresponding sequence of curves $\gamma_n:=\gamma_{\boldsymbol{x}_n,\sigma_n}$ is compact in the $W^{1,2}_{\mathrm{loc}}$ topology, and any limit point of γ_n is either a geodesic $\gamma\colon\mathbb{R}\looparrowright M$ or a constant curve $\gamma\equiv\gamma(0)$.

Proof. Since M is compact, after extracting subsequences by means of a usual diagonal procedure, we can ensure that

$$\lim_{n\to\infty} x_{n,i} = x_i, \quad \forall i \in \mathbb{Z}, \qquad\qquad \lim_{n\to\infty} \sigma_n = \sigma \in [\sigma_0, \sigma_1].$$

Since $d(x_{n,i}, x_{n,i+1}) \leq \rho/2$, we have $d(x_i, x_{i+1}) \leq \rho/2$ as well and hence

$$\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}} \in P.$$

We set $\gamma := \gamma_{\boldsymbol{x},\sigma}$ and consider the monotone increasing affine diffeomorphisms

$$\theta_{n,i} \colon [i\sigma, (i+1)\sigma] \to [i\sigma_n, (i+1)\sigma_n].$$

Each segment $\gamma_n \circ \theta_{n,i}$ converges to $\gamma|_{[\sigma i,(i+1)\sigma]}$ in the C^∞-topology, and, in particular, the whole curves γ_n converge to γ in the $W^{1,2}_{\mathrm{loc}}$-topology. Assumption (6.2) implies that

$$\dot\gamma(\sigma i^-) - \dot\gamma(\sigma i^+) = \lim_{n\to\infty} \big(\dot\gamma_n(\sigma_n i^-) - \dot\gamma_n(\sigma_n i^+)\big) = 0, \qquad \forall i \in \mathbb{Z}.$$

Thus $\gamma\colon \mathbb{R} \to M$ is a smooth solution of the geodesic equation $\nabla_t \dot\gamma \equiv 0$. □

Adopting commonly used terminology, we call *anti-gradient flow segment* of $E_{k,\sigma}$ any smooth path $u\colon [s_0, s_1] \to L_k$ satisfying the ordinary differential equation $\dot u = -\nabla E_{k,\sigma}(u)$.

Lemma 6.3 (Abstract crossing energy bound)**.** *Let $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \Pi$ be two open subsets such that*

(i) *$\overline{\mathcal{U}_1 \setminus \mathcal{U}_0}$ does not contain any geodesic or any constant curve,*
(ii) *there exists $R > 0$ such that $d_{C^0}(\gamma_0, \gamma_1) \geq R$ for all $\gamma_0 \in \mathcal{U}_0$, $\gamma_1 \in \Pi \setminus \mathcal{U}_1$.*

For each $[\sigma_0, \sigma_1] \subset (0, \infty)$, there exists $\delta > 0$ with the following property: for each $\sigma \in [\sigma_0, \sigma_1]$, $k \in \mathbb{N} \cap [2, \infty)$ and an anti-gradient flow segment $u\colon [s_0, s_1] \to L_k$ of $E_{k,\sigma}$ that crosses the shell $\mathcal{U}_1 \setminus \mathcal{U}_0$ (i.e. $\iota_\sigma(u(r_0)) \in \mathcal{U}_0$ and $\iota_\sigma(u(r_1)) \notin \mathcal{U}_1$ for some $r_0, r_1 \in [s_0, s_1]$), we have

$$E_{k,\sigma}(u(s_0)) - E_{k,\sigma}(u(s_1)) \geq \delta.$$

Proof. Assumption (i), together with Lemma 6.2, provides a constant $\lambda > 0$ such that

$$\|\nabla E_{k,\sigma}(\boldsymbol{x})\|_{g_k} \geq \lambda, \qquad \forall \sigma \in [\sigma_0, \sigma_1],\ k \in \mathbb{N} \cap [2, \infty),\ \boldsymbol{x} \in L_k \cap \mathcal{U}_1 \setminus \mathcal{U}_0.$$

Given $R > 0$ from assumption (ii), pick $r > 0$ as in Lemma 6.1. Let $u\colon [s_0, s_1] \to L_k$ be an anti-gradient flow segment of $E_{k,\sigma}$ that crosses the shell $\mathcal{U}_1 \setminus \mathcal{U}_0$. Let $[r_0, r_1] \subset [s_0, s_1]$ be a subinterval such that $\iota_\sigma(u(r_i)) \in \partial\mathcal{U}_0$ and $\iota_\sigma(u(r_{1-i})) \in \partial\mathcal{U}_1$ for some $i \in \{0, 1\}$, and $\iota_\sigma \circ u|_{(r_0,r_1)}$ is contained in $\mathcal{U}_1 \setminus \mathcal{U}_0$. By assumption (ii) and Lemma 6.1, we have

$$\int_{r_0}^{r_1} \|\dot u(s)\|_{g_k}\, ds \geq d_k(u(r_0), u(r_1)) \geq r.$$

Therefore,

$$\begin{aligned} E_{k,\sigma}(u(s_0)) - E_{k,\sigma}(u(s_1)) &= \int_{s_0}^{s_1} \|\nabla E_{k,\sigma}(u(s))\|_{g_k}^2 \, ds \\ &\geq \int_{r_0}^{r_1} \|\nabla E_{k,\sigma}(u(s))\|_{g_k} \|\dot{u}(s)\|_{g_k} \, ds \\ &\geq \lambda r =: \delta. \end{aligned}$$

□

In the next section, we will apply the abstract crossing energy bound from Lemma 6.3 to anti-gradient flow segments crossing neighborhoods of closed orbits contained in a suitable compact invariant set.

6.4. **Locally maximal, expansive, compact invariant sets.** For each $v \in SM$, let $\gamma_v \in \Pi$ be the corresponding geodesic such that $\dot{\gamma}_v(0) = v$. Notice that γ_v is parametrized by arc length, i.e., $\|\dot{\gamma}_v\|_g \equiv 1$. Let $I \subset SM$ be an invariant subset of the geodesic flow ϕ_t. We denote the space of unit-speed geodesics tangent to I by $\mathcal{G}(I) = \big\{\gamma_v \mid v \in I\big\} \subset \Pi$, the subspace of τ-periodic closed geodesics among them by $\mathcal{P}_\tau(I) = \mathcal{G}(I) \cap \Lambda_\tau$, and their union by

$$\mathcal{P}_*(I) := \bigcup_{\tau>0} \mathcal{P}_\tau(I).$$

Furthermore, we denote the open C^0-ball of radius r centered at $\gamma \in \Pi$ by

$$\mathcal{U}(\gamma, r) := \big\{\zeta \in \Pi \;\big|\; d_{C^0}(\gamma, \zeta) < r\big\}.$$

In a similar vein, we denote the open C^0-neighborhood of radius r of a subset $\mathcal{W} \subset \Pi$ by

$$\mathcal{U}(\mathcal{W}, r) := \bigcup_{\gamma \in \mathcal{W}} \mathcal{U}(\gamma, r).$$

The real line $\mathbb{R}$ acts on the free path space Π as

$$t \cdot \gamma = \gamma(t + \cdot) \in \Pi, \qquad t \in \mathbb{R},\ \gamma \in \Pi. \tag{6.3}$$

Notice that, if γ is a periodic curve (i.e., $\gamma \in \Lambda_\tau$ for some $\tau > 0$), then $\mathbb{R} \cdot \gamma$ is an embedded circle in Π.

Lemma 6.4. *Let $I \subset SM$ be a locally maximal, expansive, compact invariant subset of the geodesic flow. There exists $r > 0$ such that, for each $\gamma \in \mathcal{P}_*(I)$, the subset $\mathcal{U}(\mathbb{R} \cdot \gamma, r) \setminus \mathbb{R} \cdot \gamma$ does not contain geodesics or constant curves.*

Proof. Recall that every geodesic $\gamma \colon \mathbb{R} \looparrowright M$ has diameter at least ρ, i.e.,

$$\max_{t_1, t_2 \in \mathbb{R}} d(\gamma(t_1), \gamma(t_2)) \geq \rho.$$

Therefore, for each $x \in M$ and for each geodesic $\gamma \colon \mathbb{R} \looparrowright M$, there exists $t \in \mathbb{R}$ such that $d(x, \gamma(t)) \geq \rho/2$. In other words, $\mathcal{U}(\gamma, \rho/2)$ does not contain constant curves.

Next, arguing by contradiction, assume that there exist two sequences of geodesics

$$\gamma_n \in \mathcal{G}(I) \quad \text{and} \quad \zeta_n \in \mathcal{U}(\gamma_n, 1/n) \setminus \mathbb{R} \cdot \gamma_n.$$

We set $\lambda_n := \|\dot{\zeta}_n(0)\|_g$ and consider the associated unit tangent vectors

$$v_n := \dot{\gamma}_n(0) \in I \quad \text{and} \quad w_n := \frac{\dot{\zeta}_n(0)}{\lambda_n} \in SM.$$

Notice that

$$\phi_t(v_n) = \dot\gamma_n(t) \quad\text{and}\quad \phi_{\lambda_n t}(w_n) = \frac{\dot\zeta_n(t)}{\lambda_n}.$$

Since $d_{C^0}(\zeta_n,\gamma_n) < 1/n$ and both ζ_n and γ_n are geodesics, we readily see that $\lambda_n \to 1$ as $n\to\infty$, and

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}} \tilde d(\phi_{\lambda_n t}(w_n), \phi_t(v_n)) = 0, \tag{6.4}$$

where $\tilde d\colon SM\times SM\to[0,\infty)$ is the distance on the unit tangent bundle induced by the Riemannian metric g. In particular, if n is large enough, the whole orbit $t\mapsto\phi_t(w_n)$ lies in an isolating neighborhood $U\subset SM$ of I and hence $w_n\in I$ due to the local maximality of I. Henceforth, we assume n to be large enough so that $w_n\in I$ and $1/n<\rho/2$.

We claim that w_n does not belong to the orbit $t\mapsto\phi_t(v_n)$. Indeed, if $w_n = \phi_{t_0}(v_n)$ for some $t_0\in\mathbb{R}$, we have $\zeta_n(t)=\gamma_n(t_0+\lambda_n t)$ for all $t\in\mathbb{R}$. Since $\zeta_n\not\in\mathbb{R}\cdot\gamma_n$, we must have $\lambda_n\neq 1$. Therefore, since γ_n is periodic, there exists $t_1\in\mathbb{R}$ such that $\zeta_n(t_1)=\gamma_n(t_1+\rho/2)$. However, this would imply $d(\zeta_n(t_1),\gamma_n(t_1))=\rho/2>1/n$, contradicting the fact that $\zeta_n\in\mathcal{W}(\gamma_n,1/n)$.

The assumption that I is expansive provides, in particular, a uniform lower bound $\delta>0$ with the following property: for every $n\in\mathbb{N}$, since w_n and v_n belong to distinct orbits of the geodesic flow, there exists $t\in\mathbb{R}$ such that

$$\tilde d(\phi_{\lambda_n t}(w_n),\phi_t(v_n))\geq\delta,$$

contradicting (6.4). □

We are now ready to prove the first version of the uniform crossing energy bound.

Proposition 6.5 (Crossing energy bound)**.** *Let $I\subset SM$ be a locally maximal, expansive, compact invariant subset of the geodesic flow. Then, for each $r_1>0$ sufficiently small, $r_0\in(0,r_1)$ and $[\sigma_0,\sigma_1]\subset(0,\infty)$, there exists $\delta>0$ such that the following holds: for each $\sigma\in[\sigma_0,\sigma_1]$, $k\in\mathbb{N}\cap[2,\infty)$, $\gamma\in\mathcal{P}_*(I)$ and an anti-gradient flow segment $u\colon[s_0,s_1]\to L_k$ of $E_{k,\sigma}$ crossing the shell $\mathcal{U}(\mathbb{R}\cdot\gamma,r_1)\setminus\mathcal{U}(\mathbb{R}\cdot\gamma,r_0)$, we have*

$$E_{k,\sigma}(u(s_0)) - E_{k,\sigma}(u(s_1))\geq\delta.$$

Proof. By Lemma 6.4, there exists $r_1>0$ small enough so that, for each $\gamma\in\mathcal{P}_*(I)$, the open set $\mathcal{U}(\mathbb{R}\cdot\gamma,r_1)\setminus\mathbb{R}\cdot\gamma$ contains neither geodesics nor constant curves. Fix $r_0\in(0,r_1)$ and set

$$\mathcal{U}_0 := \mathcal{U}(\mathbb{R}\cdot\gamma,r_0)\subset\mathcal{U}_1:=\mathcal{U}(\mathbb{R}\cdot\gamma,r_1)$$

so that $\overline{\mathcal{U}_1\setminus\mathcal{U}_0}$ does not contain geodesics or constant curves. For each $\gamma_0\in\mathcal{U}_0$ and $\gamma_1\in\Pi\setminus\mathcal{U}_1$, we have $d_{C^0}(\gamma_0,\gamma_1)\geq r_1-r_0>0$. This, together with Lemma 6.3, implies the proposition. □

6.5. **Functional setting in period one.** In the previous sections, for the sake of simplicity, we worked with the spaces Λ_τ of τ-periodic curves. As we already remarked, since any reparametrization with constant speed of a geodesic is still a geodesic, in the variational theory of closed geodesics it is enough to employ the space Λ_1 of 1-periodic curves. Clearly, Λ_τ is diffeomorphic to Λ_1 via the diffeomorphism

$\psi_\tau\colon \Lambda_\tau \to \Lambda_1$, $\psi_\tau(\gamma) = \gamma(\tau\,\cdot)$. Under ψ_τ, the energy transforms as $\mathcal{E}_1 \circ \psi_\tau = \tau\mathcal{E}_\tau$. From now on, we will be working in the setting of $\tau = 1$.

In the finite-dimensional reduction, we will only need one discretization parameter, since we will set $\sigma := 1/k$. In order to simplify the notation, we will suppress 1, τ, σ, and instead write, consistently with the notation from Section 2.1,

$$\Lambda := \Lambda_1, \qquad \mathcal{E} = \mathcal{E}_1\colon \Lambda \to [0,\infty), \quad E_k = E_{k,\sigma}\colon L_k \to [0,\infty).$$

We have the already introduced $S^1 = \mathbb{R}/\mathbb{Z}$-action, $t \cdot \gamma = \gamma(t + \cdot)$, on Λ. While this action does not preserve the subspace $\iota_{1/k}(L_k)$, it does preserve the critical set $\iota_{1/k}(\mathrm{crit}(E_k)) \subset \mathrm{crit}(\mathcal{E})$.

For a closed geodesic $\gamma \in \mathrm{crit}^+(\mathcal{E})$, let us fix the discretization parameter

$$k \geq \lceil 8\mathcal{E}(\gamma)^{1/2}/\rho \rceil.$$

Then

$$d(\gamma(t), \gamma(t + \tfrac{1}{k})) \leq \rho/8, \qquad \forall t \in S^1.$$

In particular, γ belongs to the image of the embedding $\iota_{1/k}\colon L_k \hookrightarrow \Lambda$. In what follows, with a slight abuse of notation, we will occasionally treat $\iota_{1/k}$ as an inclusion and simply consider L_k as a subspace of the free loop space Λ.

From now on, we will only need to work with the C^0-neighborhoods of a closed geodesic $\gamma \in \mathrm{crit}^+(\mathcal{E})$ in the free loop space Λ. Let us denote such a neighborhood of radius $r > 0$ by

$$\mathcal{V}(\gamma, r) := \mathcal{U}(\gamma, r) \cap \Lambda = \big\{\zeta \in \Lambda \bigm| d_{C^0}(\gamma, \zeta) < r\big\},$$

and also set

$$\mathcal{V}(S^1 \cdot \gamma, r) := \bigcup_{t \in S^1} \mathcal{V}(t \cdot \gamma, r).$$

Let $I \subset SM$ be an invariant subset of the geodesic flow ϕ_t. The space $\mathcal{P}(I)$ of 1-periodic closed geodesics tangent to I, which was introduced in (2.3), is related to the space $\mathcal{P}_*(I)$ of unit speed closed geodesics tangent to I by

$$\mathcal{P}(I) := \bigcup_{\tau > 0} \psi_\tau(\mathcal{P}_\tau(I)).$$

Proposition 6.5 readily translates as follows to the setting in period one.

Proposition 6.6 (Crossing energy bound in Λ)**.** *Let $I \subset SM$ be a locally maximal, expansive, compact invariant subset of the geodesic flow. For every $r_1 > 0$ sufficiently small and $r_0 \in (0, r_1)$, there exists $\delta > 0$ such that the following holds: for each $\gamma \in \mathcal{P}(I)$ and $k := \big\lceil (\mathcal{E}(\gamma)^{1/2} + 1)8/\rho \big\rceil$, any anti-gradient flow segment $u\colon [s_0, s_1] \to L_k$ of E_k that crosses the shell $\mathcal{V}(S^1 \cdot \gamma, r_1) \setminus \mathcal{V}(S^1 \cdot \gamma, r_0)$ satisfies*

$$E_k(u(s_0)) - E_k(u(s_1)) \geq \sqrt{\mathcal{E}(\gamma)}\,\delta. \qquad \square$$

7. Persistence of the total local homology

7.1. **Local homology of closed geodesics.** Let (M, g) be a closed Riemannian manifold, $\mathcal{E}\colon \Lambda \to [0,\infty)$ be its energy functional and $\gamma \in \mathrm{crit}^+(\mathcal{E})$ be a closed geodesic. The *local homology* of the critical circle $S^1 \cdot \gamma$ with coefficient in a field $\mathbb{F}$, suppressed in the notation, is the relative homology group

$$C_*(S^1 \cdot \gamma) := H_*(\Lambda^{<c} \cup S^1 \cdot \gamma, \Lambda^{<c}),$$

where $c := \sqrt{\mathcal{E}(\gamma)}$.

A closed geodesic $\gamma \in \mathrm{crit}^+(\mathcal{E})$ is said to be isolated when there exists a neighborhood $\mathcal{U} \subset \Lambda$ of the critical circle $S^1 \cdot \gamma$ such that $\mathcal{U} \cap \mathrm{crit}(\mathcal{E}) = S^1 \cdot \gamma$. It is well known that the local homology group of any isolated closed geodesic γ is finitely generated, and there exist arbitrarily small open neighborhoods $\mathcal{W} \subset \Lambda$ of $S^1 \cdot \gamma$, sometimes called *Gromoll–Meyer neighborhoods*, [GM69a, GM69b], such that the inclusion induces an isomorphism

$$C_*(S^1 \cdot \gamma) \xrightarrow{\cong} H_*(\Lambda^{<c} \cup \mathcal{W}, \Lambda^{<c}).$$

Moreover, for each $\epsilon > 0$ small enough, the inclusion induces a monomorphism $C_*(S^1 \cdot \gamma) \hookrightarrow H_*(\Lambda^{<c+\epsilon}, \Lambda^{<c})$.

7.2. **Proofs of Theorem D and Corollary E.** Proposition 6.6 has the following consequence in terms of persistence of the total local homology of a locally maximal, expansive, compact invariant subset for the geodesic flow. Theorem D will be a rather direct application of it. In the statement, we use the notation from Section 2.3.

Lemma 7.1. *Let (M, g) be a closed Riemannian manifold and $I \subset SM$ be a locally maximal, expansive, compact invariant subset for its geodesic flow. Then, there exists $\delta > 0$ such that, for each $c \in \sigma(I)$, the following assertions hold:*

(i) *The inclusion induces a monomorphism*

$$C_*(\mathcal{P}^c(I)) \hookrightarrow H_*(\Lambda^{<c+\delta}, \Lambda^{<c}).$$

(ii) *The total local homology $C_*(\mathcal{P}^c(I))$ is contained in the image of the homomorphism*

$$H_*(\Lambda^{\leq c}, \Lambda^{<c-\delta}) \to H_*(\Lambda^{\leq c}, \Lambda^{<c})$$

induced by the inclusion.

Proof. As before, we denote by $\rho := \mathrm{injrad}(g) > 0$ the injectivity radius. By Lemma 6.4, we can fix $R \in (0, \rho/16)$ small enough so that, for each $\gamma \in \mathcal{P}(I)$, the C^0-open neighborhood $\mathcal{V}(S^1 \cdot \gamma, R)$ does not contain closed geodesics other than those in $S^1 \cdot \gamma$ or constant curves, i.e.,

$$\mathcal{V}(S^1 \cdot \gamma, R) \cap \mathrm{crit}(\mathcal{E}) = S^1 \cdot \gamma, \qquad \forall \gamma \in \mathcal{P}(I). \tag{7.1}$$

Up to replacing R with $R/2$, we even obtain

$$\mathcal{V}(S^1 \cdot \gamma, R) \cap \mathcal{V}(S^1 \cdot \zeta, R) = \varnothing, \qquad \forall \gamma, \zeta \in \mathcal{P}(I) \text{ with } S^1 \cdot \gamma \neq S^1 \cdot \zeta.$$

Take radii r_1, r_2 with $0 < r_1 < r_2 < R$, where r_2 is small enough so that, according to Proposition 6.6, there exists a constant $\delta > 0$ with the following property: for each closed geodesic $\gamma \in \mathcal{P}(I)$ and for

$$k := \lceil (\mathcal{E}(\gamma)^{1/2} + 1) 8/\rho \rceil,$$

any anti-gradient flow segment $u\colon [s_0, s_1] \to L_k$ of E_k that crosses the shell $\mathcal{V}(S^1 \cdot \gamma, r_2) \setminus \mathcal{V}(S^1 \cdot \gamma, r_1)$ satisfies the condition that

$$E_k(u(s_0)) - E_k(u(s_1)) \geq \sqrt{\mathcal{E}(\gamma)}\, \frac{2(\rho+1)}{\rho}\, \delta. \tag{7.2}$$

Compared with the statement of Proposition 6.6, here we included a multiplicative factor $2(\rho + 1)/\rho$ in front of the constant δ for aesthetic reasons, as it will simplify inequality (7.3). Recall that "*crossing the shell* $\mathcal{V}(S^1\cdot\gamma,r_2)\setminus\mathcal{V}(S^1\cdot\gamma,r_1)$" means that $u(s_0')\in\mathcal{V}(S^1\cdot\gamma,r_1)$ and $u(s_1')\not\in\mathcal{V}(S^1\cdot\gamma,r_2)$ for some $s_0',s_1'\in[s_0,s_1]$. If needed, we choose δ so that

$$0<\delta<1.$$

From now on, we fix a spectral value $c\in\sigma(I)$ and work within the sublevel set $\Lambda^{<c+1}$. We set the discretization parameter to be $k:=\lceil(c+1)8/\rho\rceil$. For each $\gamma\in\mathcal{P}^c(I)$, any anti-gradient flow segment $u\colon[s_0,s_1]\to L_k^{<c+1}$ of E_k that crosses the shell $\mathcal{V}(S^1\cdot\gamma,r_2)\setminus\mathcal{V}(S^1\cdot\gamma,r_1)$ satisfies the inequality (7.2), which can be rewritten as

$$\Big(\sqrt{E_k(u(s_0))}-\sqrt{E_k(u(s_1))}\Big)\Big(\sqrt{E_k(u(s_0))}+\sqrt{E_k(u(s_1))}\Big)\geq c\,\frac{2(\rho+1)}{\rho}\,\delta.$$

Since

$$\sqrt{E_k(u(s_0))}+\sqrt{E_k(u(s_1))}<2(c+1)$$

and $c\geq\rho$, we conclude that

$$\sqrt{E_k(u(s_0))}-\sqrt{E_k(u(s_1))}\geq\delta. \tag{7.3}$$

Next, consider Gromoll–Meyer neighborhoods $\mathcal{W}(S^1\cdot\gamma)$ of each critical circle $S^1\cdot\gamma\subset\mathcal{P}^c(I)$, whose defining property was recalled at the beginning of this section. We require such neighborhoods to be small enough so that

$$\mathcal{W}(S^1\cdot\gamma)\Subset\Lambda^{<c+\delta}\cap\mathcal{V}(S^1\cdot\gamma,r_1).$$

Here the notation $A\Subset B$ means, as usual, that $\overline{A}\subset\operatorname{int}(B)$. We also choose another neighborhood $\mathcal{Y}(S^1\cdot\gamma)\subset\mathcal{W}(S^1\cdot\gamma)$ of $S^1\cdot\gamma$ that is so small that

$$r_\gamma:=\inf\big\{d_k(\boldsymbol{x},\boldsymbol{y})\ \big|\ \boldsymbol{x}\in L_k\cap\partial\mathcal{Y}(S^1\cdot\gamma),\ \boldsymbol{y}\in L_k\cap\partial\mathcal{W}(S^1\cdot\gamma)\big\}>0. \tag{7.4}$$

We set

$$\begin{aligned}
\mathcal{Y}&:=\bigcup_{S^1\cdot\gamma}\mathcal{Y}(S^1\cdot\gamma),\\
\mathcal{W}&:=\bigcup_{S^1\cdot\gamma}\mathcal{W}(S^1\cdot\gamma),\\
\mathcal{V}_i&:=\bigcup_{S^1\cdot\gamma}\mathcal{V}(S^1\cdot\gamma,r_i)\text{ for }i=1,2,\\
\mathcal{V}&:=\bigcup_{S^1\cdot\gamma}\mathcal{V}(S^1\cdot\gamma,R),\\
\mathcal{X}&:=\Lambda^{<c+\delta}\setminus\mathcal{V}_1,
\end{aligned}$$

where the unions range over the critical circles $S^1\cdot\gamma\subset\mathcal{P}^c(I)$. Then we have

$$\mathcal{Y}\Subset\mathcal{W}\Subset\mathcal{V}_1\Subset\mathcal{V}_2\Subset\mathcal{V},\qquad \operatorname{crit}(\mathcal{E})\cap\mathcal{V}=\mathcal{P}^c(I),\qquad \overline{\mathcal{W}}\cap\overline{\mathcal{X}}=\varnothing.$$

By excision, the inclusion induces an isomorphism

$$H_*(\Lambda^{<c}\cup\mathcal{W},\Lambda^{<c})\oplus H_*(\Lambda^{<c}\cup\mathcal{X},\Lambda^{<c})\xrightarrow{\cong}H_*(\Lambda^{<c}\cup\mathcal{W}\cup\mathcal{X},\Lambda^{<c}).$$

By the defining property of Gromoll–Meyer neighborhoods, the inclusion gives rise to an isomorphism

$$C_*(\mathcal{P}^c(I))\xrightarrow{\cong}H_*(\Lambda^{<c}\cup\mathcal{W},\Lambda^{<c}).$$

All together, the inclusion induces a monomorphism

$$C_*(\mathcal{P}^c(I)) \hookrightarrow H_*(\Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X}, \Lambda^{<c}),$$

which fits into the following commutative diagram, all of whose arrows are induced by inclusions.

$$\begin{tikzcd} C_*(\mathcal{P}^c(I)) \arrow[r, hook] \arrow[dr] & H_*(\Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X}, \Lambda^{<c}) \arrow[d, "i"] \\ & H_*(\Lambda^{<c+\delta}, \Lambda^{<c}) \end{tikzcd}$$

In order to prove assertion (i) of Lemma 7.1, it remains to show that the homomorphism i is injective. To this end, it is enough to build a continuous homotopy

$$h_s\colon \Lambda^{<c+\delta} \to \Lambda^{<c+\delta},\ \ s \in [0, s_0],$$

with the following properties:

(a) $h_0 = \mathrm{id}$,
(b) $\mathcal{E} \circ h_s \leq \mathcal{E}$ for all $s \in [0, s_0]$,
(c) $h_{s_0}(\Lambda^{<c+\delta}) \subset \Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X}$.

Indeed, these three conditions readily imply that the homomorphism

$$(h_{s_0})_*\colon H_*(\Lambda^{<c+\delta}, \Lambda^{<c}) \to H_*(\Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X}, \Lambda^{<c})$$

is a left inverse of i, i.e., $(h_{s_0})_* \circ i = \mathrm{id}$, and hence i is injective. We shall build the homotopy h_s in a few steps.

We shall work with the finite-dimensional loop space L_k. As we already mentioned in Subsection 6.5, with a slight abuse of notation, we may view the embedding $\iota_{1/k}\colon L_k \hookrightarrow \Lambda$ as an inclusion, and therefore L_k as a subspace of Λ. For each $\gamma \in \Lambda^{<c+\delta}$ parametrized proportionally to arc length, we have

$$d(\gamma(t_1), \gamma(t_2)) \leq |t_1 - t_2|(c+\delta) < \frac{\rho}{8}, \qquad \forall t_1, t_2 \in \mathbb{R} \text{ with } |t_1 - t_2| \leq 1/k.$$

In particular, this holds for each closed geodesic $\gamma \in \mathrm{crit}(\mathcal{E}) \cap \Lambda^{<c+\delta}$, and hence

$$d(\zeta(t_1), \zeta(t_2)) < 2r + \tfrac{\rho}{8} < \tfrac{\rho}{4}, \qquad \forall \zeta \in \mathcal{V}(S^1 \cdot \gamma, r),\ t_1, t_2 \in \mathbb{R} \text{ with } |t_1 - t_2| \leq 1/k.$$

Since the radius R that we fixed at the beginning of the proof is smaller than $\rho/16$, this inequality implies that the closure $\overline{L_k \cap \mathcal{V}(S^1 \cdot \gamma, R)}$ is contained in the interior of L_k, i.e.,

$$\overline{L_k \cap \mathcal{V}(S^1 \cdot \gamma, R)} \subset \mathrm{int}(L_k). \tag{7.5}$$

We define the continuous homotopy $h_s\colon \Lambda^{<c+\delta} \to \Lambda^{<c+\delta}$ for $s \in [0,1]$ as follows: for each $\gamma \in \Lambda^{<c+\delta}$ and $s \in [0,1]$, the curve $\gamma_s := h_s(\gamma)$ is given by $\gamma_s|_{[s,1]} = \gamma|_{[s,1]}$, whereas $\gamma_s|_{[0,s]}$ is a constant speed reparametrization of $\gamma|_{[0,s]}$. This homotopy clearly satisfies condition (a). It also meets condition (b), for

$$\begin{aligned} \mathcal{E}(\gamma_s) &= \int_0^s \|\dot\gamma_s\|_g^2\, dt + \int_s^1 \|\dot\gamma_s\|_g^2\, dt = \frac{1}{s}\left(\int_0^s \|\dot\gamma_s\|_g\, dt\right)^2 + \int_s^1 \|\dot\gamma_s\|_g^2\, dt \\ &= \frac{1}{s}\left(\int_0^s \|\dot\gamma\|_g\, dt\right)^2 + \int_s^1 \|\dot\gamma\|_g^2\, dt \leq \int_0^s \|\dot\gamma\|_g^2\, dt + \int_s^1 \|\dot\gamma\|_g^2\, dt = \mathcal{E}(\gamma). \end{aligned}$$

The image $h_1(\Lambda^{<c+\delta}) \subset \Lambda^{<c+\delta}$ consists of curves parametrized with constant speed, and therefore

$$d(\gamma(t_1),\gamma(t_2)) < \frac{c+\delta}{k} \leq \frac{\rho}{8}, \quad \forall \gamma \in h_1(\Lambda^{<c+\delta}),\ t_1,t_2 \in [0,1] \text{ with } |t_1-t_2| \leq \frac{1}{k}.$$

We extend the homotopy $h_s\colon \Lambda^{<c+\delta} \to \Lambda^{<c+\delta}$ to $s \in [1,2]$ as follows: for each $\gamma \in \Lambda^{<c+\delta}$, $s \in [0,1]$, and $i \in \{0,\dots,k-1\}$, set

$$h_{1+s}(\gamma)|_{[(i+s)/k,(i+1)/k]} := h_1(\gamma)|_{[(i+s)/k,(i+1)/k]},$$

whereas $h_{1+s}(\gamma)|_{[i/k,(i+s)/k]}$ is the unique shortest geodesic segment (of length less than $\rho/8$) joining $h_{1+s}(\gamma)(i/k)$ and $h_{1+s}(\gamma)((i+1)/k)$. Clearly, by replacing a portion of a curve with the shortest geodesic segment we do not increase the energy and so the homotopy h_s still satisfies condition (b) for $s \in [1,2]$. Moreover,

$$h_2(\Lambda^{<c+\delta}) \subset L_k \cap \Lambda^{<c+\delta}.$$

By (7.5), we can find a smooth function $f\colon L_k \to [0,1]$ such that

$$\mathrm{supp}(f) \subset \overline{L_k \cap \mathcal{V}} \subset \mathrm{int}(L_k), \qquad f|_{L_k \cap \mathcal{V}_2} \equiv 1.$$

We denote by $\psi_s\colon L_k \to L_k$, $s \geq 0$, the flow of the vector field $-f\nabla E_k$. Since the latter vector field is supported in the interior of L_k, the flow ψ_s is defined for all times $s \in \mathbb{R}$. Moreover, within $L_k \cap \mathcal{V}_2$, the flow ψ_s coincides with the anti-gradient flow of E_k with the same time-parametrization. We extend the definition of h_s to all $s \geq 0$ by setting

$$h_{2+s} := \psi_s \circ h_2, \qquad \forall s \geq 0.$$

Clearly, condition (b) still holds for all $s \geq 0$.

It remains to find $s_0 > 0$ such that h_{2+s_0} satisfies condition (c). Indeed, we will find $s_0 > 0$ such that

$$\psi_{s_0}(L_k^{<c+\delta}) \subset \Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X}. \tag{7.6}$$

The critical subset $\mathrm{crit}(\mathcal{E}) \cap \mathcal{E}^{-1}(c) \subset \Lambda$ is compact. This, along with (7.1), implies that $\mathcal{P}^c(I)$ consists of finitely many critical circles. Therefore, considering the radii r_γ defined by (7.4), we have

$$r := \inf_{\gamma \in \mathcal{P}^c(I)} r_\gamma = \inf \big\{ d_k(\boldsymbol{x},\boldsymbol{y}) \ \big|\ \boldsymbol{x} \in L_k \cap \partial\mathcal{Y},\ \boldsymbol{y} \in L_k \cap \partial\mathcal{W} \big\} > 0.$$

We set

$$\lambda' := \inf_{L_k \cap \mathcal{V}_2 \setminus \mathcal{Y}} \|\nabla E_k\|_{g_k} > 0.$$

For any anti-gradient flow trajectory $u\colon [s_1,s_2] \to L_k^{<c+1}$ of E_k crossing the shell $\mathcal{W} \setminus \mathcal{Y}$ (i.e., such that $u(s_1') \in \mathcal{Y}$ and $u(s_2') \notin \mathcal{W}$ for some $s_1', s_2' \in [s_1,s_2]$) we have

$$E_k(u(s_1)) - E_k(u(s_2)) \geq \lambda' r,$$

and, therefore,

$$\begin{aligned} \sqrt{E_k(u(s_1))} - \sqrt{E_k(u(s_2))} &\geq \frac{\lambda' r}{\sqrt{E_k(u(s_1))} + \sqrt{E_k(u(s_2))}} \\ &\geq \frac{\lambda' r}{2(c+1)} =: \sigma. \end{aligned} \tag{7.7}$$

Let

$$\lambda := \min\Big\{\lambda', \inf_{L_k^{\geq c+\sigma}\cap\mathcal{V}_2} \|\nabla E_k\|_{g_k}\Big\} > 0,$$

where $L_k^{\geq c+\sigma} := E_k^{-1}[(c+\sigma)^2,\infty)$. The desired s_0 satisfying (7.6) is given by

$$s_0 := \frac{(c+\delta)^2 - c^2}{\lambda^2}.$$

Let us show that (7.6) indeed holds for this choice of s_0:

- For each $\boldsymbol{x} \in L_k^{<c+\delta}\setminus\mathcal{V}_2$, if $\psi_s(\boldsymbol{x}) \in \mathcal{V}_1$ for some $s > 0$, then the anti-gradient flow segment $\psi_{[0,s]}(\boldsymbol{x})$ crosses the shell $\mathcal{V}_2 \setminus \mathcal{V}_1$, and (7.3) implies

  $$\sqrt{E_k(\psi_s(\boldsymbol{x}))} \leq \sqrt{E_k(\boldsymbol{x})} - \delta < c + \delta - \delta = c.$$

  In particular,

  $$\psi_{s_0}(L_k^{<c+\delta} \setminus \mathcal{V}_2) \subset \Lambda^{<c} \cup \mathcal{X}.$$

- For each $\boldsymbol{x} \in L_k^{<c+\delta} \cap \mathcal{V}_2$, we need to consider different subcases. If there exists $s \in (0, s_0]$ such that $\psi_s(\boldsymbol{x}) \notin \mathcal{V}_2$, we can apply the previous point and infer that

  $$\psi_{s_0}(\boldsymbol{x}) \in \Lambda^{<c} \cup \mathcal{X}.$$

  Let us now consider the case where

  $$\psi_{[0,s_0]}(\boldsymbol{x}) \subset \mathcal{V}_2 \setminus \Lambda^{<c}. \tag{7.8}$$

  We claim that there exists $s_1 \in [0, s_0]$ such that

  $$\psi_{s_1}(\boldsymbol{x}) \in \mathcal{Y}^{<c+\sigma},$$

  where as usual $\mathcal{Y}^{<c+\sigma} := \mathcal{Y} \cap \Lambda^{<c+\sigma}$. Indeed, if such an s_1 did not exist, we would have $\|\nabla E_k(\psi_s(\boldsymbol{x}))\|_{g_k} \geq \lambda$ for all $s \in [0, s_0]$ and infer that

  $$\begin{aligned}\sqrt{E_k(\psi_{s_0}(\boldsymbol{x}))} &= \sqrt{E_k(\boldsymbol{x}) - \int_0^{s_0} \|\nabla E_k(\psi_s(\boldsymbol{x}))\|^2_{g_k}\, ds}\\ &< \sqrt{(c+\delta)^2 - s_0\lambda^2} = c,\end{aligned}$$

  which contradicts (7.8). We also claim that

  $$\psi_{[s_1,s_0]}(\boldsymbol{x}) \subset \mathcal{W}.$$

  Indeed, if we had $\psi_s(\boldsymbol{x}) \notin \mathcal{W}$ for some $s \in (s_1, s_0]$, then the anti-gradient flow segment $\psi_{[s_1,s_0]}(\boldsymbol{x})$ would cross the shell $\mathcal{W} \setminus \mathcal{Y}$. Then (7.7) would imply that

  $$\sqrt{E_k(\psi_{s_0}(\boldsymbol{x}))} \leq \sqrt{E_k(\psi_{s_1}(\boldsymbol{x}))} - \sigma < c + \sigma - \sigma = c,$$

  contradicting once again (7.8). Overall, we have proved that

  $$\psi_{s_0}(\boldsymbol{x}) \in \Lambda^{<c} \cup \mathcal{W} \cup \mathcal{X},$$

  and hence (7.6) holds.

This completes the proof of assertion (i) of Lemma 7.1. The proof of assertion (ii) is similar. □

Proof of Theorem D. Let $\delta > 0$ be the constant given by Lemma 7.1. For each $c \in \sigma(I)$, Lemma 7.1(i) implies that the inclusion induces a monomorphism

$$C_*(\mathcal{P}^c(I)) \hookrightarrow H_*(\Lambda^{<c+\delta}, \Lambda^{<c}). \tag{7.9}$$

In particular, we have the monomorphism

$$C_*(\mathcal{P}^c(I)) \hookrightarrow H_*(\Lambda^{\leq c}, \Lambda^{<c}),$$

where $\Lambda^{\leq c} := \mathcal{E}^{-1}[0, c^2]$. Henceforth, we will view the total local homology $C_*(\mathcal{P}^c(I))$ as a vector subspace of $H_*(\Lambda^{\leq c}, \Lambda^{<c})$. Consider the exact triangle

$$\begin{tikzcd} H_*(\Lambda^{<c}, \Lambda^0) \arrow[r, "i"] & H_*(\Lambda^{\leq c}, \Lambda^0) \arrow[d, "j"] \\ & H_*(\Lambda^{\leq c}, \Lambda^{<c}) \arrow[ul, "\partial_*"] \end{tikzcd} \tag{7.10}$$

where all arrows except the connecting homomorphism ∂_* are induced by the inclusions. Let $C \subset H_*(\Lambda^{\leq c}, \Lambda^0)$ be a complement of $\mathrm{im}(i)$, i.e., $H_*(\Lambda^{\leq c}, \Lambda^0) = \mathrm{im}(i) \oplus C$. Notice that $j|_C$ is a monomorphism and set

$$B := j|_C^{-1}(C_*(\mathcal{P}^c(I))).$$

For each $b \in (c, \infty]$, consider the homomorphism

$$j_b \colon B \to H_*(\Lambda^{<b}, \Lambda^0)$$

induced by the inclusion. If B is non-trivial, there exist $b_k > \ldots > b_1 > c$, with $b_k \in (c, \infty]$, and a direct sum decomposition $B = B_{b_k} \oplus \ldots \oplus B_{b_1}$ such that

$$\ker(j_b) = \begin{cases} \{0\} & \text{if } b \in (c, b_1], \\ B_{b_i} \oplus \ldots \oplus B_{b_1} & \text{if } b \in (b_i, b_{i+1}], \\ B & \text{if } b > b_k. \end{cases}$$

This readily implies that $([c, b_i), n_i) \in \mathcal{B}$ for all $i = 1, \ldots, k$, where $n_i \geq \dim B_{b_i}$. Moreover, the injectivity of (7.9) together with the following commutative diagram whose arrows are induced by inclusions

$$\begin{tikzcd} B \arrow[r, "j_{c+\delta}"] \arrow[d, hook, "j"'] & H_*(\Lambda^{<c+\delta}, \Lambda^0) \arrow[d] \\ C_*(\mathcal{P}^c(I)) \arrow[r, hook] & H_*(\Lambda^{<c+\delta}, \Lambda^{<c}) \end{tikzcd}$$

implies that $b_1 > c + \delta$. Therefore, all bars $[c, b_i)$ have size at least δ.

Now, consider the subgroup

$$A := \partial_*(C_*(\mathcal{P}^c(I))) \subset H_*(\Lambda^{<c}, \Lambda^0)$$

and, for each $a < b$, the homomorphisms

$$i_{b,a} \colon H_*(\Lambda^{<a}, \Lambda^0) \to H_*(\Lambda^{<b}, \Lambda^0)$$

induced by the inclusion. If A is non-trivial, there exist $a_1 < \ldots < a_h < c$ with $a_1 > 0$ and a direct sum decomposition $A = A_{a_1} \oplus \ldots \oplus A_{a_h}$ such that

$$\mathrm{im}(i_{c,a}) \cap A = \begin{cases} \{0\} & \text{if } a \in (0, a_1], \\ A_{a_1} \oplus \ldots \oplus A_{a_i} & \text{if } b \in (a_i, a_{i+1}], \\ A & \text{if } a \in (a_h, c]. \end{cases}$$

The exact triangle (7.10) implies that $A \subseteq \ker i_{b,c}$ for all $b > c$. Therefore, $([a_i, c), m_i) \in \mathcal{B}$ for all $i = 1, \ldots, h$, where $m_i \geq \dim A_{a_i}$. By Lemma 7.1(ii), $C_*(\mathcal{P}^c(I))$ is contained in the image of the homomorphism

$$H_*(\Lambda^{\leq c}, \Lambda^{<c-\delta}) \to H_*(\Lambda^{\leq c}, \Lambda^{<c})$$

induced by the inclusion. Inserting this homomorphism into the following commutative diagram, whose horizontal rows are induced by the inclusions,

$$\begin{tikzcd} H_*(\Lambda^{\leq c}, \Lambda^{<c-\delta}) \arrow[r] \arrow[d, "\partial_*"'] & H_*(\Lambda^{\leq c}, \Lambda^{<c}) \arrow[d, "\partial_*"] \\ H_*(\Lambda^{<c-\delta}, \Lambda^0) \arrow[r] & H_*(\Lambda^{<c}, \Lambda^0) \end{tikzcd}$$

we readily see that $c > a_h + \delta$. Therefore, all bars $[a_i, c)$ have size at least δ. Finally, the exact triangle (7.10) also implies that $C_*(\mathcal{P}^c(I)) \cong B \oplus A$, and hence

$$n_1 + \ldots + n_k + m_1 + \ldots + m_h \geq \dim B + \dim A = \dim C_*(\mathcal{P}^c(I)). \qquad \square$$

Proof of Corollary E. Since the compact invariant subset $I \subset SM$ is locally maximal and expansive, every closed geodesic $\gamma \in \mathcal{P}(I)$ is isolated. For each $c \in \sigma(I)$, the total local homology of $\mathcal{P}(I)$ splits as a direct sum

$$C_*(\mathcal{P}^c(I)) \cong \bigoplus_{S^1 \cdot \gamma \subset \mathcal{P}^c(I)} C_*(S^1 \cdot \gamma),$$

due to the excision property of singular homology.

If a critical circle $S^1 \cdot \gamma \subset \mathrm{crit}^+(\mathcal{E})$ is non-degenerate and prime, its local homology with any coefficient has dimension $\dim C_*(S^1 \cdot \gamma) = 2$. This is well known to the experts (see, e.g., [BL10, Prop. 3.8(i)]), but we sketch the argument for the reader's convenience. Let $\mathbb{E}^- \subset T_\gamma \Lambda$ be a vector subspace of maximal dimension over which the Hessian $d^2\mathcal{E}(\gamma)$ is negative definite. The dimension of $\mathbb{E}^-$ is equal to Morse index of γ, which is finite. Consider an embedded ball $\mathcal{N}_\gamma \subset \Lambda$ of dimension $\dim \mathcal{N}_\gamma = \dim \mathbb{E}^-$ containing γ in its interior and having tangent space $T_\gamma \mathcal{N}_\gamma = \mathbb{E}^-$. Up to shrinking $\mathcal{N}_\gamma$ around γ, we have $\mathcal{E}(\zeta) < c := \mathcal{E}(\gamma)$ for all $\zeta \in \mathcal{N}_\gamma \setminus \{\gamma\}$, and the map $S^1 \times \mathcal{N}_\gamma \to \Lambda$, $(t, \gamma) \mapsto t \cdot \gamma$ is a homeomorphism onto a neighborhood $\mathcal{N} \subset \Lambda$ of the critical circle $S^1 \cdot \gamma$. The inclusion $(\mathcal{N}, \mathcal{N}^{<c}) \hookrightarrow (\Lambda^{<c} \cup S^1 \cdot \gamma, \Lambda^{<c})$ induces an isomorphism in homology, and hence

$$\begin{aligned} C_*(S^1 \cdot \gamma) &\cong H_*(\mathcal{N}, \mathcal{N}^{<c}) \\ &\cong H_*(S^1 \times \mathcal{N}_\gamma, S^1 \times \mathcal{N}_\gamma \setminus \{\gamma\}) \\ &\cong H_{*-\dim \mathbb{E}^-}(S^1) \\ &\cong \mathbb{F} \oplus \mathbb{F}. \end{aligned}$$

All together, we have proved that $\dim C_*(\mathcal{P}^c(I)) \geq 2n_c(I)$, where $n_c(I)$ denotes the number of non-degenerate, prime, critical circles in $\mathcal{P}^c(I)$. Combined with

Theorem D, this implies Corollary E: there exists $\delta > 0$ such that every $c \in \sigma(I)$ is a boundary point of at least $2n_c(I)$ bars of size at least δ in the closed geodesics barcode $\mathcal{B}$. $\square$

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