ON THE BARCODE ENTROPY OF REEB FLOWS

ERMAN ÇİNELİ, VIKTOR L. GINZBURG, BAŞAK Z. GÜREL,
AND MARCO MAZZUCCHELLI

Abstract. In this paper we continue investigating connections between Floer theory and dynamics of Hamiltonian systems, focusing on the barcode entropy of Reeb flows. Barcode entropy is the exponential growth rate of the number of not-too-short bars in the Floer or symplectic homology persistence module. The key novel result is that the barcode entropy is bounded from below by the topological entropy of any hyperbolic invariant set. This, combined with the fact that the topological entropy bounds the barcode entropy from above, established by Fender, Lee and Sohn, implies that in dimension three the two types of entropy agree. The main new ingredient of the proof is a variant of the Crossing Energy Theorem for Reeb flows.

Contents

Date: January 2, 2024.

2020 *Mathematics Subject Classification.* 53D40, 37B40, 37J12, 37J55.

Key words and phrases. Periodic orbits, Reeb flows, Floer homology, topological entropy, barcode entropy, persistence modules.

The work is partially supported by the NSF grants DMS-2304207 (BG) and DMS-2304206 (VG), Simons Foundation grants 855299 (BG) and MP-TSM-00002529 (VG), ERC Starting Grant 851701 via a postdoctoral fellowship (EÇ), and the ANR grants CoSyDy, ANR-CE40-0014 and COSY, ANR-21-CE40-0002 (MM).

1. Introduction and main results

1.1. **Introduction.** In this paper we continue investigating connections between Floer theory and dynamics of Hamiltonian systems, focusing on the relation of barcode entropy to topological entropy for Reeb flows.

Barcode entropy is an invariant associated with the sequence of Floer homology persistence modules for the iterates of a Hamiltonian diffeomorphism or the symplectic homology persistence module for a Reeb flow. In both cases it measures the exponential growth rate of the number of not-so-short bars in the barcode. It is closely related to the topological entropy of the underlying Hamiltonian system.

For compactly supported Hamiltonian diffeomorphisms $\varphi\colon M \to M$, barcode entropy was introduced in [ÇGG21]. We showed there that the barcode entropy $\hbar(\varphi)$ of φ is bounded from above by the topological entropy $\mathrm{h_{top}}(\varphi)$ and conversely that $\mathrm{h_{top}}(\varphi|_K) \leq \hbar(\varphi)$ whenever K is a (locally maximal) compact hyperbolic invariant set of φ. As a consequence, $\hbar(\varphi) = \mathrm{h_{top}}(\varphi)$ when M is a surface by the results of Katok, [Kat80]. (However, this equality does not hold in general in higher dimensions, [Çi].) For geodesic flows, barcode entropy was defined in [GGM] where similar inequalities were established. For the Reeb flow $\varphi^t = \varphi^t_\alpha$ of a contact form α on the boundary M of a Liouville domain, the barcode entropy $\hbar(\alpha)$ was introduced in [FLS]. There, a contact version of the first inequality was also proved: $\hbar(\alpha) \leq \mathrm{h_{top}}(\varphi^t)$. Here we establish an analogue of the second inequality: $\mathrm{h_{top}}(\varphi|_K) \leq \hbar(\varphi)$, where again K is a locally maximal compact hyperbolic set, and hence $\hbar(\alpha) = \mathrm{h_{top}}(\varphi^t)$ when $\dim M = 3$ by the results from [LY, LS] extending Katok's work to flows on three-manifolds. In particular, as in the Hamiltonian case, barcode entropy can and does detect topological entropy coming from localized sources such as a hyperbolic set contained, for maps, in a small ball or, for flows, in a thin mapping torus.

There is, of course, an immense body of work connecting dynamics, e.g., topological entropy, of a Hamiltonian system, broadly understood, with features of the underlying variational principle, e.g., Morse or Floer homology. What distinguishes our approach to the question is that it does not rely on unconditional global (symplectic) topological properties of the map coming from, say, the exponential growth of the Floer or Morse homology, in turn, determined by the (symplectic) isotopy class of the map or topology of the phase/configuration space.

In the setting of compactly supported Hamiltonian diffeomorphisms of symplectic manifolds, this comes for granted: such maps are (Hamiltonian) isotopic to the identity and the Floer homology is independent of the map. Hence there cannot be any global topological or Floer homological growth. However, in the setting of geodesic or Reeb flows there are more possibilities. For instance, the Morse or Floer homology can grow exponentially fast, forcing the flow to have positive topological entropy. This growth is also captured by barcode entropy but the phenomenon has been quite well understood independently of this concept.

For instance, connections between topological entropy of a geodesic flow and topology of the underlying manifold have been studied in, e.g., [Di, Kat82, Pa]. A variety of generalizations of these classical results to Reeb flows have been obtained in [A^{2}S^2, Al16, Al19, ACH, AM, ADM^2, AP, MS], relating topological entropy of Reeb flows to their Floer theoretic invariants (e.g., symplectic or contact homology). Again, the global contact topology of the underlying manifold is central to these results but plays little role in our approach.

The key technical ingredient of the proof of our main theorem is the Crossing Energy Theorem for Reeb flows, roughly speaking asserting that for an admissible Hamiltonian H, any Floer trajectory asymptotic to a periodic orbit of its Hamiltonian flow φ_H^t corresponding to a periodic orbit of $\varphi_\alpha^t|_K$ has energy bounded from below by some constant $\sigma > 0$ independent of the orbit and its period. (An admissible Hamiltonian is specifically tailored for recasting Reeb dynamics in Hamiltonian terms; in particular, it is autonomous and its flow on every positive level is a reparametrization of the Reeb flow.) For locally maximal hyperbolic sets of Hamiltonian diffeomorphisms, several variants of this theorem were originally proved in [ÇGG21, GG14, GG18]. At the center of the argument is the observation that all circles $u(s, \cdot)$ in a low energy Floer cylinder u for an iterated Hamiltonian diffeomorphism are ϵ-pseudo-orbits with $\epsilon > 0$ independent of the period. The proofs, however, do not directly translate to admissible Hamiltonians. Indeed, one of the main difficulties arising in the contact setting is that the invariant set of φ_H^t corresponding to a locally maximal hyperbolic set of φ_α^t is neither hyperbolic nor locally maximal, while both conditions are essential.

A proof of the Crossing Energy Theorem in the Hamiltonian setting for $\mathbb{CP}^n$ using generating function was given in [Al] and its counterpart for geodesic flows, also using finite-dimensional reduction, can be found in [GGM]. In [ÇG^{2}M] the theorem was proved for an isolated hyperbolic periodic orbit of a Reeb flow in a setting fairly close to the one adopted here. Finally, a variant of the theorem for holomorphic curves in the symplectization is established in [CGP] and a version with Lagrangian boundary conditions is proved in [Me24].

All known to date lower bounds on barcode entropy type invariants rely on a combination of hyperbolicity and crossing energy bounds. However, crossing energy theorems have other applications outside the subject of barcode entropy, ranging from multiplicity results for periodic orbits in a variety of settings to Le Calvez–Yoccoz type theorems to lower bounds on the spectral norm; see [Ba15, Ba17, ÇGG23, ÇG^{2}M, GG14, GG16, GG18].

1.2. **Main definitions and results.** Let $(W, d\alpha)$ be a Liouville domain. We will also use the notation α for the contact form $\alpha|_M$ on the boundary $M = \partial W$.

Fix a ground field $\mathbb{F}$, which we suppress in the notation, and denote the (non-equivariant) filtered symplectic homology of W over $\mathbb{F}$ for the action interval $[0, \tau)$ by $\mathrm{SH}^\tau(\alpha)$. Throughout this paper, the grading of symplectic homology plays no role and we view $\mathrm{SH}^a(\alpha)$ as an ungraded vector space over $\mathbb{F}$. We make no assumptions on the first Chern class $c_1(TW)$.

Together with the natural maps $\mathrm{SH}^{\tau_0}(\alpha) \to \mathrm{SH}^{\tau_1}(\alpha)$ for $\tau_0 \leq \tau_1$, the symplectic homology forms a persistence module; see Sections 2.1 and 3.3. For $\epsilon > 0$, we denote by $\mathfrak{b}_\epsilon(\alpha, \tau)$ or just $\mathfrak{b}_\epsilon(\tau)$, when α is clear from the context, the number of bars of length greater than ϵ, beginning in the range $[0, \tau)$ in the barcode $\mathcal{B}(\alpha)$ of this persistence module. This is an increasing function in τ and $1/\epsilon$, locally constant as a function of τ in the complement to $\mathcal{S}(\alpha) \cup \{0\}$, where $\mathcal{S}(\alpha)$ is the action spectrum of α.

The barcode entropy of α, denoted by $\hbar(\alpha)$, measures the exponential growth rate of $\mathfrak{b}_\epsilon(\tau)$ and is defined as follows.

Definition 1.1. The *ϵ-barcode entropy* of α is

$$\hbar_\epsilon(\alpha) := \limsup_{\tau\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(\tau)}{\tau}, \tag{1.1}$$

where log is taken base 2, $\log 0 = -\infty$ and $\log^+ := \max\{0, \log\}$, and the *barcode entropy* of α is

$$\hbar(\alpha) := \lim_{\epsilon\to 0^+} \hbar_\epsilon(\alpha) \in [0, \infty]. \tag{1.2}$$

A few comments are due at this point. First of all, since $\mathfrak{b}_\epsilon(\tau)$ is an increasing function of τ, we can replace the upper limit in (1.1) as $\tau \to \infty$ by the upper limit over an increasing sequence $\tau_i \to \infty$ as long as this sequence is not too sparse. Namely, it is not hard to see that

$$\hbar_\epsilon(\alpha) = \limsup_{i\to\infty} \frac{\log^+ b_\epsilon(\tau_i)}{\tau_i}$$

whenever $\tau_i \to \infty$ and $\tau_{i+1}/\tau_i \to 1$; cf. Proposition 4.3 and Theorem 4.4. Second, $\hbar_\epsilon(\alpha)$ increases as $\epsilon \to 0^+$, and hence the limit in (1.2) does exist. Clearly,

$$\hbar_\epsilon(\alpha) \leq \hbar(\alpha),$$

and, as is easy to see, for any $a > 0$,

$$\hbar_\epsilon(a\alpha) = a^{-1}\hbar_\epsilon(\alpha) \text{ and } \hbar(a\alpha) = a^{-1}\hbar(\alpha).$$

Remark 1.2. Hypothetically, $\hbar_\epsilon(\alpha)$ and $\hbar(\alpha)$ might depend on the entire Liouville domain $(W, d\alpha)$ rather than just the contact form α on $M = \partial W$. However, we are not aware of any examples where this happens. (Corollary C below implies, in particular, that $\hbar(\alpha)$ is completely determined by (M, α) when $\dim M = 3$.) Moreover, according to [FLS, Sect. 4.2], $\hbar(\alpha)$ is independent of the filling at least when $c_1(TW) = 0$ and we tend to think in general as long as (M, α) is the boundary of the Liouville domain and hence $\hbar(\alpha)$ is defined.

Denote by $\mathrm{h}_{\mathrm{top}}(\alpha)$ the topological entropy of the Reeb flow φ_α^t of α. The next three theorems relating barcode entropy and topological entropy are contact counterparts of similar results for Hamiltonian diffeomorphisms and geodesic flows: see [ÇGG21] and [GGM].

Theorem A ([FLS])**.** *For any Liouville domain* $(W, d\alpha)$*, we have*

$$\hbar(\alpha) \leq \mathrm{h_{top}}(\alpha).$$

This theorem is originally proved in [FLS]. (Strictly speaking, the theorem is stated there under the additional conditions that $c_1(TW) = 0$. However, this condition appears to be immaterial for the argument.) We will comment on the proof in Remark 4.5. Note that $\hbar(\alpha) < \infty$ by Theorem A.

The next theorem, which is the main new result of this paper, shows that the barcode entropy can be positive and is related to the hyperbolic invariant sets of φ_α^t. We refer the reader to, e.g., [KH] for the definition of such sets and Section 5 for a further discussion.

Theorem B. *Let* K *be a compact hyperbolic invariant set of the Reeb flow* φ_α^t*. Then*

$$\hbar(\alpha) \geq \mathrm{h_{top}}(K),$$

where we set $\mathrm{h_{top}}(K) := \mathrm{h_{top}}(\varphi_\alpha^t|_K)$.

Next, recall that when $\dim M = 3$,

$$\mathrm{h_{top}}(\alpha) = \sup_K \mathrm{h_{top}}(K),$$

where K ranges over all hyperbolic invariant sets, [LY, LS]. (This is a generalization to flows on three-manifolds of a theorem originally proved in [Kat80] for diffeomorphisms of surfaces.) Combining this fact with Theorems A and B, we obtain the following.

Corollary C. *Assume that* $\dim M = 3$*. Then* $\hbar(\alpha) = \mathrm{h_{top}}(\alpha)$.

This corollary is a Reeb counterpart of a similar result for Hamiltonian diffeomorphisms of surfaces; see [ÇGG21, Thm. C]. The latter theorem does not generalize to higher dimensions as the counterexamples constructed in [Çi] show. While this construction does not readily extend to Reeb flows, we do not expect Corollary C to hold in higher dimensions either.

Furthermore, in dimension three $\mathrm{h_{top}}(\alpha)$ is C^0 lower-semicontinuous in α at a C^1-open and dense set in the space of all contact forms on M as is proved in [ADMP]; see also [ADM²]. By Corollary C, the same is true for $\hbar(\alpha)$ whenever M bounds a Liouville domain.

Remark 1.3 (Other types of barcode entropy). The barcode entropy in the Hamiltonian setting and for geodesic flows also has a relative counterpart associated with the filtered Floer or Morse homology for Lagrangian or geodesic chords, [ÇGG21] and [GGM, Sec. 4.4]. A variant of relative barcode entropy for Reeb flows can easily be defined via wrapped Floer homology and one would expect an analogue of Theorem A to also hold in this setting; see [Fe].

Furthermore, yet two different versions of barcode entropy, both introduced in [ÇGG22] in the Hamiltonian framework, can also be defined in the Reeb setting and again we expect the above results to hold for them. These are sequential barcode entropy and total persistence entropy. Both entropies have properties similar to barcode entropy and, by construction, bound the ordinary barcode entropy from above. Hence, in both cases, Theorem B follows from its counterpart for the ordinary barcode entropy. In the Hamiltonian setting, Theorem A for these variants of barcode entropy was proved in [ÇGG22]. We conjecture that these refinements of Theorem A, and hence Corollary C, also hold in the contact setting.

For the sake of brevity we do not consider here these generalizations or modifications of barcode entropy, focusing instead on the proof of Theorem B.

Remark 1.4 (Barcode entropy for geodesic flows). As we have pointed out in the introduction, barcode entropy for geodesic flows was defined in [GGM], where the analogues of Theorems A and B and Corollary C were also proved. In that specific case, the barcode entropy is equal to the barcode entropy considered here. This is essentially a consequence of the equality of the filtered Morse homology and the symplectic homology, although some attention needs to be paid to the definition of the latter; see [AS, SW, Vi], [GGM, Rmk. 4.5] and also [FLS, Rmk. 4.10].

The paper is organized as follows. In Section 2 we set our conventions and notation and also discuss the class of (semi-)admissible Hamiltonians used throughout the paper. The relevant facts from Floer theory are assembled in Section 3. In Section 4 we revisit the definition of barcode entropy, and reformulate it in a way more suitable for dynamics applications and prove equivalence of the definitions. We derive Theorem B from the Crossing Energy Theorem in Section 5, which is then proved in Section 6.

2. Preliminaries, conventions, and notation

2.1. **Persistence modules.** Persistence modules play a central role in the definition of barcode entropy. In this section we define the class of persistence modules suitable for our goals and briefly touch upon their properties. We refer the reader to [PRSZ] for a general introduction to persistence modules, their applications to geometry and analysis and further references, although the class of modules they consider is somewhat more narrow than the one we deal with here, and also to [BV, CB] for some of the more general results.

Fix a field $\mathbb{F}$ which we will suppress in the notation. Recall that a *persistence module* (V, π) is a family of vector spaces V_s over $\mathbb{F}$ parametrized by $s \in \mathbb{R}$ together with a functorial family π of structure maps. These are linear maps $\pi_{st}\colon V_s \to V_t$, where $s \leq t$ and functoriality is understood as that $\pi_{sr} = \pi_{tr}\pi_{st}$ whenever $s \leq t \leq r$ and $\pi_{ss} = id$. In what follows we often suppress π in the notation and simply refer to (V, π) as V. In such a general form the concept is not particularly useful and usually one imposes additional conditions on the spaces V_t and the structure maps π_{st}. These conditions vary depending on the context. Below we spell out the framework most suitable for our purposes.

Namely, we require that there is a closed, bounded from below, nowhere dense subset $\mathcal{S} \subset \mathbb{R}$, which is called the *spectrum* of V, and the following four conditions are met:

(i) The persistence module V is *locally constant* outside $\mathcal{S}$, i.e., π_{st} is an isomorphism whenever $s \leq t$ are in the same connected component of $\mathbb{R} \setminus \mathcal{S}$.
(ii) The persistence module V is *q-tame*: π_{st} has finite rank for all $s < t$.
(iii) *Left-semicontinuity*: For all $t \in \mathbb{R}$,

$$V_t = \varinjlim_{s<t} V_s. \tag{2.1}$$

(iv) *Lower bound*: $V_s = 0$ when $s < s_0$ for some $s_0 \in \mathbb{R}$. (Throughout the paper we will assume that $s_0 = 0$.)

A few comments on this definition are in order. First, note that as a consequence of (i) and (ii), V_s is finite-dimensional and (iii) is automatically satisfied when $s \notin \mathcal{S}$. Furthermore, in several instances which are of interest to us, V_s is naturally defined only for $s \notin \mathbb{R}$; then the definition is extended to all $s \in \mathbb{R}$ by (2.1). By (iv), we can always assume that $s_0 \leq \inf \mathcal{S}$, i.e., $\mathcal{S}$ is bounded from below. We emphasize, however, that $\mathcal{S}$ is not assumed to be bounded from above and it is actually not in many examples we are interested in. In what follows it will sometimes be convenient to include $s = \infty$ by setting

$$V_\infty = \varinjlim_{s\to\infty} V_s.$$

Finally, in all examples we encounter here $\mathcal{S}$ has zero measure and, in fact, zero Hausdorff dimension, but this fact is never used in the paper.

A basic example motivating requirements (i)–(iv) is that of the sublevel homology of a smooth function.

Example 2.1 (Homology of sublevels). Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ be a proper smooth function bounded from below. Set $V_s := H_*\big(\{f < s\}; \mathbb{F}\big)$ with the structure maps induced by inclusions. No other requirements are imposed on f, e.g., f need not be Morse. However, it is not hard to see that conditions (i)–(iv) are met with $\mathcal{S}$ being the set of critical values of f. We note that one can have $\dim V_s = \infty$ for $s \in \mathcal{S}$ already when $M = S^1$, unless f meets some additional conditions on f, e.g., that f is real analytic or the critical points of f are isolated.

Recall furthermore that an *interval persistence module* $\mathbb{F}_{(a,\,b]}$, where $-\infty < a < b \leq \infty$, is defined by setting

$$V_s := \begin{cases} \mathbb{F} & \text{when } s \in (a,\, b], \\ 0 & \text{when } s \notin (a,\, b], \end{cases}$$

and $\pi_{st} = id$ if $a < s \leq t \leq b$ and $\pi_{st} = 0$ otherwise. Interval modules are examples of simple persistence modules, i.e., persistence modules that cannot be decomposed as a (non-trivial) direct sum of other persistence modules.

A key fact that we will use in the paper is the normal form or structure theorem asserting that every persistence module meeting the above conditions can be decomposed as a direct sum of a countable collection (i.e., a countable multiset) of interval persistence modules. Moreover, this decomposition is unique up to reordering of the sum. (In fact, conditions (i)–(iv) are far from optimal and can be considerably relaxed.) We refer the reader [BV, CB] for proofs of this theorem for the class of persistence modules considered here and further references, and also, e.g., to [CZCG, ZC] for previous or related results.

This multiset $\mathcal{B}(V)$ of intervals entering this decomposition is referred to as the *barcode* of V and the intervals occurring in $\mathcal{B}(V)$ as *bars*. For $\epsilon > 0$ we denote by $\mathfrak{b}_\epsilon(V, s)$ or just $\mathfrak{b}_\epsilon(s)$ the number of bars $(a, b]$ in $\mathcal{B}(V)$ with $a < s$ of length $b - a > \epsilon$, counted with multiplicity. This is the only numerical invariant of persistence modules used in this paper. It is not hard to show that $\mathfrak{b}_\epsilon(s) < \infty$ for all $\epsilon > 0$ and $s < \infty$ under our conditions on V (see Remark 2.2 below), even though the total number of bars beginning below s can be infinite.

Remark 2.2 (Locally finite approximations). Every persistence module V as above can be approximated with respect to the interleaving distance by a locally finite persistence module V'. The construction of V' amounts to throwing away short bars and then adjusting V_s for $s \in \mathcal{S}$. Alternatively, in our case one can simply perturb the Hamiltonians or contact forms to ensure non-degeneracy. Then $\mathcal{B}(V)$ and $\mathcal{B}(V')$ are close with respect to the bottleneck distance, and for any $\delta > 0$

$$\mathfrak{b}_{\epsilon-\delta}(V', s) \geq \mathfrak{b}_\epsilon(V, s) \geq \mathfrak{b}_{\epsilon+\delta}(V', s)$$

when V' is sufficiently close to V. Utilizing this fact, we could have worked with a more narrow class of locally finite persistence modules and used small perturbations to define $\mathfrak{b}_\epsilon(V, s)$. This is essentially the approach taken in [ÇGG21]. However, here we find working with a broader class of persistence modules more convenient.

2.2. **Semi-admissible Hamiltonians, periodic orbits and the action functional.** In this section we spell out our conventions and notation on the symplectic dynamics side, which are essentially identical to the ones used in [ÇG^{2}M, GG20], and also recall several elementary properties of (semi-)admissible Hamiltonians to be used later.

Let, as in Section 1.2, α be the contact form on the boundary $M = \partial W$ of a Liouville domain $W^{2n\geq 4}$. We will also use the same notation α for a primitive of the symplectic form ω on W. The grading of Floer or symplectic homology is inessential for our purposes and we make no assumptions on $c_1(TW)$. As usual, denote by $\widehat{W}$ the symplectic completion of W, i.e.,

$$\widehat{W} = W \cup_M M \times [1, \infty)$$

with the symplectic form $\omega = d\alpha$ extended from W to $M \times [1, \infty)$ as

$$\omega := d(r\alpha),$$

where r is the coordinate on $[1, \infty)$. Sometimes it is convenient to have the function r also defined on a collar of $M = \partial W$ in W. Thus we can think of $\widehat{W}$ as the union of W and $M \times [1 - \eta, \infty)$ for small $\eta > 0$ with $M \times [1 - \eta, 1]$ lying in W and the symplectic form given by the same formula.

Unless specifically stated otherwise, most of the Hamiltonians $H : \widehat{W} \to \mathbb{R}$ considered in this paper are constant on W and depend only on r outside W, i.e., $H = h(r)$ on $M \times [1, \infty)$, where the C^∞-smooth function $h : [1, \infty) \to \mathbb{R}$ is required to meet the following three conditions:

- h is strictly monotone increasing;
- h is convex, i.e., $h'' \geq 0$, and $h'' > 0$ on $(1, r_{\max})$ for some $r_{\max} > 1$ depending on h;
- $h(r)$ is linear, i.e., $h(r) = ar - c$, when $r \geq r_{\max}$.

In other words, the function h changes from a constant on W to convex in r on $M \times [1, r_{\max}]$, and strictly convex on the interior, to linear in r on $M \times [r_{\max}, \infty)$.

We will refer to a as the *slope* of H (or h) and write $a = \mathit{slope}(H)$. The slope is often, but not always, assumed to be outside the action spectrum of α, i.e., $a \notin \mathcal{S}(\alpha)$. We call H *admissible* if $H|_W = \mathit{const} < 0$ and *semi-admissible* when $H|_W \equiv 0$. (This terminology differs from the standard usage, and we emphasize that *admissible Hamiltonians are not semi-admissible.*) When H satisfies only the last of the three conditions, we call it *linear at infinity*.

The difference between admissible and semi-admissible Hamiltonians is just an additive constant: $H - H|_W$ is semi-admissible when H is admissible. Hence the two Hamiltonians have the same filtered Floer homology up to an action shift. For our purposes, semi-admissible Hamiltonians are notably more suitable due to the $H|_W \equiv 0$ normalization.

The Hamiltonian vector field X_H is determined by the condition

$$\omega(X_H, \cdot) = -dH,$$

and, on $M \times [1, \infty)$,

$$X_H = h'(r)R_\alpha,$$

where R_α is the Reeb vector field. We denote the Hamiltonian flow of H by φ_H^t, the Reeb flow of α by φ_α^t, where $t \in \mathbb{R}$, and the Hamiltonian diffeomorphism generated by H by $\varphi_H := \varphi_H^1$.

Throughout the paper, by a *τ-periodic orbit* of H we will mean one of several closely related but distinct objects. It can be a τ-periodic orbit of φ_H and then $\tau \in \mathbb{N}$. Alternatively, it can stand for a τ-periodic orbit of the flow φ_H^t with $\tau \in (0, \infty)$. Furthermore, working with periodic orbits of flows or maps, we might or not have the initial condition fixed. For instance, without an initial condition fixed, a non-constant 1-periodic orbit of the flow of H gives rise to a whole circle of 1-periodic orbits (aka fixed points) of φ_H. Likewise, a prime τ-periodic orbit of φ_H comprises τ τ-periodic points. In most cases the exact meaning should be clear from the context and is often immaterial; when the difference is essential we will specify whether an orbit is of the flow or the diffeomorphism and if the initial condition is fixed or not.

Every T-periodic orbit z of the Reeb flow with $T < a = \mathit{slope}(H)$ gives rise to a 1-periodic orbit $\tilde{z} = (z, r_*)$ of the flow of H with r_* determined by the condition

$$h'(r_*) = T. \tag{2.2}$$

Clearly, $\tilde{z}$ lies in the shell $1 < r < r_{\max}$, and we have a one-to-one correspondence between 1-periodic orbits of H and the periodic orbits of φ_α^t with period $T < a$ whenever $a \notin \mathcal{S}(\alpha)$. In the pair $\tilde{z} = (z, r_*)$, we usually view $\tilde{z}$ as a 1-periodic orbit of the flow φ_H^t of H or a circle of fixed points of the Hamiltonian diffeomorphism φ_H, while z, contrary to what the notation might suggest, is parametrized by the

Reeb flow but not as a projection of $\tilde{z}$ to M. (By (2.2), the two parametrizations of z differ by the factor of $h'(r_*) = T$.) Fixing an initial condition on z determines an initial condition on $\tilde{z}$, and the other way around. In particular, z gives rise to a whole circle $\tilde{z}(S^1)$ of fixed points of φ_H.

We say that a T-periodic orbit z of the Reeb flow is *isolated* (as a periodic orbit) if for every $T' > T$ it is isolated among periodic orbits with period less than T'. Clearly, all periodic orbits of α are isolated if and only if for every T' the number of periodic orbits with period less than T' is finite. For instance, a non-degenerate periodic orbit is isolated. Note that $\tilde{z}$ is isolated as a 1-periodic orbit of the flow of H if z is isolated. No fixed point of φ_H on $\tilde{z}(S^1)$ is isolated, but $\tilde{z}$ is Morse–Bott non-degenerate, as the set of fixed points $\tilde{z}(S^1)$, if and only if z is non-degenerate; cf. [Bo].

The action functional $\mathcal{A}_H$ is defined by

$$\mathcal{A}_H(\gamma) = \int_\gamma \hat{\alpha} - \int_{S^1} H(\gamma(t))\, dt,$$

where $\gamma\colon S^1 = \mathbb{R}/\mathbb{Z} \to \widehat{W}$ is a smooth loop in $\widehat{W}$ and $\hat{\alpha}$ is the Liouville primitive α of ω on W and $\hat{\alpha} = r\alpha$ on $M \times [1-\eta, \infty)$ for a sufficiently small $\eta > 0$. More explicitly, when $\gamma\colon S^1 \to M \times [1, \infty)$, we have

$$\mathcal{A}_H(\gamma) = \int_{S^1} r(\gamma(t))\alpha\big(\gamma'(t)\big)\, dt - \int_{S^1} h\big(r(\gamma(t))\big)\, dt.$$

Thus when $\gamma = \tilde{z} = (z, r_*)$ is a 1-periodic orbit of H, the action can be expressed as a function of r_* only:

$$\mathcal{A}_H(\tilde{z}) = A_h(r_*),$$

where

$$A_h\colon [1, \infty) \to [0, \infty) \text{ is given by } A_h(r) = rh'(r) - h(r). \tag{2.3}$$

Sometimes we will also denote this *action function* by A_H. This is a monotone increasing function, for

$$A_h'(r) = h'(r) + rh''(r) - h'(r) = rh''(r) \geq 0.$$

It is not hard to show that

$$\max A_h = A_h(r_{\max}) = c \geq a; \tag{2.4}$$

see [ÇG^{2}M, Sect. 2.1]. For this reason, we will in some instances limit the domain of this function to $[1, r_{\max}]$.

While the function A_h expresses the Hamiltonian action as a function of r, we will also need another variant $\mathfrak{a}_H$ (or $\mathfrak{a}_h$) of the action function, expressing the Hamiltonian action as a function of the period T, i.e., the contact action. In other words, the function $\mathfrak{a}_H$ translates the contact action to the Hamiltonian action. Thus

$$\mathfrak{a}_H = A_h \circ (h')^{-1}\colon [0, a] \to [0, \max A_h = A_h(r_{\max})]$$

is more specifically defined by the condition

$$\mathfrak{a}_H(T) = A_h(r), \text{ where } h'(r) = T. \tag{2.5}$$

Since H is semi-admissible, h' is one-to-one on $[1, r_{\max}]$, and the inverse $(h')^{-1}$ is defined on $[0, a]$. Then, using the chain rule, we have

$$\mathfrak{a}'_H(T) = r := (h')^{-1}(T) \text{ and } 1 \leq \mathfrak{a}'_H \leq r_{\max}. \tag{2.6}$$

Thus $\mathfrak{a}_H$ is a strictly monotone increasing, convex C^1-function, which is C^∞ on $(0, a)$, with $\mathfrak{a}''_h = \infty$ at $T = 0$ and $T = a$. Furthermore,

$$\mathfrak{a}_{H_1} \leq \mathfrak{a}_{H_0} \text{ on } [0, \mathit{slope}(H_0)] \text{ whenever } H_1 \geq H_0. \tag{2.7}$$

A simple way to prove (2.7) is as follows, [ÇG^{2}M, Sect. 2.1]. First, note that that $-A_h(r)$ is the ordinate of the intersection of the tangent line to the graph of h at $(r, h(r))$ with the vertical axis. Furthermore, $\mathfrak{a}_{H_1}(T) = A_{h_1}(r_1)$ and $\mathfrak{a}_{H_0}(T) = A_{h_0}(r_0)$, where $h'_1(r_1) = T = h'_0(r_0)$. Hence, the two tangent lines have the same slope T. The tangent line to the graph of h_1 lies above the tangent line to the graph of h_0; for it passes through the point $(r_1, h_1(r))$ which is above the graph of $h_0 \leq h_1$. Therefore, $A_{h_0}(r_0) \geq A_{h_1}(r_1)$.

Furthermore, it is not hard to see that

$$\mathfrak{a}_{sH}(T) \to T \text{ as } s \to \infty \tag{2.8}$$

uniformly on compact sets in $[0, \infty)$ whenever H is semi-admissible.

3. Filtered Floer and symplectic homology

In this section we recall basic definitions and results from Floer theory used in the proof of Theorem B. Many, but not all, of the constructions here are quite standard and go back to [CFH, Vi] and can also be found in numerous subsequent accounts.

3.1. **Floer equation.** Fix an almost complex structure J on $\widehat{W}$ satisfying the following conditions:

- J is compatible with ω, i.e., $\omega(\cdot, J\cdot)$ is a Riemannian metric,
- $Jr\partial/\partial r = R_\alpha$ on the cylinder $M \times [1, \infty)$,
- J preserves $\ker(\alpha)$.

The last two conditions are equivalent to that

$$dr \circ J = -r\alpha. \tag{3.1}$$

We call such almost complex structures *admissible*. If the first condition still holds on $\widehat{W}$, and the second and the third conditions are met only outside a compact set while within a compact set J can be time-dependent and 1-periodic in time, we call J *admissible at infinity*.

Next, let H be a Hamiltonian linear at infinity and let J be an admissible at infinity almost complex structure. Following [ÇG^{2}M, GG20], it is convenient for our purposes to adopt the L^2-anti-gradient of $\mathcal{A}_H$,

$$\partial_s u = -\nabla_{L^2} \mathcal{A}_H(u), \tag{3.2}$$

as the Floer equation, where $u \colon \mathbb{R} \times S^1 \to \widehat{W}$ and (s, t) are the coordinates on $\mathbb{R} \times S^1$ with $S^1 = \mathbb{R}/\mathbb{Z}$. Hence the function $s \mapsto \mathcal{A}_H\big(u(s, \cdot)\big)$ is decreasing. Explicitly, this equation reads

$$\partial_s u - J\big(\partial_t u - X_H(u)\big) = 0. \tag{3.3}$$

Note that the leading term of this equation is the ∂-operator, as opposed to the $\bar\partial$-operator as in the standard conventions. In other words, when $H \equiv 0$, solutions of (3.3) are anti-holomorphic rather than holomorphic curves. Nonetheless the standard properties of the solutions of the Floer equation readily translate to our setting, e.g., via the change of variables $s \mapsto -s$. We will often refer to solutions u of the Floer equation as *Floer cylinders*. Recall that the energy of u is by definition

$$E(u) = \int_{S^1\times\mathbb{R}} \|\partial_s u\|^2 \, dt\, ds.$$

Let us assume from now on that J is admissible and H is (semi-)admissible. Then the Floer equation is translation and rotation invariant since, J and H are independent of t (autonomous) and s. Thus, whenever u is a Floer cylinder, $(s,t) \mapsto u(s+s_0, t+t_0)$ is also a Floer cylinder for all $(s_0,t_0) \in \mathbb{R}\times S^1$. In particular, u is never regular unless u is independent of t. Recall, however, that in the notation from Section 2.2, $\tilde z = (z, r_*)$ is Morse–Bott non-degenerate if and only if z is non-degenerate; cf. [Bo].

Let $u\colon \mathbb{R}\times S^1 \to \widehat{W}$ be a Floer cylinder for H. We say that u is *asymptotic* to $\tilde z$ at ∞ if there exists a sequence $s_i \to \infty$ such that $u(s_i,\cdot) \to \tilde z$ in the C^1-sense, up to the choice of the initial condition on $\tilde z$ which might depend on s_i. (In other words, here we view $\tilde z$ as a 1-periodic orbit of the flow of H without fixing an initial condition and the choice of the initial condition turns it into a 1-periodic orbit of φ_H.) This definition is equivalent to that $u(s,\cdot) \to \tilde z$ in the C^∞-sense as $s \to +\infty$ when z is non-degenerate, and hence $\tilde z$ is Morse–Bott non-degenerate. Moreover, in this case $u(0,s)$ converges as $s \to \infty$, [Bo]. (Likewise, u is said to be asymptotic to $\tilde z$ at $-\infty$ when $s_i \to -\infty$, etc.)

In general, u can be asymptotic to more than one orbit $\tilde z$ at the same end. However, $\mathcal{A}_H(\tilde z) = \lim \mathcal{A}_H\big(u(s_i,\cdot)\big)$, and hence $\mathcal{A}_H(\tilde z)$ is independent of the choice of $\tilde z$. Furthermore, (3.4) below holds: $E(u)$ is the difference of actions of the orbits which u is asymptotic to at $\pm\infty$. It is a standard fact that u is asymptotic to some 1-periodic orbits of H at $\pm\infty$ if and only if $E(u) < \infty$; see [Sa99, Sec. 1.5].

Next, assume that u is asymptotic to $\tilde z$ at ∞ and z is isolated or, equivalently, $\tilde z$ is isolated as a 1-periodic orbit of the flow of H. Then, as is easy to see, $\tilde z$ is unique (as a 1-periodic orbit of φ_H^t) and $u(s,\cdot) \to \tilde z$ as $s \to \infty$ in the C^1-sense, up to the choice of an initial condition on $\tilde z$ which might depend on s. This is a consequence of the fact that

$$E\big(u|_{[s_i,\infty)\times S^1}\big) \to 0 \text{ as } s_i \to \infty$$

since $\mathcal{A}_H\big(u(s,\cdot)\big)$ is a monotone function of s and of the argument in [Sa99, Sec. 1.5].

Let u be asymptotic to $\tilde x = (x, r^+)$ at $-\infty$ and $\tilde y = (y, r^-)$ at $+\infty$. Then

$$E(u) = \mathcal{A}_H(\tilde x) - \mathcal{A}_H(\tilde y) = A_H(r^+) - A_H(r^-). \tag{3.4}$$

Here $r^+ \geq r^-$ – hence the notation – since (3.2) is an anti-gradient Floer equation and A_H is an increasing function.

We will make extensive use of two standard properties of Floer cylinders u for admissible or semi-admissible Hamiltonians H and admissible almost complex structures J.

The first one is the maximum principle asserting that the function $r \circ u$ cannot attain a local maximum in the domain in $\mathbb{R} \times S^1$ mapped by u into $M \times [1, \infty)$, i.e., where r is defined. (We refer the reader to, e.g., [Vi] and also [FS, Sec. 2] for a direct proof of this fact.) The same is true for H linear and J admissible at infinity, in the domain where $H = ar - c$ and J is admissible. Moreover, the maximum principle also holds for continuation Floer trajectories when $h(r) = a(s)r - c(s)$ and $a(s)$ is a non-decreasing function of s, with no constraints on the function $c(s)$. This version of the maximum principle is crucial to having Floer cylinders and continuation solutions of the Floer equation contained in a compact region of $\widehat{W}$, and hence the Floer homology and continuation maps for homotopies with non-decreasing slope are defined.

In particular, let, as above, u be a solution of the Floer equation asymptotic to 1-periodic orbits $\tilde{x} = (x, r^+)$ at $-\infty$ and $\tilde{y} = (y, r^-)$ at $+\infty$. Then, by the maximum principle,

$$\sup_{\mathbb{R}\times S^1} r\big(u(s,t)\big) \leq r^+ = r\big(u(-\infty,t)\big).$$

The second fact we will use is that $E(u) > \epsilon$ when u enters η-deep into W, i.e., u is not entirely contained in $M \times [1 - \eta, \infty)$, for some $\epsilon > 0$ depending on J and η but independent of u and a (semi-)admissible Hamiltonian H. This is an immediate consequence of monotonicity since $H = \mathit{const}$ in W, and hence u is an (anti-)holomorphic curve; see, e.g., [Si].

3.2. **Floer homology and continuation maps.** Fix a ground field $\mathbb{F}$ which we will suppress in the notation. Let H be a Hamiltonian H linear at infinity. Assume first that $\mathit{slope}(H) \notin \mathcal{S}(\alpha)$. Then, regardless of whether H is non-degenerate or not, the filtered (contractible) Floer homology $\mathrm{HF}^\tau(H)$ over $\mathbb{F}$ is readily defined as long as $\tau \in \mathbb{R}$ is outside the action spectrum $\mathcal{S}(H)$ of H. This is simply the homology $\mathrm{HF}^\tau(\tilde{H})$ of the Floer complex of a small non-degenerate perturbation $\tilde{H}$ of H with $\mathit{slope}(\tilde{H}) = \mathit{slope}(H)$, generated by the 1-periodic orbits with action less than τ. (Here we treat $\mathrm{HF}^\tau(H)$ as an ungraded vector space over $\mathbb{F}$.) It is easy to see that $\mathrm{HF}^\tau(\tilde{H})$ is independent of $\tilde{H}$ when $\tilde{H}$ is sufficiently close to H. The total Floer homology $\mathrm{HF}(H)$ is $\mathrm{HF}^\infty(H)$ or, more precisely, $\mathrm{HF}^\tau(H)$ where $\tau > \operatorname{supp}\mathcal{S}(H)$.

Clearly, for $\tau_1 \leq \tau_2$ we have the "inclusion" map

$$\mathrm{HF}^{\tau_1}(H) \to \mathrm{HF}^{\tau_2}(H). \tag{3.5}$$

As in Section 2.1, we use (2.1) to extend this definition of $\mathrm{HF}^\tau(H)$ to all $\tau \in \mathbb{R}$. Namely, for any $\tau \in \mathbb{R}$ which is now allowed to be in $\mathcal{S}(H)$ or $\tau = \infty$, we set

$$\mathrm{HF}^\tau(H) := \varinjlim_{\tau' \leq \tau} \mathrm{HF}^{\tau'}(H), \tag{3.6}$$

where we require that $\tau' \notin \mathcal{S}(H)$. The "inclusion" maps naturally extend to these homology spaces and with this definition the family of spaces $\tau \mapsto \mathrm{HF}^\tau(H)$ becomes a persistence module. These maps are isomorphisms as long as the interval $[\tau_1, \tau_2)$ contains no points of $\mathcal{S}(H)$. (We changed the notation for the persistence module parameter s to τ, for s is taken by the homotopy parameter below.) In what follows we will be interested in the number $\mathfrak{b}_\epsilon^{Dyn}(H)$ of bars of length greater than $\epsilon > 0$ in the barcode of this persistence module beginning below $\mathfrak{a}_H(a) - \epsilon$, where $a = \mathit{slope}(H)$.

Note that

$$\mathrm{HF}^\tau(H) = 0 \text{ for } \tau \leq 0 \tag{3.7}$$

when H is semi-admissible.

Remark 3.1. Alternatively, in a more *ad hoc* fashion, one could have set

$$\mathrm{HF}^\tau(H) := \varinjlim_{\tilde{H} \leq H} \mathrm{HF}^\tau(\tilde{H}),$$

where $\tilde{H}$ is again non-degenerate, $\mathit{slope}(\tilde{H}) = \mathit{slope}(H)$, but now $\tilde{H} \leq H$ pointwise and $\tau \notin \mathcal{S}(\tilde{H})$. However, with our conventions, this definition would *not* be literally equivalent to the one above and (3.6) would not hold in general. In other words, $\tau \mapsto \mathrm{HF}^\tau(H)$ would not be a persistence module in the sense of Section 2.1: Left-semicontinuity, (iii) and (2.1), would break down. For instance, assume that $H \equiv \tau$ on W and $H > \tau$ on $M \times (0, \infty)$, e.g., $\tau = 0$ and H is semi-admissible. Then we would have $\mathrm{HF}^\tau(H) = \mathrm{H}_*(W, M) \neq 0$ but $\mathrm{HF}^{\tau'}(H) = 0$ for all $\tau' < \tau$.

Let H_s, $s \in \mathbb{R}$, be a homotopy between two linear at infinity Hamiltonians H_0 and H_1, i.e., H_s is a family of linear at infinity Hamiltonians such that $H_s = H_0$ when s is close to $-\infty$ and $H_s = H_1$ when s is close to $+\infty$. (In what follows we will take the liberty to have homotopies parametrized by $s \in [0, 1]$ or some other finite interval rather than $\mathbb{R}$.) There are two cases where a homotopy gives rise to a map in Floer homology.

The first one is when all Hamiltonians H_s have the same slope. Then the homotopy induces a continuation map

$$\mathrm{HF}^\tau(H_0) \to \mathrm{HF}^{\tau+C}(H_1)$$

shifting the action filtration by

$$C = \int_{-\infty}^{\infty} \max_{z \in \widehat{W}} \max\{0, -\partial_s H_s(z)\}\, ds.$$

Moreover, it is well-known and not hard to show that $\mathrm{HF}(H)$ does not change as long as $\mathit{slope}(H)$ stays outside of $\mathcal{S}(\alpha)$.

The second case is when H_s is monotone increasing, i.e., the function $s \mapsto H_s(z)$ is monotone increasing for all $z \in \widehat{W}$. In particular, the function $s \mapsto \mathit{slope}(H_s)$ is also monotone increasing. Note that while $\mathit{slope}(H_0)$ and $\mathit{slope}(H_1)$ are still required to be outside $\mathcal{S}(\alpha)$, the intermediate slopes $\mathit{slope}(H_s)$ can pass through the points of $\mathcal{S}(\alpha)$. Such a homotopy induces a map

$$\mathrm{HF}^\tau(H_0) \to \mathrm{HF}^\tau(H_1)$$

preserving the action filtration.

In both cases the fact that the continuation Floer trajectories are confined to a compact set is a consequence of the maximum principle; see Section 3.1.

The Floer homology is insensitive to perturbations of the Hamiltonian H and τ as long as *slope*$(H) \notin \mathcal{S}(\alpha)$ and τ remains outside $\mathcal{S}(H)$. To be more precise, fix a linear at infinity Hamiltonian H and τ meeting these conditions. Assume that the slope of H' is sufficiently close to the slope of H and H' is C^0-close to H on the complement of the domain where they both are linear functions of r, and that τ' is close to τ. Then there is a natural isomorphism of the Floer homology groups

$$\mathrm{HF}^{\tau}(H) \cong \mathrm{HF}^{\tau'}(H'). \tag{3.8}$$

Our next goal is to eliminate the assumption that $a := \mathit{slope}(H) \notin \mathcal{S}(\alpha)$. The most important case in our setting is that of the total Floer homology, i.e., $\tau > \sup \mathcal{S}(H)$ for a semi-admissible Hamiltonian H, and we will focus on this case. Then $V_s := \mathrm{HF}(sH)$ is defined for all $s > 0$ with $sa \notin \mathcal{S}(\alpha)$. Moreover, the homotopy $H_s := sH$ is monotone increasing and these spaces are connected by continuation maps. (This would not be true if H were admissible rather than semi-admissible.) In what follows, it is essential to extend the definition of V_s to all $s \in \mathbb{R}$ and turn V_s into a persistence module. To this end, we set $V_s = 0$ for $s \leq 0$. When $s > 0$ and $sa \in \mathcal{S}(\alpha)$ we use (2.1) as in Section 2.1:

$$V_s := \varinjlim_{s'<s} \mathrm{HF}(s'H), \tag{3.9}$$

where $s'a \notin \mathcal{S}(\alpha)$. Clearly, $\{V_s\}$ is indeed a persistence module. We will denote the number of bars of length greater than $\epsilon > 0$ beginning below s in the barcode of V by $\mathfrak{b}_{\epsilon}^{Fl}(s, H)$.

Remark 3.2. Alternatively and more generally, for any Hamiltonian H linear at infinity, we could have defined the filtered Floer homology as

$$\mathrm{HF}^{\tau}(H) := \varinjlim_{H' \leq H} \mathrm{HF}^{\tau}(H'),$$

where the limit is taken over Hamiltonians $H' \leq H$ linear at infinity with *slope*$(H') \notin \mathcal{S}(\alpha)$. One can show that, when H is (semi-)admissible, we may require H' to be (semi-)admissible and that this definition agrees with (3.9) in the sense that

$$\mathrm{HF}(H) := \varinjlim_{0<s<1} \mathrm{HF}(sH)$$

when H is semi-admissible; cf. Remark 3.3. However, for our purposes, the definition, (3.9), is more convenient as it fits better in the general framework of persistence modules; see Section 2.1.

3.3. **Symplectic homology.** In this section we review the definition and properties of filtered symplectic homology, focusing on its relations to the filtered Floer homology of semi-admissible Hamiltonians. These relations are somewhat less standard than the material from the previous two sections. Our treatment of the question has some overlaps with, e.g., [AM, Me18], although the setting and emphasis there are different, and also more directly with [ÇG²M].

The *symplectic homology* $\mathrm{SH}^\tau(\alpha)$, where $\tau > 0$, is defined as

$$\mathrm{SH}^\tau(\alpha) := \varinjlim_{H} \mathrm{HF}^\tau(H), \tag{3.10}$$

where traditionally the limit is taken over all Hamiltonians linear at infinity and such that $H|_W < 0$. Since admissible (but not semi-admissible) Hamiltonians form a co-final family, we can limit H to this class. Furthermore, we set

$$\mathrm{SH}^\tau(\alpha) := 0 \text{ when } \tau \leq 0. \tag{3.11}$$

When working with this definition, it is useful to keep in mind that, by (2.8),

$$\mathcal{S}(H) \to \{0\} \cup \mathcal{S}(\alpha) \tag{3.12}$$

uniformly on compactly intervals.

Remark 3.3 (Cofinal sequences). In (3.10), with (3.11) in mind, we could have required that $H|_W \leq 0$ rather than that $H|_W < 0$, or equivalently allowed H to be semi-admissible or admissible. This would result in the same groups $\mathrm{SH}^\tau(\alpha)$. Indeed, let H be a semi-admissible Hamiltonian. Pick two sequences of positive numbers: $s_i \to \infty$ and $\epsilon_i \to 0$. Then the sequence $H_i = s_i H - \epsilon_i$ is co-final in the class of admissible Hamiltonians.

The definition of symplectic homology via a direct limit, (3.10), over admissible or even semi-admissible Hamiltonians is quite inconvenient for our purposes. In fact, the limit over a much smaller class of Hamiltonians is sufficient:

Lemma 3.4. *Let H be any semi-admissible Hamiltonian. Then we have*

$$\mathrm{SH}^\tau(\alpha) = \varinjlim_{s\to\infty} \mathrm{HF}^\tau(sH) \tag{3.13}$$

for any $\tau \leq \infty$.

Clearly, similar statement holds for any action interval. We will prove the lemma a bit later in this section. In fact, passing to a limit in the definition of the symplectic homology is not needed at all if one is willing to make concessions of restricting the action range from above and also slightly reparametrizing the action.

Theorem 3.5. *Let H be a semi-admissible Hamiltonian with $\mathrm{slope}(H) = a$. Then, for every $\tau \leq a$, there exists an isomorphism*

$$\Phi_H^\tau\colon \mathrm{SH}^\tau(\alpha) \xrightarrow{\cong} \mathrm{HF}^{\mathfrak{a}_H(\tau)}(H),$$

where the function $\mathfrak{a}_H$ turning the Reeb period (aka the contact action) into the Hamiltonian action is defined by (2.5) in Section 2.2.

Moreover, these isomorphisms are natural in the sense that they commute with the "inclusion" and monotone continuation maps. To be more precise, for any $\tau' \leq \tau \leq a$

and two semi-admissible Hamiltonians $H' \leq H$, the diagrams

$$\begin{array}{ccc} \mathrm{SH}^{\tau'}(\alpha) & \xrightarrow{\Phi_H^{\tau'}} & \mathrm{HF}^{\mathfrak{a}_H(\tau')}(H) \\ \big\downarrow & & \big\downarrow \\ \mathrm{SH}^{\tau}(\alpha) & \xrightarrow{\Phi_H^{\tau}} & \mathrm{HF}^{\mathfrak{a}_H(\tau)}(H) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{SH}^{\tau}(\alpha) & \xrightarrow{\Phi_{H'}^{\tau}} & \mathrm{HF}^{\mathfrak{a}_{H'}(\tau)}(H') \\ \big\downarrow{\scriptstyle id} & & \big\downarrow \\ \mathrm{SH}^{\tau}(\alpha) & \xrightarrow{\Phi_H^{\tau}} & \mathrm{HF}^{\mathfrak{a}_H(\tau)}(H) \end{array}$$

commute, where the vertical arrows are the "inclusion" maps in the first diagram and the right vertical arrow is the monotone continuation map in the second.

The first consequence of the theorem, central to this paper, is the fact that the filtered symplectic homology defined as above is a persistence module. (This does not directly follow the definition.)

Corollary 3.6. *The family of vector spaces $\tau \mapsto \mathrm{SH}^{\tau}(\alpha)$ is a persistence module in the sense of Section 2.1 with structure maps defined as the direct limit of the "inclusion" maps (3.5).*

Proof. Conditions (i), (ii) and (iv) from Section 2.1 are obviously satisfied with $s_0 = 0$ and $\mathcal{S} := \mathcal{S}(\alpha) \cup \{0\}$. Left-semicontinuity, (iii), follows from the first commutative diagram together with the facts that the function $\mathfrak{a}_H$ is continuous and $t \mapsto \mathrm{HF}^t(H)$ is a persistence module. ☐

The second consequence of the theorem is that to obtain the filtered symplectic homology for a finite range of action it suffices to take a semi-admissible Hamiltonian with an appropriate slope without passing to the limit; cf. [Vi]. Indeed, setting $\tau = \infty$ or just $\tau > \sup \mathcal{S}(H)$ we have the following.

Corollary 3.7. *For any semi-admissible Hamiltonian H with slope$(H) = a$,*

$$\mathrm{SH}^{a}(\alpha) \cong \mathrm{HF}(H).$$

Moreover, whenever $H' \leq H$ are semi-admissible with $a' =$ slope(H') and $a =$ slope(H), the diagram

$$\begin{array}{ccc} \mathrm{SH}^{a'}(\alpha) & \xrightarrow{\cong} & \mathrm{HF}(H') \\ \big\downarrow & & \big\downarrow \\ \mathrm{SH}^{a}(\alpha) & \xrightarrow{\cong} & \mathrm{HF}(H) \end{array}$$

commutes, where the left vertical arrow is the structure or "inclusion" map and the right vertical arrow is the continuation map.

With the corollaries stated, we conclude this section by proving Lemma 3.4 and Theorem 3.5.

Proof of Lemma 3.4. The lemma is essentially a consequence of Remark 3.3 and the definitions. When $\tau \leq 0$, the statement follows immediately from (3.7) and (3.11). Ditto for $\tau = \infty$. Hence, we will assume that $0 < \tau < \infty$ throughout the rest of the proof.

Clearly, in (3.13) we can replace the direct limit as $s \to \infty$ by the direct limit over any monotone increasing sequence $s_i \to \infty$, and the limit is independent of this sequence. Fix such a sequence and any monotone decreasing sequence $\epsilon_i \to 0^+$. Then $H_i := s_i H - \epsilon_i$ is a cofinal sequence, which we can use in (3.10). On the other hand,

$$\mathrm{HF}^\tau(H_i) = \mathrm{HF}^{\tau-\epsilon_i}(s_i H).$$

Assume first that $\tau \notin \mathcal{S}(\alpha)$. Then $\tau - \epsilon_i \notin \mathcal{S}(\alpha)$ for all large i and, by (3.12), $\tau - \epsilon_i$ and τ are in the same connected component of the complement to $\mathcal{S}(s_i H) \to \mathcal{S}(\alpha) \cup \{0\}$. (This is the point where it is essential that $\tau \neq 0$.) Therefore,

$$\mathrm{HF}^{\tau-\epsilon_i}(s_i H) = \mathrm{HF}^\tau(s_i H)$$

due to (3.8), and the statement again follows by passing to the limit as $i \to \infty$. It is essential for the next step that here the sequences $s_i \to \infty$ and $\epsilon_i \to 0^+$ are arbitrary.

Now, assume that possibly $\tau \in \mathcal{S}(\alpha)$. Pick a sequence $s_i \to \infty$ so that $\tau \notin \mathcal{S}(s_i H)$. Then

$$\mathrm{HF}^\tau(s_i H) = \mathrm{HF}^t(s_i H)$$

for all $t \in [\tau - \delta_i,\, \tau + \delta_i]$ and for some $\delta_i > 0$ depending on i. Next, chose a sequence $\epsilon_i \to 0^+$ such that $\epsilon_i < \delta_i$. Then, by (3.8),

$$\mathrm{HF}^\tau(s_i H) = \mathrm{HF}^{\tau-\epsilon_i}(s_i H) = \mathrm{HF}^\tau(H_i)$$

for all large i. Passing to the limit as $i \to \infty$, we obtain (3.13). ☐

Proof of Theorem 3.5. Let $H_0 \leq H_1$ be two semi-admissible Hamiltonians. For the sake of simplicity we will assume that they have the same $r_{\max}$. (This assumption is not essential.) Consider the function

$$f = f_{H_0,H_1} := \mathfrak{a}_{H_1} \circ \mathfrak{a}_{H_0}^{-1} \colon [0,\, A_{H_0}(r_{\max})] \to [0,\, A_{H_1}(r_{\max})].$$

The function f is monotone as a composition of two monotone increasing functions and gives rise to a one-to-one correspondence between the action spectra as long as the target is in the range of f. Furthermore, $f(\tau) \leq \tau$ for all τ; see [ÇG²M]. The proof of the theorem hinges on the following result.

Lemma 3.8 (Prop. 3.1, [ÇG²M])**.** *For all $\tau < A_{H_0}(r_{\max})$, there are isomorphisms of the Floer homology groups*

$$\mathrm{HF}^\tau(H_0) \xrightarrow{\cong} \mathrm{HF}^{f(\tau)}(H_1). \tag{3.14}$$

These isomorphisms are natural in the sense that they commute with "inclusion" maps and monotone homotopies.

Set $H_0 = H$ and $H_1 = sH$ for $s \geq 1$ and $f_s := f_{H,sH}$. Then, for any $\tau \leq a$ we have isomorphisms of the Floer homology groups

$$\mathrm{HF}^{\tau}(H) \xrightarrow{\cong} \mathrm{HF}^{f_s(\tau)}(sH).$$

It is not hard to see from (2.8) that $f_s(\tau) \to \mathfrak{a}_H^{-1}(\tau)$ as $s \to \infty$. Passing to the limit as $s \to \infty$ and applying Lemma 3.4, we obtain the inverse of the desired isomorphism $\Phi_H^{\mathfrak{a}_H^{-1}(\tau)}$. Naturality of these isomorphisms readily follows from that the isomorphisms (3.14) are natural. ☐

4. Barcode entropy revisited

While Definition 1.1 is simple and intuitive, it is not very convenient to work with; for it is not directly connected to the dynamics of the Reeb flow or the Hamiltonian flow of a (semi-)admissible Hamiltonian. Nor is it obviously related to the Floer homology of such Hamiltonians. In this section we rephrase the definition of barcode entropy in several different ways to remedy this shortcoming. While completely self-contained, the discussion below has substantial overlaps with [FLS], although our treatment of the question is more brief. We start with some simple algebraic observations.

4.1. **Barcode entropy of a persistence module.** The definition of barcode entropy of Reeb flows extends to general persistence modules in a straightforward way. Namely, let $V = \{V_s\}$ be a persistence module. Denote by $\mathfrak{b}_\epsilon(s, V)$ or just $\mathfrak{b}_\epsilon(s)$ the number of bars of length greater than $\epsilon > 0$ and beginning below s. The *barcode entropy* of V is then defined as

$$\hbar_\epsilon(V) = \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(s, V)}{s} \quad \text{and} \quad \hbar(V) = \lim_{\epsilon \searrow 0} \hbar_\epsilon(V). \tag{4.1}$$

We say that a persistence module $W = \{W_s\}$ is a *reparametrization* of V if $W_s = V_{\xi(s)}$, where the function $\xi\colon \mathbb{R} \to \mathbb{R}$ is continuous, strictly monotone increasing and onto. (The structure maps in the persistence module W come from the structure maps in V.) It is not hard to see that

$$\mathfrak{b}_{c\epsilon}(s, W) \leq \mathfrak{b}_\epsilon\big(\xi(s), V\big) \tag{4.2}$$

when ξ^{-1} is Lipschitz with (global) Lipschitz constant c or, equivalently,

$$\mathfrak{b}_{c'\epsilon}\big(\xi(s), V\big) \leq \mathfrak{b}_\epsilon(s, W),$$

where c' is the (global) Lipschitz constant of ξ. As a consequence,

$$\hbar(W) = a\hbar(V) \tag{4.3}$$

whenever ξ is bi-Lipschitz, i.e., both ξ and ξ^{-1} are Lipschitz on $\mathbb{R}$, and $a = \lim \xi(s)/s$ as $s \to \infty$ assuming that the limit exists. For instance, (4.3) holds when $\xi(s) = as$ for some $a > 0$.

Next, given a persistence module V, let us define a family of persistence modules $V(s)$, $s \in \mathbb{R}$, by truncating V at s, i.e., $V(s)_\tau = V_\tau$ when $\tau \leq s$ and $V(s)_\tau = V_s$ when $\tau \geq s$. (Warning: this is not the standard notion of truncation; cf. Example

4.2.) In other words, the finite bars in V ending below s give rise to the finite bars of $V(s)$, the infinite bars or finite bars of V containing s become the infinite bars of $V(s)$, and the other bars disappear. In particular, all bars in $V(s)$ begin in the interval $[0,\, s)$.

Then, by analogy with the barcode entropy of a Hamiltonian diffeomorphism (see [ÇGG21]), we associate to V the *dynamics barcode entropy* as follows. Set $\mathfrak{b}_\epsilon^{Dyn}(s,V)$ to be the number of bars of length greater than $\epsilon > 0$ beginning below $s-\epsilon$ in the barcode of $V(s)$. Thus $\mathfrak{b}_\epsilon^{Dyn}(s,V) = \mathfrak{b}_\epsilon(s-\epsilon, V)$ is almost to the total number of bars longer than ϵ in $V(s)$, up to an error coming from the bars beginning in $[s-\epsilon,\, s)$. Then the dynamics barcode entropy of V is defined by

$$\hbar_\epsilon^{Dyn}(V) = \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon^{Dyn}(s,V)}{s} \quad \text{and} \quad \hbar^{Dyn}(V) = \lim_{\epsilon \searrow 0} \hbar_\epsilon^{Dyn}(V). \tag{4.4}$$

Remark 4.1. The reason that the bars are required to begin below $s-\epsilon$ is that the bars of V containing s, no matter how short, give rise to infinite bars in $V(s)$. The results from [As, Kal] indicate that in the Hamiltonian case the number of such bars can grow arbitrarily fast for some sequence $s_k \to \infty$ without any clear relation to the topological entropy of the underlying system. The requirement on the beginning of the bars keeps such bars from affecting the count.

Example 4.2. In the setting we are interested in, V is the persistence module $\mathrm{SH}^s(\alpha)$, and, by Corollary 3.7, the truncated module $V(sa)$ is isomorphic to a reparametrization of the Floer homology persistence module $\tau \mapsto \mathrm{HF}^\tau(sH)$ when H is semi-admissible with $a = \mathit{slope}\, H$. This example motivates the choice of our rather uncommon truncation procedure. An alternative would be a more standard variant of truncation where $V(s)_\tau := 0$ for $\tau > s$. This variant would also be suitable for our purposes and adopting it we would count the bars longer than ϵ beginning below s (rather than $s-\epsilon$), although the version we use is more intuitive from the Floer theoretic perspective.

Note that the dynamics barcode entropy can be defined for any family of persistence modules. For the family of truncated persistence modules $V(s)$, the dynamics barcode entropy coincides with the ordinary barcode entropy as the following formal and nearly obvious proposition shows.

Proposition 4.3. *For any persistence module V and any $\epsilon > 0$, we have*

$$\hbar_\epsilon(V) = \hbar_\epsilon^{Dyn}(V) \quad \textit{and hence} \quad \hbar(V) = \hbar^{Dyn}(V). \tag{4.5}$$

Furthermore, in (4.1) and (4.4), replacing the upper limits as $s \to \infty$ by the upper limit over any monotone increasing sequence $s_k \to \infty$ such that $s_{k+1}/s_k \to 1$ does not affect the definitions and (4.5) holds already on the level of ϵ-entropy.

Hence, in what follows, we need not distinguish between these two types of barcode entropy and will use the notation $\hbar$ for both of them.

Proof. By definition, for any $s \geq \epsilon > 0$, we have

$$\mathfrak{b}_\epsilon(s-\epsilon) = \mathfrak{b}_\epsilon^{Dyn}(s) \leq \mathfrak{b}_\epsilon(s),$$

where we suppressed V in the notation, and (4.5) follows. To prove the moreover part, we focus on (4.1). Clearly,

$$\limsup_{k\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(s_k)}{s_k} \le \hbar_\epsilon$$

for any sequence $s_k \to \infty$ and we only need to prove the opposite inequality.

Let the sequence s_k be as in the proposition. Let $t_i \to \infty$ be such that

$$\lim_{i\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(t_i)}{t_i} = \hbar_\epsilon.$$

For every i, pick k_i so that $s_{k_i} \le t_i < s_{k_i+1}$. Then

$$\frac{s_{k_i+1}}{s_{k_i}} \frac{\log^+ \mathfrak{b}_\epsilon(s_{k_i+1})}{s_{k_i+1}} = \frac{\log^+ \mathfrak{b}_\epsilon(s_{k_i+1})}{s_{k_i}} \ge \frac{\log^+ \mathfrak{b}_\epsilon(t_i)}{t_i}.$$

Passing to the (upper) limit and using the fact that $s_{k+1}/s_k \to 1$, we see that

$$\limsup_{k\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(s_k)}{s_k} \ge \hbar_\epsilon,$$

which concludes the proof of the proposition. □

4.2. **Barcode entropy via Floer homology.** Let us now apply the observations from the previous section to Floer homology. Let H be a semi-admissible Hamiltonian with slope a. As in Section 3.2, consider the persistence module $s \mapsto V_s$ formed by the Floer homology spaces $V_s := \mathrm{HF}(sH)$ along with the continuation maps $\mathrm{HF}(s'H) \to \mathrm{HF}(sH)$ when $s' \le s$. By Corollary 3.7, V is isomorphic to the symplectic homology persistence module $\mathrm{SH}^{as}(\alpha)$. Hence, the number $\mathfrak{b}_\epsilon^{Fl}(s, H)$ of bars of length greater than $\epsilon > 0$ beginning below s in the barcode of V is equal to $\mathfrak{b}_\epsilon(as)$. As a consequence,

$$\hbar_\epsilon(\alpha) = \frac{1}{a} \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon^{Fl}(s, H)}{s}. \tag{4.6}$$

From now on we will not distinguish the persistence modules $\mathrm{SH}^{as}(\alpha)$ and $\mathrm{HF}(sH)$.

Alternatively, as has been pointed out in Section 3.2, for a fixed Hamiltonian H with slope a, we can view the filtered Floer homology $\tau \mapsto \mathrm{HF}^\tau(H)$ as a persistence module. Note that all bars in this persistence module begin below $\mathfrak{a}_H(a)$. Let as in Section 4.1 $\mathfrak{b}_\epsilon^{Dyn}(H)$ be the number of bars of length greater than $\epsilon > 0$ in its barcode beginning below $\mathfrak{a}_H(a) - \epsilon$. Then, analogously to the definition of barcode entropy for Hamiltonian diffeomorphisms in [ÇGG21], we set the *dynamics barcode entropy* of H to be

$$\hbar_\epsilon^{Dyn}(H) = \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon^{Dyn}(sH)}{s}, \tag{4.7}$$

and

$$\hbar^{Dyn}(H) = \lim_{\epsilon\searrow 0} \hbar_\epsilon^{Dyn}(H).$$

As readily follows from the definition,

$$\hbar_\epsilon^{Dyn}(cH) = c \cdot \hbar_\epsilon^{Dyn}(H) \quad \text{and} \quad \hbar^{Dyn}(cH) = c \cdot \hbar^{Dyn}(H)$$

for every $c > 0$.

As in Section 4.1, denote by $V(s)$ be the persistence module obtained from the persistence module $s \mapsto V_s := \mathrm{SH}^{as}(\alpha) = \mathrm{HF}(sH)$ by truncating at s. It is clear that $V(s)_\infty = \mathrm{HF}(sH)$. However, as persistence modules, $V(s)$ and $\mathrm{HF}(sH)$ are in general different: $V(s)_\tau \neq \mathrm{HF}^\tau(sH)$. Yet the two families of persistence modules have the same barcode entropy as the following observation along the lines of Proposition 4.3 asserts.

Theorem 4.4. *For any semi-admissible Hamiltonian H with slope a, we have*

$$\hbar(\alpha) = \frac{\hbar^{Dyn}(H)}{a}. \tag{4.8}$$

Furthermore, in (4.7), replacing the upper limits as $s \to \infty$ by the upper limit over any monotone increasing sequence $s_k \to \infty$ such that $s_{k+1}/s_k \to 1$ does not affect the definition and hence (4.8).

Due to this theorem we need not distinguish the barcode entropies $\hbar$ and $\hbar^{Dyn}$. Note however that in contrast with Proposition 4.3 or (4.6) we do not claim here the equality on the level of ϵ-barcode entropy, and we believe that $\hbar_\epsilon(\alpha) \neq \hbar_\epsilon^{Dyn}(H)/a$ in general.

Remark 4.5 (On the proof of Theorem A)*.* Assume that $ka \notin \mathcal{S}(\alpha)$ for all $k \in \mathbb{N}$. Then taking $s_k = k$ in (4.7) we arrive at the definition of barcode entropy for a semi-admissible Hamiltonian H which is completely analogous to the definition of the barcode entropy for compactly supported Hamiltonian diffeomorphisms since $\varphi_{kH} = \varphi_H^k$. Furthermore, the proof of [ÇGG21, Thm. A] carries over word-for-word to semi-admissible Hamiltonians and, as a consequence, we arrive at Theorem A in a way somewhat different from [FLS]. To be more specific, in this variant of the proof the tomograph construction is to be applied to the shell $U = \{1 \leq r \leq r_{\max}\}$ resulting in the inequality

$$\hbar^{Dyn}(H) \leq \mathrm{h_{top}}\big(\varphi_H|_U\big) = a\, \mathrm{h_{top}}(\alpha).$$

Proof of Theorem 4.4. By Theorem 3.5, for any semi-admissible Hamiltonian H with slope a there exists a natural isomorphism of vector spaces

$$\mathrm{HF}^{\mathfrak{a}_H(\tau)}(H) \to \mathrm{SH}^\tau(\alpha)$$

as long as $\tau \leq a$, and the function $\mathfrak{a}$ is bi-Lipschitz with $1 \leq \mathfrak{a}'_H \leq r_{\max}$. Hence, we also have isomorphisms for the family of persistence modules

$$\mathrm{HF}^{\mathfrak{a}_{sH}(\tau)}(sH) \to V(s)_\tau,$$

where as above $V(s)$ is the persistence module $V_s := \mathrm{SH}^{as}(\alpha) = \mathrm{HF}(sH)$ truncated at s. The reparametrizations $\mathfrak{a}_{sH}$ are bi-Lipschitz uniformly in s: $1 \leq \mathfrak{a}'_{sH} \leq r_{\max}$ for all $s > 0$ by (2.6).

Therefore, similarly to (4.2), we have

$$\mathfrak{b}_\epsilon^{Dyn}(sH) \geq \mathfrak{b}_\epsilon\big(V(s)\big) \geq \mathfrak{b}_{r_{\max}\epsilon}^{Dyn}(sH).$$

Passing to the upper limit as $s \to \infty$ and then to the limit as $\epsilon \to 0$, we see that $\hbar^{Dyn}(H) = \hbar^{Dyn}(V)$. Now (4.8) follows from Proposition 4.3 and (4.3). The proof of the "Furthermore" part is nearly identical to its counterpart in the proof of Proposition 4.3 and we omit it. □

5. Crossing energy

The key new ingredient of the proofs is the Crossing Energy Theorem (Theorem 5.1). In this section we state the Crossing Energy Theorem, which is proved in Section 6, and use it to establish Theorem B.

5.1. **Crossing Energy Theorem.** Recall that a compact invariant set K of the Reeb flow is said to be *locally maximal* or isolated if there exists a neighborhood $U \supset K$ such that every invariant set contained in U must be a subset of K or, equivalently, for every $x \in U \setminus K$ the integral curve through x is not entirely contained in U. We call U an *isolating neighborhood*. For instance, a hyperbolic periodic orbit is locally maximal. In general, a periodic orbit can be isolated as a periodic orbit but not as an invariant set. We refer the reader to [KH, Sect. 17.4] or [FH] for the definition of a hyperbolic invariant set.

Theorem 5.1 (Crossing Energy Theorem, I)**.** *Let K be a hyperbolic, locally maximal compact invariant set of the Reeb flow of α. Fix an interval*

$$I = [r_-, r_+] \subset (1, r_{\max})$$

and let $H(r, x) = h(r)$ be a semi-admissible Hamiltonian with slope$(H) =: a \notin \mathcal{S}(\alpha)$ such that

$$h''' \geq 0 \text{ on } [1, r_+ + \delta] \tag{5.1}$$

for some $\delta > 0$ with $r_+ + \delta < r_{\max}$. Fix an admissible almost complex structure J. Furthermore, let z be a T-periodic orbit of α in K and $\tilde{z} := (z, r^*)$ be the corresponding 1-periodic orbit of the flow of sH. (Hence, $sa \geq T$ and r^* depends on s.) Assume that $r^* \in I$.

Then there exists $\sigma > 0$ such that $E(u) \geq \sigma$, independent of s and z, for any Floer cylinder $u \colon \mathbb{R} \times S^1 \to \widehat{W}$ of sH asymptotic, at either end, to $\tilde{z}$.

This theorem is a partial generalization of [ÇG²M, Thm. 4.1] where K is just one locally maximal periodic orbit of the Reeb flow. A remark is due regarding the statement of Theorem 5.1 which we will prove in Section 6.

Remark 5.2. The point that σ is independent of s is crucial for our purposes. Moreover, without it, for a fixed s the theorem would follow immediately from a suitable variant of Gromov compactness theorem under no assumptions on K other than that for every T all T-periodic orbits in K are isolated.

Secondly, fix $0 < \sigma' < \sigma$ and $s \geq 0$. Consider a C^∞-small s-periodic in time, non-degenerate perturbation $\tilde{H}$ of sH and a C^∞-small compactly supported generic s-periodic perturbation $\tilde{J}$ of J. The 1-periodic orbit $\tilde{z}$ of sH from Theorem 5.1 splits into several non-degenerate periodic orbits of $\tilde{H}$ contained in a small tubular neighborhood of $\tilde{z}$. It follows again from a suitable version of the Gromov compactness theorem (see, e.g., [Fi]) that every Floer cylinder of $\tilde{H}$ asymptotic to any of these orbits at either end has energy greater than σ'.

5.2. **Proof of Theorem B.** The proof comprises four steps.

Step 1: The Hamiltonian. Throughout the proof, we treat the barcode entropy of α in the sense of Theorem 4.4. Thus, let $H(x, r) = h(r)$ be a semi-admissible Hamiltonian. Without loss of generality, we may assume that $a := \mathit{slope}(H) = 1$.

We will require in addition that (5.1) is satisfied and $a = 1 \notin \mathcal{S}(\alpha)$ so that we can apply Theorem 5.1 to H. Furthermore, we can make $h'(r_+) < 1$ arbitrarily close to 1. To be more precise, it is not hard to show that for any $\eta > 0$, there exists a semi-admissible Hamiltonian H with $a = 1$ such that (5.1) holds and

$$1 - \eta \leq h'(r_+).$$

Recall that $\mathfrak{b}_\epsilon^{Dyn}(sH)$ is the number of bars of length greater than $\epsilon > 0$ beginning below $s - \epsilon$ in the barcode of $V_\tau := \mathrm{HF}^\tau(sH)$; see Section 4.2. By Theorem 4.4 and in particular (4.8), we have

$$\hbar(\alpha) = \lim_{\epsilon \to 0^+} \limsup_{s \to \infty} \frac{\log^+ \mathfrak{b}_\epsilon^{Dyn}(sH)}{s}.$$

Here we may assume that $s \notin \mathcal{S}(\alpha)$, i.e., $\mathit{slope}(sH) \notin \mathcal{S}(\alpha)$ since $a = 1$, and hence $\mathrm{HF}(sH)$ is defined directly, without passing to the limit.

Step 2: Reduction to locally maximal sets. Recall that K is a compact hyperbolic invariant set of the Reeb flow φ_α^t. The goal of this step is to show that without loss of generality we can assume that K is locally maximal in addition to being hyperbolic. (Here we closely follow the first step in the proofs of [GGM, Thm. B] and [ÇGG21, Thm. B].) Note first that, by the variational principle for topological entropy, [FH, Cor. 4.3.9], for every $\delta > 0$ there exists an invariant probability measure μ supported in K such that $h_\mu \geq \mathrm{h_{top}}(K) - \delta$, where h_μ is the metric entropy of the Reeb flow $\varphi_\alpha^t|_K$. Moreover, the measure μ can be chosen ergodic; see [Wa, Thm. 8.4]. Then, by [LY, Thm. D'] extending [KH, Thm. S.5.9.(1)] to flows, whenever $h_\mu > 0$, there exists a locally maximal hyperbolic set K contained in a neighborhood of $\operatorname{supp} \mu$, and hence in a neighborhood of K, such that

$$\mathrm{h_{top}}(K') \geq h_\mu - \delta \geq \mathrm{h_{top}}(K) - 2\delta.$$

Since $\delta > 0$ is arbitrary, replacing K by K' we may assume that K is locally maximal and hyperbolic. Now Theorem 5.1 can be applied to H and K, which we will do in the last step of the proof.

Step 3: Periodic orbits in K with action constraint. Denote by $p(s)$ the number of periodic orbits of the flow $\varphi_\alpha^t|_K$ with period $T \leq s$. By [FH, Thm. 5.4.22], we have

$$\mathrm{h_{top}}(K) = \limsup_{s \to \infty} \frac{\log^+ p(s)}{s} =: L$$

since K is hyperbolic. Furthermore, fix r_- in the range $(1, r^+)$ and set $I' = (r_-, r_+]$. Recall that in Theorem 5.1, $I = [r_-, r_+]$, and hence $I' \subset I$.

Let $p^H(s)$ be the number of s-periodic orbits $\tilde{z} = (z, r)$ of the flow φ_H^t with z in K and $r \in I'$. Equivalently, $p^H(s)$ is the number of non-constant 1-periodic orbits $\tilde{z}$ of the flow of sH with $r \in I'$. The goal of this step is to show that

$$h'(r_+) \cdot L \leq \limsup_{s\to\infty} \frac{\log^+ p^H(s)}{s} \leq L. \tag{5.2}$$

In fact, we will just need the first inequality, and the nearly obvious second one is included only for the sake of completeness.

Note that since $a = 1$ and $X_H = h'(r)R$, where R is the Reeb vector field, there is a one-to-one correspondence between closed Reeb orbits z in K with period $T \leq s$ and s-periodic orbits $\tilde{z} = (z, r)$ of the flow of H. In particular, $p^H(s) \leq p(s)$ and the second inequality follows.

Let $\tilde{z} = (z, r^*)$ be an s-periodic orbit of the flow of H and T be the period of z as an orbit of the Reeb flow. Then, as in (2.2), T and r^* are related by the condition

$$sh'(r^*) = T.$$

Set

$$a_- := h'(r_-) \text{ and } a_+ := h'(r_+).$$

Then, for r^* to be in $I' = (r_-, r_+]$ we must have

$$a_- s \leq T < a_+ s,$$

and thus

$$p^H(s) = p(a_+ s) - p(a_- s). \tag{5.3}$$

(The reason that we took I' to be a semi-open interval rather than the closed interval I is to ensure that this equality holds literally.)

Clearly,

$$\begin{aligned} p(a_+ s) &= \big(p(a_+ s) - p(a_- s)\big) + p(a_- s) \\ &\leq \max\big\{2\big(p(a_+ s) - p(a_- s)\big),\, 2p(a_- s)\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_+ \cdot L &= \limsup_{s\to\infty} \frac{\log^+ p(a_+ s)}{s} \\ &\leq \limsup_{s\to\infty} \frac{1}{s} \log^+ \max\big\{2\big(p(a_+ s) - p(a_- s)\big),\, 2p(a_- s)\big\} \\ &= \limsup_{s\to\infty} \frac{1}{s} \max\Big\{ \log^+ \Big(2\big(p(a_+ s) - p(a_- s)\big)\Big),\, \log^+ \big(2p(a_- s)\big)\Big\} \\ &= \max\left\{ \limsup_{s\to\infty} \frac{\log^+ \Big(2\big(p(a_+ s) - p(a_- s)\big)\Big)}{s},\, \limsup_{s\to\infty} \frac{\log^+ \big(2p(a_- s)\big)}{s} \right\} \\ &= \max\left\{ \limsup_{s\to\infty} \frac{\log^+ p^H(s)}{s},\, a_- \cdot L \right\}. \end{aligned}$$

Here, in the last equality, we have used (5.3). The second term in the last line is strictly smaller than a_+L since $a_- < a_+$, and hence the first term must be greater than on equal to a_+L. This proves proves the first inequality in (5.2).

Step 4: Punchline. Let, as in Remark 5.2, $(G, \tilde{J})$ be a compactly supported, regular C^∞-small perturbation of the pair (sH, J). Under this perturbation, every non-constant non-degenerate s-periodic orbit $\tilde{z} = (z, r^*)$ of the flow of H splits into at least two 1-periodic orbits of G. Furthermore, by Theorem 5.1 and again Remark 5.2, whenever $r^* \in I = [r_-, r_+]$ a Floer cylinder asymptotic to any of these orbits of G at either end has energy greater than some constant $\sigma' > 0$ which is independent of s and $\tilde{z}$. (However, the upper bound on the size $\|sH - G\|_{C^\infty}$ of the perturbation may depend on s.) Note also that $a_+s < s - \epsilon$ when s is sufficiently large, and hence all such orbits have action below $s - \epsilon$.

Assume now that $2\epsilon < \sigma'$. It follows from, e.g., [ÇGG21, Prop. 3.8] that the Floer persistence module $\mathrm{HF}^\tau(G)$ has at least $p^H(s)/2$ bars of length greater than 2ϵ. Since sH and G are C^∞-close, the same is true for the Floer persistence module $\mathrm{HF}^\tau(sH)$ with 2ϵ replaced by ϵ. (Here, the perturbation G is chosen after ϵ is fixed.). In other words,

$$\mathfrak{b}_\epsilon^{Dyn}(sH) \geq p^H(s)/2.$$

By (4.8) and (5.2),

$$\begin{aligned}\hbar(\alpha) &\geq \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon^{Dyn}(sH)}{s} \\ &\geq \limsup_{s\to\infty} \frac{\log^+ p^H(s)}{s} \\ &\geq h'(r_+) \cdot \mathrm{h_{top}}(K) \\ &\geq (1-\eta)\, \mathrm{h_{top}}(K).\end{aligned}$$

Thus

$$\hbar(\alpha) \geq (1-\eta)\, \mathrm{h_{top}}(K).$$

As was pointed out in Step 1, we can take $\eta > 0$ arbitrarily close to 0. It follows that

$$\hbar(\alpha) \geq \mathrm{h_{top}}(K),$$

which concludes the proof of Theorem B. □

6. Proof of the Crossing Energy Theorem

The goal of this section is to prove the Crossing Energy Theorem – Theorem 5.1. The proof follows the same general line of reasoning as several other arguments of this type. The key new ingredient which makes the proof work for Reeb flows is a location constraint theorem from [ÇG²M]. We state this result in the next section.

6.1. **Refinement and location constraints.** We start this section with a refinement of Theorem 5.1, which better reflects the logical structure of the proof.

In what follows, the Hamiltonian H is assumed to be semi-admissible and J is admissible. In particular, H and J are independent of time. We will also assume that all Floer cylinders u we consider have sufficiently small energy, and hence, by monotonicity, are contained $M \times (1-\eta, \infty)$ for some small $\eta > 0$; see Section 3.1. In particular, the projection of u to M is defined.

Theorem 6.1 (Crossing Energy Theorem, II). *Let K be a compact invariant set of the Reeb flow. Fix an admissible almost complex structure J and an interval*

$$I = [r_-, r_+] \subset (1, r_{\max}).$$

Let $H(r,x) = h(r)$ be a semi-admissible Hamiltonian with slope $a \notin \mathcal{S}(\alpha)$ such that (5.1) is satisfied for some $\delta > 0$ with $r_+ + \delta < r_{\max}$. Let $\tau > 0$ and $u\colon \mathbb{R} \times S^1 \to \widehat{W}$ be a Floer cylinder for τH asymptotic, at either end, to a 1-periodic orbit in $K \times I$.

(i) *Assume that K is locally maximal and let U be an isolating neighborhood of K. Then*
$$E(u) \geq \sigma \tag{6.1}$$
for some constant $\sigma > 0$ independent of τ and u, whenever u is not entirely contained in $\widehat{U} := U \times [1, \infty) \subset \widehat{W}$.

(ii) *Assume that K is hyperbolic and $U \setminus K$ contains no periodic orbits of the Reeb flow for some neighborhood U of K. Then, when U is sufficiently small and u is entirely contained in $\widehat{U}$, the energy lower bound (6.1) holds again with $\sigma > 0$ independent of τ and u.*

Theorem 5.1 readily follows from Theorem 6.1, for the requirements of both parts of the theorem are satisfied when K is locally maximal and hyperbolic. Part (i) of Theorem 6.1 is already sufficient for many purposes. For instance, when K comprises just one locally maximal periodic orbit, Part (ii) is void and Part (i) generalizes [ÇG^{2}M, Thm. 4.1]. For a fixed τ, the lower bound, (6.1), is again stable under small perturbations of H and J as in Remark 5.2. Note also that the parameter s in sH is renamed here as τ since s serves as the $\mathbb{R}$-coordinate in the domain $\mathbb{R} \times S^1$ of a Floer cylinder u in the proof of Theorem 6.1.

The central component of the proof of the Crossing Energy Theorem is the following result which, under a minor additional condition on H, along the lines of (5.1), limits the range of $r \circ u$.

Theorem 6.2 (Location Constraint – Thm. 6.1, [ÇG^{2}M]). *Let $H(r,x) = h(r)$ be a semi-admissible Hamiltonian. Assume that $1 < r_*^- \leq r_*^+$ and $\delta > 0$ are such that*

$$1 < r_*^- - \delta \text{ and } r_*^+ + \delta \leq r_{\max},$$

and

$$h''' \geq 0 \text{ on } [1, r_*^+ + \delta] \subset [1, r_{\max}). \tag{6.2}$$

Fix an admissible almost complex structure J. Then there exists $\sigma_0 > 0$ such that for any $\tau > 0$ and any Floer cylinder $u\colon \mathbb{R} \times S^1 \to \widehat{W}$ for τH with energy $E(u) \leq \sigma_0$

and asymptotic, at either end, to a periodic orbit in $M \times [r_*^-, r_*^+]$*, the image of* u *is contained in* $M \times (r_*^- - \delta, r_*^+ + \delta)$.

In other words, a small energy Floer cylinder for τH asymptotic at either end to a periodic orbit in the shell $M \times [r_*^-, r_*^+]$ must be entirely contained in a slightly larger shell $M \times (r_*^- - \delta, r_*^+ + \delta)$. The key non-trivial part of this theorem is the lower bound $r_*^- - \delta$ and this is the part we will actually use. The upper bound is quite standard and we included it only for the sake of completeness. We refer the reader to [ÇG^{2}M] for the proof of the theorem. (Strictly speaking Theorem 6.2 is proved there for $\tau \in \mathbb{N}$. However, the argument carries over word-for-word to the case of $\tau \in (0, \infty)$.)

Remark 6.3. As in Remark 5.2, by the target compactness theorem from [Fi], the assertion of the theorem still holds with perhaps a smaller value of σ_0 when sH and J are replaced by their compactly supported, τ-periodic in time C^∞-small perturbations. (The size of the perturbation may depend on s.) We also note that in Theorems 5.1, 6.1 and 6.2, we could have required H to be admissible rather than semi-admissible.

Remark 6.4. The analogue of Theorem 5.1 and Part (ii) of Theorem 6.1 for geodesic flows, [GGM, Thm. D], holds when the hyperbolicity condition is replaced by a weaker requirement that K is expansive; see, e.g., [KH, Def. 3.2.11] or [FH, Sec. 1.7 and 5.3] for the definition. It is conceivable that this is also true in the present setting although the proof from [GGM] does not extend to general Reeb flows. However, as is easy to see from the proof below and [FH, Sec. 5.3], in Part (ii) hyperbolicity can be replaced by expansivity and shadowing. See also [ÇG^{2}M, Thm. 4.1].

6.2. **Energy bounds – proof of Theorem 6.1.** There are several sufficiently different approaches to the proof of crossing energy type results. Historically, the first one was based on target Gromov compactness, [Fi], and used in the proof of the original Crossing Energy Theorem in [GG14] and then in [GG18, ÇGG21] and more recently and in a more sophisticated form in [CGP]. The second approach relies on the upper bound (6.3) below from, e.g., [Sa90, Sa99], and on the conceptual level is also closely related to (the proof of) Gromov compactness. This method is pointed out in [ÇGG21, Rmk. 6.4] and then developed in [ÇG^{2}M]; see also [Me24]. The third approach is technically quite different. It uses finite-dimensional approximations and is fundamentally based on a "classical" form of Morse theory and the existence of the gradient flow; see [Al, GGM]. This approach does not fit well in the general Floer theory setting, but either of the first two methods can be employed to prove Theorem 6.1. Here, following [ÇG^{2}M], we have chosen the second one, which is more hands-on and direct, albeit somewhat less general. The proof again comprises four steps.

Step 1: From energy to L^∞*-upper bounds.* Throughout the proof, it is convenient to adopt a different Hamiltonian iteration procedure. Namely, rather than looking at 1-periodic orbits of the Hamiltonian τH, we will look at the τ-periodic orbits of H, changing the time range from $S^1 = \mathbb{R}/\mathbb{Z}$ to $S^1_\tau = \mathbb{R}/\tau\mathbb{Z}$. We will refer to the resulting

Hamiltonian as $H^{\sharp\tau}$; cf. [GG14, GG18]. This modification does not affect the Floer complexes, the action and the action filtration, the energy of a Floer trajectory, etc., with an isomorphism given by an (s,t)-reparametrization.

With this in mind, by [Sa99, Sec. 1.5] and [Sa90], there exist constants $C_H > 0$ and $\sigma_H > 0$ such that, for any Floer cylinder $u\colon \mathbb{R}\times S^1_\tau \to \widehat{W}$ for $H^{\sharp\tau}$ with $E(u) < \sigma_H$, we have the point-wise upper bound

$$\big\|\partial_s u(s,t)\big\| < C_H \cdot E(u)^{1/4}, \tag{6.3}$$

where s is the coordinate on $\mathbb{R}$ and the norm on the left is L^∞; see also [Br]. The constants σ_H and C_H depend on H via its first and second derivatives and J, but not on τ or u. (This is one instance where it is more convenient to work with $H^{\sharp\tau}$ than τH; for the proof of (6.3) is local in the domain of u.)

Throughout the rest of the proof we will assume that $E(u) < \sigma_H$, and hence u satisfies (6.3).

Step 2: Pseudo-orbits. Recall that a map γ from an interval or a circle is an η-*pseudo-orbit* or just a *pseudo-orbit* of the flow of X if

$$\big\|\dot\gamma(t) - X\big(\gamma(t)\big)\big\| < \eta$$

for all t in the domain of γ; see [KH, Def. 18.1.5] or [FH]. Then, when $\eta > 0$ is small, γ is close to the integral curve of the flow through $x = \gamma(0)$. To be more precise, as is easy to see, whenever a closed interval $\mathcal{I}$ in the domain of γ is fixed and $\eta > 0$ is sufficiently small, $\gamma|_{\mathcal{I}}$ is pointwise close to the integral curve of the flow through $\gamma(0)$ on the same interval $\mathcal{I}$.

By (6.3) and the Floer equation, (3.3), Floer circles

$$u(s)\colon t \mapsto u(s,t)$$

are pseudo-orbits of the Hamiltonian flow of H with $\eta = C_H \cdot E(u)^{1/4}$, provided that $E(u) < \sigma_H$ as we have assumed. Hence, when $E(u)$ is small, $u(s)$ approximates the integral curve of the Hamiltonian flow through $u(0,0)$ over the interval $[-\tau/2,\, \tau/2]$.

The goal of this step is to translate this fact from the Hamiltonian flow of H to the Reeb flow of α by suitably reparametrizing $u(s)$.

Let u be a Floer cylinder for $H^{\sharp\tau}$ with $E(u) < \sigma_H$ such that $\inf r(u) \geq r_{\min}$ for some constant $r_{\min} > 1$. Note that, by the maximum principle, u is contained in the shell $M \times [r_{\min}, r_{\max}]$. Set $\epsilon := E(u)$ and $C := C_H$. Then, by (6.3),

$$\|\partial_s u - X_H\| < C\epsilon^{1/4}$$

for all $s \in \mathbb{R}$, where $X_H = h' R_\alpha$. (This is another point where it is more convenient to work with $H^{\sharp\tau}$ than with τH; for $X_{H^{\sharp\tau}} = X_H$ is independent of τ.) Denote by v the projection of u to M. Then we also have

$$\big\|\partial_t v - h'(r(u))R_\alpha\big\| < C\epsilon^{1/4}. \tag{6.4}$$

Fixing $s \in \mathbb{R}$, let us reparametrize the map $t \mapsto v(s,t)$ by using the change of variables $t = t(\xi)$ so that

$$t'(\xi) = \frac{1}{h'\big(r(u(s,t))\big)},$$

and set

$$\gamma(\xi) = \gamma_s(\xi) := v\big(s, t(\xi)\big). \tag{6.5}$$

Then γ is parametrized by the circle S^1_T with

$$T \geq \tau \min_{t \in S^1_\tau} h'\big(r(u(0,t))\big) \geq \tau h'(r_{\min}). \tag{6.6}$$

Furthermore, by (6.4) and Theorem 6.2,

$$\big\|\dot\gamma(\xi) - R_\alpha\big(\gamma(\xi)\big)\big\| < \frac{C\epsilon^{1/4}}{\min_{t \in S^1_\tau} h'\big(r(u(s,t))\big)} \leq \frac{C\epsilon^{1/4}}{h'(r_{\min})} =: \eta, \tag{6.7}$$

where the dot stands for the derivative with respect to ξ. (Note that in this inequality we have used Theorem 6.2 in a crucial way.) In other words, γ is an η-pseudo-orbit of the Reeb flow with η completely determined by ϵ and other auxiliary data, e.g., H, but independent of s and u.

Step 3: Proof of Part (i). Arguing by contradiction, assume that there exists a sequence $\tau_k \to \infty$ and a sequence of Floer cylinders $u_k \colon \mathbb{R} \times S^1_{\tau_k} \to \widehat{W}$ of $H^{\sharp\tau_k}$ satisfying the requirements of the theorem and such that

$$\epsilon_k := E(u_k) \to 0.$$

In particular, u_k is asymptotic to a τ_k-periodic orbit $\tilde z_k = (z_k, r^*_k)$ of the flow of H where z_k is a periodic orbit of the Reeb flow in K and $r^*_k \in I$. Moreover, u_k is not entirely contained in $\widehat{U} = U \times [1, \infty)$, where U is an isolating neighborhood of K. By shrinking U if necessary, we can guarantee that the closure $\bar U$ is a closed isolating neighborhood, i.e, $K \subset U$ is a maximal invariant set in $\bar U$, and that u_k is not entirely contained in $\bar U \times [1, \infty)$.

Let $\delta > 0$ be as in (5.1). Setting $r^+_* = r^-_* = r^*_k$ in Theorem 6.2, we have (6.2) satisfied. Without loss of generality, we may assume that $\epsilon_k < \sigma_0$ and hence that theorem applies to u_k. It follows that u_k is contained in the shell $M \times (r_- - \delta,\, r_+ + \delta)$, where

$$1 < r_{\min} := r_- - \delta < r_+ + \delta < r_{\max}.$$

Then u_k is also contained in a bigger shell $M \times [r_{\min},\, r_{\max}]$ independent of u_k.

As in Steps 1 and 2, we assume that $\epsilon_k < \sigma_H$ and hence (6.3) holds. Finally, we may also require all u_k to be asymptotic to $\tilde z_k$ at the same end, say, $+\infty$. (For $-\infty$ the argument is identical.)

Suppressing k in the notation, we set $u := u_k$ and $\tau := \tau_k$ and $\epsilon := \epsilon_k$, etc.

The requirement that u is not entirely contained in $\bar U \times [1, \infty)$ is equivalent to that v is not entirely contained in $\bar U$. Since u is asymptotic to $\tilde z$, for some $s_0 \in \mathbb{R}$ and $t_0 \in S^1_\tau$, the curve $v(s_0, S^1_\tau)$ is contained in $\bar U$ while $v(s_0, t_0) \in \partial U := \bar U \setminus U$. By translation and rotation invariance of the Floer equation, without loss of generality, we may assume that $s_0 = 0$ and $t_0 = 0$. Thus, $v(0,t) \in \bar{U}$ for all $t \in S^1_\tau$ and $x := v(0,0) \in \partial U$.

Let $\gamma\colon S^1_T \to \bar{U}$ be defined by (6.5) with $s = s_0 = 0$ and T satisfying (6.6). By (6.7), γ is an η-pseudo-orbit of the Reeb flow passing through x.

Let us now reintroduce the subscript k in the notation. To summarize, we have found a sequence of η_k-pseudo-orbits $\gamma_k\colon S^1_{T_k} \to \bar{U}$ of the Reeb flow with $\eta_k \to 0$ by (6.7) and $T_k \to \infty$ by (6.6) and $x_k = \gamma_k(0) \in \partial U$. We can view γ_k as defined on the interval $[-T_k/2,\, T_k/2]$ rather than on the circle $S^1_{T_k}$. By passing if necessary to a subsequence and using the diagonal process, we can ensure that $x_k \to x \in \partial U$ and γ_k converges uniformly on compact sets. Then the limit is the integral curve $\xi \mapsto \varphi^\xi_\alpha(x)$, $\xi \in \mathbb{R}$, of the Reeb flow passing through $x \in \partial U \subset \bar{U} \setminus K$ and contained in $\bar{U}$. This is impossible since $\bar{U}$ is a closed isolating neighborhood of K.

Step 4: Proof of Part (ii). Let us fix a neighborhood U of K so small that the Shadowing Lemma applies to pseudo-orbits in U; see, e.g., [KH, Thm. 18.1.] or [FH, Sec. 5.3].

As in Step 3, since u is asymptotic at at least one end to a periodic orbit in $K \times I$, it is entirely contained in $U \times [r_{\min}, r_{\max}]$ whenever $\epsilon := E(u) > 0$ is sufficiently small which we will require through the rest of the proof. Thus v is contained in U. Furthermore, u is asymptotic to periodic orbits $\tilde{z}_\pm = (z_\pm, \rho_\pm)$ of $H^{\sharp\tau}$ at $\pm\infty$ where $z_\pm$ are periodic orbits of the Reeb flow in K due to the condition that all closed Reeb orbits of α in U are contained in K. Let $T_\pm$ be the period of $z_\pm$.

Then, by (3.4),

$$A_{\tau H}(\rho_+) - A_{\tau H}(\rho_-) = \epsilon > 0,$$

and hence $\rho_+ > \rho_-$. At the same time, by (2.2),

$$\tau h'(\rho_\pm) = T_\pm.$$

Therefore, $T_+ > T_-$ because h' is strictly increasing on I. It follows that $z_\pm$, up to the initial condition, are distinct as periodic orbits of the Reeb flow. In other words, we have an alternative: either $z_\pm$ are geometrically distinct, i.e., $z_+(\mathbb{R}) \neq z_-(\mathbb{R})$, or z_+ is a multiple cover of z_-.

Let γ_s be as in (6.5). By (6.7), γ_s is an η-pseudo-orbit with $\eta = C\epsilon^{1/4}/h'(r_{\min})$. Furthermore, as $s \to \pm\infty$, the loops γ_s converge to $z_\pm$. Due to the Shadowing Lemma, when ϵ is sufficiently small, depending only on U and the Reeb flow but not u, for every s there is a periodic orbit $\hat{\gamma}_s$ of the Reeb flow shadowing γ_s. This periodic orbit is unique, depends continuously on s and converges to $z_\pm$ as $s \to \pm\infty$. In particular, the image $\Gamma_s := \hat{\gamma}_s(\mathbb{R})$ depends continuously on s in Hausdorff topology and converges to $z_+(\mathbb{R})$ and $z_-(\mathbb{R})$ as $s \to \pm\infty$. Periodic orbits in K are isolated and, therefore, we must have

$$\Gamma_s = z_+(\mathbb{R}) = z_-(\mathbb{R})$$

for all s. Furthermore, z_- is homotopic to z_+ in the circle $z_+(\mathbb{R}) = z_-(\mathbb{R})$. This is, however, impossible since at the same time z_+ must then be a multiple cover of z_- by the above alternative.

This contradiction concludes the proof of Theorem 5.1. □

References

[A^{2}S^2] A. Abbondandolo, M.R.R. Alves, M. Sağlam, F. Schlenk, Entropy collapse versus entropy rigidity for Reeb and Finsler flows, Preprint arXiv:2103.01144.

[AS] A. Abbondandolo, M. Schwarz, On the Floer homology of cotangent bundles, *Comm. Pure Appl. Math.*, **59** (2006), 254–316.

[Al] S. Allais, On periodic points of Hamiltonian diffeomorphisms of $\mathbb{CP}^d$ via generating functions, *J. Symplectic Geom.*, **20** (2022), 1–48.

[Al16] M.R.R. Alves, Cylindrical contact homology and topological entropy, *Geom. Topol.*, **20** (2016), 3519–35692.

[Al19] M.R.R. Alves, Legendrian contact homology and topological entropy, *J. Topol. Anal.*, **11** (2019), 53-108.

[ACH] M.R.R. Alves, V. Colin, K. Honda, Topological entropy for Reeb vector fields in dimension three via open book decompositions, *J. Ec. polytech. Math.*, **6** (2019), 119–148.

[ADM^2] M.R.R. Alves, L. Dahinden, M. Meiwes, L. Merlin, C^0-Robustness of topological entropy for geodesic flows, Preprint arXiv:2109.03917.

[ADMP] M.R.R. Alves, L. Dahinden, M. Meiwes, A. Pirnapasov, C^0-stability of topological entropy for Reeb flows in dimension 3, Preprint arXiv:2311.12001.

[AM] M.R.R. Alves, M. Meiwes, Dynamically exotic contact spheres in dimensions ≥ 7, *Comment. Math. Helv.*, **94** (2019), 569–622.

[AP] M.R.R. Alves, A. Pirnapasov, Reeb orbits that force topological entropy, Preprint arXiv:2004.08106.

[As] M. Asaoka, Abundance of fast growth of the number of periodic points in 2-dimensional area-preserving dynamics, *Comm. Math. Phys.*, **356** (2017), 1–17.

[ACW] A. Avila, S. Crovisier, A. Wilkinson, C^1 density of stable ergodicity, *Adv. Math.*, **379** (2021), Paper No. 107496, 68 pp.

[Ba15] M. Batoréo, On hyperbolic points and periodic orbits of symplectomorphisms, *J. Lond. Math. Soc. (2)*, **91** (2015), 249–265.

[Ba17] M. Batoréo On non-contractible hyperbolic periodic orbits and periodic points of symplectomorphisms, *J. Symplectic Geom.*, **15** (2017), 687–717.

[Bo] F. Bourgeois, A Morse–Bott approach to contact homology, In *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, Fields Inst. Commum., AMS, **35**, (2003), 55–77.

[Br] B. Bramham, Pseudo-rotations with sufficiently Liouvillean rotation number are C^0-rigid, *Invent. Math.*, **199** (2015), 561–580.

[BV] P. Bubenik, T.Vergili, Topological spaces of persistence modules and their properties, *J. Appl. Comput. Topol.*, **2** (2018), 233–269.

[CZCG] G. Carlsson, A. Zomorodian, A. Collins, L. Guibas, Persistence barcodes for shapes, *Int. J. Shape Model.*, **11** (2005), 149–187.

[CFH] K. Cieliebak, A. Floer, H. Hofer, Symplectic homology II: A general construction, *Math. Zeit.*, **218** (1995), 103–122.

[Çi] E. Çineli, A generalized pseudo-rotation with positive topological entropy, Preprint arXiv:2310.14761.

[ÇGG21] E. Çineli, V.L. Ginzburg, B.Z. Gürel, Topological entropy of Hamiltonian diffeomorphisms: a persistence homology and Floer theory perspective, Preprint arXiv:2111.03983.

[ÇGG22] E. Çineli, V.L. Ginzburg, B.Z. Gürel, On the growth of the Floer barcode, Preprint arXiv:2207.03613.

[ÇGG23] E. Çineli, V.L. Ginzburg, B.Z. Gürel, On the generic behavior of the spectral norm, Preprint arXiv:2310.00470.

[ÇG^{2}M] E. Çineli, V.L. Ginzburg, B.Z. Gürel, M. Mazzucchelli, Invariant sets and hyperbolic closed Reeb orbits, Preprint arXiv:2309.04576.

[CB] W. Crawley-Boevey, Decomposition of pointwise finite-dimensional persistence modules, *J. Algebra Appl.*, **14** (2015), no. 5, 1550066, 8 pp.

[CGP] D. Cristofaro-Gardiner, R. Prasad, The dense existence of compact invariant sets I: Reeb flows, Hamiltonian diffeomorphisms and the Le Calvez–Yoccoz property, in preparation.

[Di] E.I. Dinaburg, A connection between various entropy characterizations of dynamical systems, *Izv. Akad. Nauk SSSR Ser. Mat.*, **35** (1971), 324–366.

[FLS] E. Fender, S. Lee, B. Sohn, Barcode entropy for Reeb flows on contact manifolds with exact Liouville fillings, Preprint arXiv:2305.04770.

[Fe] R. Fernandes, work in progress.

[Fi] J.W. Fish, Target-local Gromov compactness, *Geom. Topol.*, **15** (2011), 765–826.

[FH] T. Fisher, B. Hasselblatt, *Hyperbolic Flows*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Berlin, 2019.

[FS] U. Frauenfelder, F. Schlenk, Hamiltonian dynamics on convex symplectic manifolds, *Israel J. Math.*, **15** (2006), 1–56.

[GG14] V.L. Ginzburg, B.Z. Gürel, Hyperbolic fixed points and periodic orbits of Hamiltonian diffeomorphisms, *Duke Math. J.*, **163** (2014), 565–590.

[GG16] V.L. Ginzburg, B.Z. Gürel, Non-contractible periodic orbits in Hamiltonian dynamics on closed symplectic manifolds, *Compos. Math.*, **152** (2016), 1777–1799.

[GG18] V.L. Ginzburg, B.Z. Gürel, Hamiltonian pseudo-rotations of projective spaces, *Invent. Math.*, **214** (2018), 1081–1130.

[GG20] V.L. Ginzburg, B.Z. Gürel, Lusternik–Schnirelmann theory and closed Reeb orbits, *Math. Z.*, **295** (2020), 515–582.

[GGM] V.L. Ginzburg, B.Z. Gürel, M. Mazzucchelli, Barcode entropy of geodesic flows, Preprint arXiv:2212.00943; to appear in *J. Eur. Math. Soc. (JEMS)*.

[Kal] V. Kaloshin, An extension of the Artin-Mazur theorem, *Ann. of Math.* (2), **150** (1999), 729–741.

[Kat80] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.*, **51** (1980), 137–173.

[Kat82] A. Katok, Entropy and closed geodesics, *Ergodic Theory Dynam. Systems*, **2** (1982), 339–365.

[KH] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems.* With a supplementary chapter by A. Katok and Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[LY] Z. Lian, L.-S. Young, Lyapunov exponents, periodic orbits, and horseshoes for semiflows on Hilbert spaces, *J. Amer. Math. Soc.*, **25** (2012), 637–665.

[LS] Y. Lima, O.M. Sarig, Symbolic dynamics for three-dimensional flows with positive topological entropy, *J. Eur. Math. Soc. (JEMS)*, **21** (2019), 199–256.

[MS] L. Macarini, F. Schlenk, Positive topological entropy of Reeb flows on spherizations, *Math. Proc. Cambridge Philos. Soc.*, **151** (2011), 103–128.

[Me18] M. Meiwes, *Rabinowitz Floer Homology, Leafwise Intersections, and Topological Entropy*, Inaugural Dissertation zur Erlangung der Doktorwürde der Naturwissenschaftlich – Mathematischen Gesamtfakultät der Ruprecht – Karls – Universität Heidelberg, 2018; available at http://www.ub.uni-heidelberg.de/archiv/24153.

[Me24] M. Meiwes, Relative barcode entropy and horseshoes, in preparation.

[Pa] G.P. Paternain, *Geodesic Flows*, Progress in Mathematics, vol. 180, Birkhaüser Boston, Inc., Boston, MA, 1999.

[PRSZ] L. Polterovich, D. Rosen, K. Samvelyan, J. Zhang, *Topological Persistence in Geometry and Analysis*, University Lecture Series, vol. 74, Amer. Math. Soc., Providence, RI, 2020.

[Sa90] D.A. Salamon, Morse theory, the Conley index and Floer homology, *Bull. L.M.S.*, **22** (1990), 113–140.

[Sa99] D.A. Salamon, Lectures on Floer homology. In *Symplectic Geometry and Topology*, IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, 143–229.

[SW] D.A. Salamon, J. Weber, Floer homology and the heat flow, *Geom. Funct. Anal.*, **16** (2006), 1050–1138.

[Si] J.-C. Sikorav, Some properties of holomorphic curves in almost complex manifolds, in *Holomorphic curves in symplectic geometry*, pp. 165–189; Progr. Math., **117** Birkhäuser Verlag, Basel, 1994.

[Vi] C. Viterbo, Functors and computations in Floer cohomology, I, *Geom. Funct. Anal.*, **9** (1999), 985–1033.

[Wa] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

[ZC] A. Zomorodian, G. Carlsson, Computing persistent homology, *Discrete Comput. Geom.*, **33** (2005), 249–274.

EÇ: Institut de Mathématiques de Jussieu - Paris Rive Gauche (IMJ-PRG), 4 place Jussieu, Boite Courrier 247, 75252 Paris Cedex 5, France
Email address: erman.cineli@imj-prg.fr

VG: Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
Email address: ginzburg@ucsc.edu

BG: Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
Email address: basak.gurel@ucf.edu

MM: CNRS, UMPA, École Normale Supérieure de Lyon, 69364 Lyon, France
Email address: marco.mazzucchelli@ens-lyon.fr