A GENERAL SUPPORT THEOREM FOR ANALYTIC DOUBLE FIBRATION TRANSFORMS

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Abstract. We develop a systematic approach for resolving the analytic wave front set for a class of integral geometry transforms appearing in various tomography problems. Combined with microlocal analytic continuation, this leads to uniqueness and support theorems for analytic integral transforms which are in the microlocal double fibration framework introduced by Guillemin.

For the case of ray transforms, we show that the double fibration setup has a concrete interpretation in terms of curve families obtained by projecting integral curves of a fixed vector field on some fiber bundle down to the base. This setup includes geodesic X-ray type transforms, null bicharacteristic ray transforms and transforms related to real principal type systems. We also study transforms integrating over submanifolds of any codimension, and give geometric characterizations for the Bolker condition required for recovering singularities.

Our approach is based on a general result related to recovering the analytic wave front set of a function from its transform given by a suitable analytic elliptic Fourier integral operator. This approach extends and unifies a number of previous works. We use wave packet transforms to extrapolate the geometric features of wave front set propagation for such operators when their canonical relation satisfies the Bolker condition.

1. Introduction

This article was originally motivated by the recent work [OSSU20] that studied inverse problems for real principal type differential operators. It was shown there that certain null bicharacteristic ray transforms of the coefficient functions can be determined from boundary measurements. A special case is the light ray transform studied e.g. in [Ste17, LOSU20, FIO21], but in the general case the invertibility of these transforms was left as an open question in [OSSU20].

In this work we observe that the transforms mentioned above fall under the double fibration approach to integral geometry as in [Hel66, GGŠ69, GS77, Gui85]. We will thus study the much more general class of transforms arising from a double fibration satisfying a Bolker condition. In particular, this includes ray transforms arising from the flow of a vector field on some fiber bundle over the base manifold, and generalized Radon transforms where one has access to integrals over a suitable family of k-dimensional submanifolds (the case of ray transforms corresponds to $k = 1$).

The recovery of C^∞ singularities for such transforms R follows from the standard clean intersection calculus of Fourier integral operators (FIOs) applied to the normal operator R^*R as discussed in [GS77, Gui85]. As the main contribution of this work we show that if the underlying structures are analytic, then certain analytic singularities of a function can be recovered from the analytic singularities of its transform. The strength of this approach is that unlike in the C^∞ case, recovery of analytic singularities combined with microlocal analytic continuation (also known as the microlocal Holmgren or Sato-Kawai-Kashiwara theorem, [Hör85, Theorem 8.5.6']) leads to local uniqueness results, global uniqueness results under foliation conditions, and Helgason type support theorems for our transforms. Such results were known for the weighted Radon transform [BQ87], geodesic X-ray transform [SU08, FSU08, Kri09], light ray transform [Ste17], and certain generalized Radon

transforms [HZ17]. Our approach hinges on a general result that allows one to recover the analytic wave front set of a function f from the knowledge of Rf, where R is a special type of analytic elliptic FIO. We hope that this result could be useful in other contexts as well.

1.1. **Ray transforms.** We first discuss our results for ray transforms. Let M be a manifold with smooth boundary, let Ξ be a fiber bundle over M whose fibers are manifolds without boundary, and let $\pi : \Xi \to M$ be the projection (then $\partial\Xi = \pi^{-1}(\partial M)$). Let Y be a vector field in Ξ with flow Φ_t, and consider its "horizontal projection"

$$Y^h = d\pi \circ Y : \Xi \to TM.$$

Given any $z \in \Xi$ we consider the maximally extended integral curve $\gamma_z : [-\tau_-(z), \tau_+(z)] \to \Xi$, $\gamma_z(t) = \Phi_t(z)$, so that $\dot\gamma_z(t) = Y(\gamma_z(t))$ and $\gamma_z(0) = z$. In general one may have $\tau_\pm(z) = +\infty$, but we will only consider curves that are not trapped and satisfy $\tau_\pm(z) < \infty$. There are corresponding base space curves $x_z(t) = \pi(\gamma_z(t))$ with tangent vector

$$\dot x_z(t) = Y^h(\Phi_t(z))).$$

We consider the integral geometry problem where we know the integrals of a function f over all curves $x_z(t)$ for $z \in \mathcal{G}$, where $\mathcal{G}$ is a submanifold of Ξ describing the set of admissible curves. We can also include a weight $\kappa \in C^\infty(\mathcal{G} \times M)$ that is nowhere vanishing. The related integral transform is

$$R : C_c^\infty(M^{\mathrm{int}}) \to C^\infty(\mathcal{G}), \quad Rf(z) = \int_{-\tau_-(z)}^{\tau_+(z)} \kappa(z, x_z(t)) f(x_z(t))\, dt.$$

The inverse problem is to determine the function f, or some properties of f, from the knowledge of Rf.

We require that $\mathcal{G}$ is chosen so that Y is never tangent to $\mathcal{G}$ (to remove redundancy in parametrizing the curves) and that $\dim(\mathcal{G}) \geq \dim(M)$ (to ensure that the inverse problem is not formally underdetermined). These conditions hold e.g. if $\mathcal{G}$ is an open subset of the inward pointing boundary $\partial_+\Xi$ of Ξ, defined by

$$\partial_+\Xi = \{z \in \Xi \mid \pi(z) \in \partial M,\ Y^h(z) \text{ is transverse to } \partial M, \text{ and } Y^h(z) \text{ points inside } M\}.$$

In general $\mathcal{G}$ could also be lower dimensional submanifold of Ξ, which corresponds to a ray transform with restricted data.

Examples of operators that arise as R include the following:

- The geodesic X-ray transform, with $\Xi = SM$ and $Y = X_g$ where SM is the unit sphere bundle and X_g is the geodesic vector field on SM (see e.g. [PSU23]). One can take $\mathcal{G} = \partial_+SM$ (full data case) or $\mathcal{G}$ could be a strict open subset of ∂_+SM (partial data case). One could also take $\mathcal{G}$ to be a lower dimensional submanifold of ∂_+SM, one example being the X-ray transform with sources on a curve [FLU03].
- The transform in [FSU08] involving a family of curves γ satisfying an equation $\ddot\gamma = G(\gamma, \dot\gamma)$, with the choice $\Xi = TM$ and $Y(x, y) = y^j\partial_{x_j} + G^j(x, y)\partial_{y_j}$. These curve families include e.g. Riemannian geodesics, magnetic geodesics [DPSU07], and thermostats [AZ17].
- The null bicharacteristic ray transform for a differential operator P with real principal symbol p, where $\Xi = p^{-1}(0) \subset T^*M$ and Y is the Hamilton vector field H_p [OSSU20]. If P is the Lorentzian wave operator, this is just the light ray transform. This case also includes ray transforms for real principal type systems such as fully anisotropic elasticity, where p is a simple eigenvalue of the Christoffel matrix [dHILS23].

The first question is to determine whether R above is covered by the double fibration framework. This will be true under certain minimal conditions, one of which is phrased in terms of *variation fields* that reduce to Jacobi fields in the case of geodesic curves.

Definition 1.1. Let z_s be a smooth curve through z_0 in $\mathcal{G}$ with $\partial_s z_s|_{s=0} = w$. The family (x_{z_s}) is called a *variation* of the curve x_{z_0}. The associated *variation field* $J_w : (-\tau_-(z_0), \tau_+(z_0)) \to TM$ along x_{z_0} is given by

$$J_w(t) = \partial_s x_{z_s}(t)|_{s=0} = d\pi(d\Phi_t(w)).$$

The space of all possible directions of variation at $x_{z_0}(t)$ is

$$V_{z_0}(t) = \{J_w(t) \mid w \in T_{z_0}\mathcal{G}\}.$$

Definition 1.2. We say that R is a *double fibration ray transform* if the curves $x_z(t)$ satisfy the following:

(i) (No tangential intersections) $x_z(t) \in M^{\text{int}}$ for $z \in \mathcal{G}$ and $t \in (-\tau_-(z), \tau_+(z))$;
(ii) (No self-intersections) $t \mapsto x_z(t)$ is injective for all $z \in \mathcal{G}$;
(iii) (No singular points) $\dot{x}_z(t) \neq 0$ for all $z \in \mathcal{G}$ and all $t \in (-\tau_-(z), \tau_+(z))$;
(iv) (Nontrapping) $\tau_\pm(z) < \infty$ for all $z \in \mathcal{G}$.

If $\dim(\mathcal{G}) \leq \dim(\Xi) - 2$, we also require that

(v) (Enough variations) $V_z(t) + \mathbb{R}Y^h = T_{x_z(t)}M$ for all $z \in \mathcal{G}$ and all $t \in (-\tau_-(z), \tau_+(z))$.

Conditions (i)–(iv) are easy to visualize. Condition (v) ensures that the curve family is so large that one can vary $x_z(t)$ within this family in such a way that the infinitesimal directions of variation cover all possible directions at any point. By Lemma 4.3, (v) is automatically satisfied if $\mathcal{G}$ is an open subset of $\partial_+\Xi$, or if $\dim(\mathcal{G}) = \dim(\Xi) - 1$, so this condition is only relevant if $\mathcal{G}$ is a lower dimensional manifold. It is proved in Section 4 that assuming (i)–(v), R is indeed a transform arising from a double fibration in the sense of Definition 1.5. Conversely, under certain orientability assumptions, any transform arising from a double fibration with one dimensional left fibers is related to a vector field Y as above though the integral curves might be periodic. We remark that conditions (i) and (ii) are not always necessary and could be removed in certain cases by extension procedures (see e.g. [Dai06, SU08]), but we assume them for simplicity.

As observed in [GS77], and also stated in Theorem 2.2 below, a double fibration ray transform R is an FIO and its canonical relation is given by

$$C = \{(z, -A(z,t)\eta, x_z(t), \eta) \mid z \in \mathcal{G},\ t \in (-\tau_-(z), \tau_+(z)),\ \eta \in T^*_{x_z(t)}M \setminus 0,\ \eta \perp \dot{x}_z(t)\},$$

where $A(z,t)\eta = (\pi \circ \Phi_t|_{\mathcal{G}})^*\eta$. Here the suggestive notation $\eta \perp \dot{x}_z(t)$ means that $\eta(\dot{x}_z(t)) = 0$. It follows from basic properties of FIOs [Hör85, vol. IV] that R is well defined on $\mathcal{E}'(M^{\text{int}})$, has certain Sobolev mapping properties, and that one has the wave front set relation

$$\text{WF}(Rf) \subset C \circ \text{WF}(f).$$

Thus any singularity of Rf at (z, ζ) must come from some singularity of f at (x, η), where $x = x_z(t)$ and $\zeta = -A(z,t)\eta$ for some t.

In order to recover singularities of f from those of Rf we need a converse statement, and this is where the Bolker condition [GS77, Gui85] comes in. The Bolker condition states that the left projection $\pi_L : C \to T^*\mathcal{G} \setminus 0$ is an injective immersion. For R as above we can give a geometric characterization. This requires two notations. Below, we say that η annihilates V if $\eta(v) = 0$ for all $v \in V$.

Definition 1.3. Given $x \in M$ let H_x be the submanifold of $\mathcal{G}$ defined by

$$H_x = \{z \in \mathcal{G} \mid x_z(t) = x \text{ for some } t\}.$$

Given $z \in \mathcal{G}$ and $t, s \in (-\tau_-(z), \tau_+(z))$ let $V_z(t,s)$ be the space of all possible directions of variation at $x_z(t)$ among variations that keep $x_z(s)$ fixed in the normal direction, i.e.

$$V_z(t,s) = \{J_w(t) \mid w \in T_z\mathcal{G},\ J_w(s) \parallel \dot{x}_z(s)\}.$$

Definition 1.4. We say that the Bolker condition holds at $(z, \zeta, x, \eta) \in C$ with $x = x_z(t)$ if

(vi) $V_z(t,s)$ is not annihilated by η for any $s \neq t$; and
(vii) $d(Y^h(x, \xi(\,\cdot\,)))(T_z H_x)$ is not annihilated by η, where $(x, \xi(\tilde{z}))$ is the unique point over x on the integral curve $\gamma_{\tilde{z}}$ through $\tilde{z} \in H_x$.

Condition (vi) means that for any $s \neq t$, there is a variation of x_z that keeps $x_z(s)$ fixed in the normal direction, and the infinitesimal variation of $x_z(t)$ has a nonzero component in the η direction. A sufficient condition for (vi) to hold in the case of the geodesic X-ray or null bicharacteristic ray transform is that the curve x_z has no conjugate points (for geodesics this holds for arbitrary η, but for null bicharacteristic rays one needs a structural condition $\eta \nparallel \xi$). Condition (vii) means that the infinitesimal variations of $\dot{x}_z(t)$, among all curves that go through $x_z(t)$, should include a vector with nonzero component in the η direction. When $\mathcal{G}$ is an open subset of $\partial_+\Xi$, condition (vii) always holds for the geodesic X-ray transform, and it also holds for the null bicharacteristic ray transform provided that the Hessian $\nabla^2_\xi p$ is nondegenerate and $\eta \nparallel \xi$. See Section 4 for detailed statements.

We say that a double fibration transform R is *analytic* if all related objects (M, Ξ, $\mathcal{G}$, Y and κ) are analytic (i.e. real-analytic). Let WF_a denote the analytic wave front set [Sjö82]. We can now state our first main theorem.

Theorem 1.1. *Let R be an analytic double fibration ray transform, and assume that the Bolker condition holds at $(z, \zeta, x, \eta) \in C$. Then for any $f \in \mathcal{E}'(M^{\mathrm{int}})$, we have*

$$(z, \zeta) \notin \mathrm{WF}_a(Rf) \implies (x, \eta) \notin \mathrm{WF}_a(f).$$

1.2. **General double fibration transforms.** We will now consider the case of transforms that integrate over k-dimensional submanifolds where $1 \leq k \leq \dim(X) - 1$. Here and below we write X instead of M^{int}, to emphasize that X can be any manifold without boundary. The results will be stated in the abstract double fibration setting.

Definition 1.5. Let X and $\mathcal{G}$ be smooth oriented manifolds with no boundary, and let Z be an oriented embedded submanifold of $\mathcal{G} \times X$ with projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$, $\pi_X : Z \to X$ and with

$$\dim(\mathcal{G}) + \dim(X) > \dim(Z) > \dim(\mathcal{G}) \geq \dim(X).$$

(a) We say that Z is a *double fibration* if $\pi_{\mathcal{G}}$ and π_X are submersions. Then the sets

$$G_z := \pi_X(\pi_{\mathcal{G}}^{-1}(z)), \qquad H_x := \pi_{\mathcal{G}}(\pi_X^{-1}(x))$$

are embedded submanifolds, with orientation forms ω_{G_z} and ω_{H_x} induced by some fixed orientation forms on $\mathcal{G}, X, Z$ and by the submersions $\pi_{\mathcal{G}}$ and π_X (see Lemma 2.1). One has $\dim(G_z) = \dim(Z) - \dim(\mathcal{G})$.

(b) Let $\kappa \in C^\infty(\mathcal{G} \times X)$ be nowhere vanishing. The linear map $R : C_c^\infty(X) \to C^\infty(\mathcal{G})$ given by

$$Rf(z) = \int_{G_z} \kappa(z,x) f(x)\, d\omega_{G_z}(x), \qquad z \in \mathcal{G},$$

is called a *double fibration transform*.

(c) We say that R is an *analytic double fibration transform* if R is as in (b) and all the related objects (i.e. $\mathcal{G}$, X, Z, ω_{G_z}, κ) are analytic.

(d) If R is a double fibration transform, we consider its canonical relation

$$C := (N^*Z \setminus 0)' = \{(z, \zeta, x, \eta) \mid (z, \zeta, x, -\eta) \in N^*Z \setminus 0\}$$

and define the left and right projections $\pi_L : C \to T^*\mathcal{G} \setminus 0$ and $\pi_R : C \to T^*X \setminus 0$.

(e) We say that the *Bolker condition* holds at $(z, \zeta, x, \eta) \in C$ if $\pi_L^{-1}(z, \zeta) = \{(z, \zeta, x, \eta)\}$ and if $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective.

One has the related diagrams

$$\begin{tikzcd} & Z \arrow[dl, "\pi_{\mathcal{G}}"'] \arrow[dr, "\pi_X"] & \\ \mathcal{G} & & X \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ T^*\mathcal{G} \setminus 0 & & T^*X \setminus 0 \end{tikzcd} \tag{1.1}$$

As discussed in Section 1.1, any double fibration ray transform according to Definition 1.2 is a double fibration transform, with Z given by

$$Z = \{(z, x_z(t)) \mid z \in \mathcal{G},\ t \in (-\tau_-(z), \tau_+(z))\}.$$

The k-plane transform that encodes the integrals of $f \in C_c^\infty(\mathbb{R}^n)$ over all k-planes in $\mathbb{R}^n$, where $1 \le k \le n-1$, is another example. Generalized Radon transforms studied in [Bey84, HZ17] where one integrates over hypersurfaces $\{b(x, \theta) = s\}$, for a defining function b satisfying suitable conditions, are also in this class. We extend this setup to any codimension in Section 1.3.3.

For double fibration ray transforms, the Bolker condition holds in the sense of Definition 1.5 iff conditions (vi) and (vii) in Definition 1.4 hold. Similar geometric conditions that characterize the validity of the Bolker condition in the general case are proved in Section 3.

We then have the following analytic regularity result that generalizes Theorem 1.1.

Theorem 1.2. *Let R be an analytic double fibration transform, and assume that the Bolker condition holds at $(z, \zeta, x, \eta) \in C$. Then for any $f \in \mathcal{E}'(X)$, we have*

$$(z, \zeta) \notin \mathrm{WF}_a(Rf) \implies (x, \eta) \notin \mathrm{WF}_a(f).$$

1.3. **Applications.** As the first application of the above results, we state a local uniqueness result for analytic double fibration transforms. This is an immediate consequence of Theorem 1.2 and microlocal analytic continuation.

Theorem 1.3. *Suppose that Σ is a C^2 hypersurface in X such that $(x_0, \eta_0) \in N^*\Sigma$ and that $f \in \mathcal{E}'(X)$ vanishes on one side of Σ near x_0. Let R be an analytic double fibration transform, and assume that the Bolker condition holds at $(z_0, \zeta_0, x_0, \eta_0) \in C$. If $Rf(z) = 0$ for z near z_0, then $f = 0$ near x_0.*

Let us give a few applications to specific transforms.

1.3.1. *Geodesic X-ray transform.* We begin with the geodesic X-ray transform, defined on a compact oriented Riemannian manifold (M, g) with smooth boundary. We say that (M, g) is strictly convex if the second fundamental form of ∂M is positive definite, and that (M, g) is nontrapping if any geodesic reaches the boundary in finite time. The geodesic X-ray transform encodes the integrals of a function $f \in C(M)$ over all maximal geodesics. For an account of various properties of this transform see [PSU23].

It was conjectured in [PSU13] that the geodesic X-ray transform of a compact strictly convex nontrapping manifold is injective (and that the corresponding transform on tensor fields is solenoidal injective). This is known if additionally there are no conjugate points, i.e. (M, g) is simple [Muh77], or if $\dim(M) \geq 3$ and (M, g) can be foliated by strictly convex hypersurfaces [UV16]. There are also results in the case of hyperbolic trapping [Gui17] and microlocal results in the presence of conjugate points [MSU15, HU18]. When $\dim(M) = 2$ we can verify the conjecture provided that M and g are real-analytic.

Theorem 1.4. *Let (M, g) be a compact, strictly convex, nontrapping two-dimensional Riemannian manifold. If M and g are analytic, then the geodesic X-ray transform, possibly with an analytic nowhere vanishing weight, acting on functions on (M, g) is injective.*

This theorem follows by combining the strictly convex foliation given by [BGL02] with a layer stripping argument based on the local uniqueness result in Theorem 1.3. In the present case of geodesics the local uniqueness result is already contained in [SU08].

1.3.2. *Null bicharacteristic ray transform.* Let us next consider the null bicharacteristic ray transform. Let M be a manifold with smooth boundary, let $p \in C^\infty(T^*M \setminus 0)$ be homogeneous in ξ, and let $\Xi = p^{-1}(0) \subset T^*M \setminus 0$. We assume that

$$\nabla_\xi p \neq 0 \text{ and } \nabla_\xi^2 p \text{ is nondegenerate everywhere on } \Xi.$$

The basic example is the light ray transform where $p(x, \xi) = g_x(\xi, \xi)$ with g a Lorentzian metric on M. More generally, if g is a pseudo-Riemannian metric, one obtains the pseudo-Riemannian geodesic X-ray transform studied in [Ilm18] for product manifolds. The geodesic X-ray transform of a Riemannian metric g, for geodesics on the cosphere bundle, also arises in this way by taking $p(x, \xi) = |\xi|_g^2 - 1$ (this p is not homogeneous but it can still be included, see Section 4.4).

Let $Y = H_p$ be the Hamilton vector field of p on Ξ, and for any $z \in \Xi$ let $x_z(t) = \pi(\Phi_t(z))$ be the projection to M of the null bicharacteristic through z. We assume that the admissible curves are parametrized by an open subset $\mathcal{G}$ of $\partial_+\Xi$, and that the following conditions hold for all $z \in \mathcal{G}$:

- (No tangential intersections) $x_z(t) \in M^{\text{int}}$ for $t \in (0, \tau_+(z))$;
- (No self-intersections) $t \mapsto x_z(t)$ is injective; and
- (Nontrapping) $\tau_+(z) < \infty$.

If $\kappa \in C^\infty(\mathcal{G} \times M)$ is nowhere vanishing, the weighted null bicharacteristic ray transform is given by

$$Rf(z) = \int_0^{\tau_+(z)} \kappa(z, x_z(t)) f(x_z(t))\, dt.$$

Under the above conditions, R is a double fibration ray transform. Hence, if all the related quantities are analytic, one has a local uniqueness theorem as in Theorem 1.3.

Theorem 1.5. *Assume that M, p, and κ are analytic. Suppose $x_0 = x_{z_0}(t_0)$ where the curve x_{z_0} has no conjugate points, and $\eta_0 \in T^*_{x_0}M$ satisfies $\eta_0 \perp \dot{x}_{z_0}(t_0)$ and $\eta_0 \nparallel \Phi_{t_0}(z_0)$. If $f \in \mathcal{E}'(M^{\text{int}})$ satisfies $Rf(z) = 0$ for $z \in \mathcal{G}$ near z_0, then*

$$(x_0, \eta_0) \notin \mathrm{WF}_a(f).$$

Moreover, if Σ is a hypersurface in M^{int} such that x_{z_0} is tangent to Σ at x_0, and if $f = 0$ on one side of Σ near x_0, then f vanishes near x_0.

For the light ray transform, it was proved in [LOSU20] that one can always recover C^∞ spacelike singularities in the absence of conjugate points and that one can never recover any timelike singularities. In this case requiring that $\eta_0 \perp \dot{x}_{z_0}(t_0)$ and η_0 is not null (which implies $\eta_0 \nparallel \Phi_{t_0}(z_0)$) is equivalent with saying that η_0 is spacelike, so Theorem 1.5 implies the recovery of spacelike analytic singularities for the light ray transform. It does not give information about lightlike singularities since for lightlike η_0 the conditions $\eta_0 \perp \dot{x}_{z_0}(t_0)$ and $\eta_0 \nparallel \Phi_{t_0}(z_0)$ are incompatible. Results on lightlike C^∞ singularities may be found in [Wan18, Wan21], and the light ray transform of wave equation solutions is studied in [VW21]. Support theorems for analytic light ray transforms are given in [Ste17].

We will next formulate a more global uniqueness theorem, based on a layer stripping argument that iterates Theorem 1.5. In words, the result is as follows: if smooth hypersurfaces Γ_s for $s \in (0,1]$ foliate a subset U of M^{int}, if $\Gamma_1 \cap \mathrm{supp}(f) = \emptyset$, and if for any $y \in \Gamma_s$ with conormal ν_y of Γ_s at y there is $z = z_y$ and a nontrapped curve x_z with $x_z(t_y) = y$ and $\dot{x}_z(t_y) \perp \nu_y$ such that x_z has no conjugate points, self-intersections or tangential intersections and $\Phi_{t_y}(z) \nparallel \nu_y$, then $f = 0$ in U provided that $Rf = 0$ in some open set containing all z_y arising in this way.

A simple way to rule out conjugate points, self-intersections and tangential intersections is to assume that the hypersurfaces Γ_s are strictly convex in a suitable sense. Then it is possible to only use short curves x_z in the layer stripping argument. Short curves do not have conjugate points under the nondegeneracy assumption for $\nabla^2_\xi p$ (see Lemma 4.10). We will formulate our uniqueness theorem in a way similar to [Ste17] for the light ray transform. We first need two definitions.

Definition 1.6. For any $x \in M^{\mathrm{int}}$, the set of *potentially visible singularities* is

$$\mathrm{PVS}(x) = \bigcup_{\xi \in \Xi_x} \Big[Y^h(x,\xi)^\perp \setminus (\mathbb{R}\xi) \Big] \subset T^*_x M \setminus 0.$$

If $U \subset M^{\mathrm{int}}$, a smooth function $F : U \to \mathbb{R}$ is said to be *strictly pseudoconvex* with respect to p in U if

$$\{p, \{p, F\}\} > 0 \text{ whenever } p = \{p, F\} = 0 \text{ in } T^*U \setminus 0.$$

The condition $\eta \in \mathrm{PVS}(x)$ means that there is some curve x_z with $x_z(t_0) = x$, $\dot{x}_z(t_0) \perp \eta$ and $\Phi_{t_0}(z) \nparallel \eta$. Thus for $\eta \in \mathrm{PVS}(x)$ it is possible to recover a singularity of f at (x,η) from the knowledge of Rf if additionally a no conjugate points condition holds, whereas for $\eta \notin \cup_{\xi \in \Xi_x} Y^h(x,\xi)^\perp$ the singularity at (x,η) can never be recovered. The notion of strict pseudoconvexity appears in connection with pseudoconvexity conditions in unique continuation problems [Hör85, Chapter 28]. Geometrically it means that curves x_z tangent to a level set $F^{-1}(s)$ stay strictly on one side of the level set near the point of tangency.

The uniqueness theorem can now be stated as follows.

Theorem 1.6. *Assume that M, p, and κ are analytic and let $f \in \mathcal{E}'(M^{\mathrm{int}})$. Let F be a smooth real-valued function near* $\mathrm{supp}(f)$*, write $\Gamma_s = F^{-1}(s)$ and $U = \cup_{s \in (0,1]} \Gamma_s$. Suppose that F satisfies the following conditions:*

(a) $\mathrm{supp}(f) \cap \Gamma_1 = \emptyset$.
(b) $dF(x) \in \mathrm{PVS}(x)$ *for all* $x \in U$.
(c) F *is strictly pseudoconvex with respect to* p *in* U.

If $Rf(z) = 0$ in some open set containing all z corresponding to curves that are tangent to some Γ_s for $s \in (0,1]$, then $f = 0$ in U.

For the light ray transform this result was proved in [Ste17]. In that case, the condition $dF(x) \in \mathrm{PVS}(x)$ translates to the level sets Γ_s being timelike (i.e. their conormals are spacelike), and some examples of pseudoconvex foliations are given in [Ste17]. In other cases, such as $p(x,\xi) = |\xi|_g^2 - 1$ (Riemannian geodesic X-ray transform) or $p(x,\xi) = g_x(\xi,\xi)$ for a pseudo-Riemannian metric of signature (n_1, n_2) with $n_1, n_2 \geq 2$, any nonzero covector is in $\mathrm{PVS}(x)$ and condition (b) always holds. We also stress that the pseudoconvexity condition in (c) could be weakened to a condition that some (possibly long) curves tangent to Γ_s have no conjugate points.

When combined with [OSSU20, Theorem 1.2], the above theorem gives a uniqueness result for Calderón type problems for real principal type operators.

Theorem 1.7. *Let M be a compact analytic manifold with smooth boundary, let P be a real principal type differential operator of order $m \geq 2$ on M with analytic coefficients, and let $V_1, V_2 \in C^\infty(M)$ with $V_1 = V_2$ to infinite order on ∂M. If the Cauchy data sets for $P + V_1$ and $P + V_2$ agree, and if the geometric assumptions of Theorem 1.6 hold with $f = V_1 - V_2$, then $V_1 = V_2$ in U.*

1.3.3. *Transforms over codimension k submanifolds.* The previous examples were in the setting of ray transforms. Next we state a result for generalized Radon transforms that integrate over codimension k submanifolds. This extends earlier results for the Euclidean Radon transform with analytic weights [BQ87] and for Radon type transforms over hypersurfaces $\{b(x,\theta) = s\}$ where b is a suitable defining function [HZ17].

Let X be a manifold without boundary. We will study a transform that integrates a function $f \in \mathcal{E}'(X)$ over codimension k submanifolds $G_{\theta,s}$ in X, given explicitly as

$$G_{\theta,s} = \{x \in X \mid b(x,\theta) = s\},$$

where $b : X \times V' \to \mathbb{R}^k$ is a smooth function, $\theta \in V'$ and $s \in V''$, where V' and V'' are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Here $m \geq 1$ and $1 \leq k \leq n-1$ where $n = \dim(X)$. We assume that we are considering integrals over $G_{\theta,s}$ with (θ, s) close to a fixed (θ_0, s_0), so we can work in local coordinates and take $\mathcal{G} = V' \times V'' \subset \mathbb{R}^{m+k}$. Moreover, if we let $U \subset X$ be a neighborhood of G_{θ_0,s_0} such that $G_{\theta,s} \subset U$ for $(\theta, s) \in \mathcal{G}$, it will be enough to have everything defined for $x \in U$.

We assume that the defining function b is such that the Jacobi matrix b_x satisfies

$$b_x(x,\theta) \text{ is surjective for } x \in U \text{ and } \theta \in V'. \tag{1.2}$$

Then $Z = \{(\theta, s, x) \mid b(x,\theta) = s\}$ will be a double fibration. If we fix orientation forms on Z and $\mathcal{G}$ (we can use the Lebesgue measure on $\mathcal{G} \subset \mathbb{R}^{m+k}$), we obtain orientation forms on each $G_{\theta,s}$ and consider the weighted double fibration transform

$$Rf(\theta,s) = \int_{G_{\theta,s}} \kappa(\theta,s,x) f(x)\, d\omega_{G_{\theta,s}}(x), \qquad (\theta,s) \in \mathcal{G},$$

where κ is smooth and nowhere vanishing.

The Bolker condition can be described as follows. The map $d\pi_L$ is injective if for any $x \in U$, $\theta \in V'$ and $\zeta \in \mathbb{R}^k \setminus 0$, the linear map

$$\big(b_x(x,\theta)^T, \quad \partial_\theta(b_x(x,\theta)^T\zeta)\big) \text{ is surjective.} \tag{1.3}$$

The map π_L is injective if for any $\theta \in V'$ and $\zeta \in \mathbb{R}^k \setminus 0$, the map

$$x \mapsto (b(x,\theta), b_\theta(x,\theta)^T\zeta) \text{ is injective on } U. \tag{1.4}$$

We remark that (1.3) can only be valid when $m \geq n - k$, which is precisely the condition that the inverse problem for R is not formally underdetermined. We also remark that the weighted

Euclidean Radon transform is obtained by taking $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ b(x,\theta) = x \cdot \theta$, which satisfies (1.2)–(1.4).

Theorem 1.2 and microlocal unique continuation yield the following uniqueness theorem.

Theorem 1.8. *Let X, b, κ and $\omega_{G_{\theta,s}}$ above be analytic. Suppose that Σ is a hypersurface in X such that $(x_0, \eta_0) \in N^*\Sigma \cap N^*G_{\theta_0,s_0}$ and that $f \in \mathcal{E}'(X)$ vanishes on one side of Σ near x_0. Assume that (1.2)–(1.4) hold. If $Rf(\theta, s) = 0$ for (θ, s) near (θ_0, s_0), then $f = 0$ near x_0.*

Let us compare the above setup to that in [HZ17]. In [HZ17], one considers a function $b : X \times (\mathbb{R}^n \setminus 0) \to \mathbb{R}$ such that $b(x,\theta)$ is positively homogeneous of degree 1 in θ, b_x is never zero, and the matrix $(\partial_{x_j\theta_k} b)_{j,k=1}^n$ is positive definite. One also assumes the following form of a global Bolker condition: for any fixed x and θ,

$$\text{the map } b_\theta(\,\cdot\,,\theta) \text{ is injective, and the map } b_x(x,\,\cdot\,) \text{ is surjective.}$$

We see that our setup generalizes this in several ways:

- the submanifolds G_z can have any codimension k with $1 \le k \le n-1$;
- one can have any dimension $m \ge n-k$ for space of parameters θ;
- the homogeneity of b is not needed;
- the positivity of $(\partial_{x_j\theta_k} b)_{j,k=1}^n$ is replaced by the weaker condition (1.3); and
- the injectivity of $x \mapsto b_\theta(x,\theta)$ is replaced by the weaker condition (1.4).

1.4. **Methods.** In our proof of the main theorem, instead of going through the normal operator R^*R as in the C^∞ case [Gui85], we will work directly with the FIO R. In fact the adjoint R^* is never used in this article. This avoids the issue of having to consider compositions of analytic FIOs. Instead we will employ the FBI transform characterization of the analytic wave front set in [Sjö82], both in the setting of Gaussian wave packets and generalized FBI transforms. Our argument also avoids the use of special cutoffs that are typical in analytic microlocal analysis and were used e.g. in the related works [SU08, Ste17].

Here is a more detailed outline of the proof of Theorem 1.2. For clarity we will write $(\hat{z}, \hat{\zeta}, \hat{x}, \hat{\eta})$ for the point of interest below.

1. We first localize matters near $\hat{x}$ and choose $\chi \in C_c^\infty(X)$ with $0 \le \chi \le 1$ and $\chi = 1$ near $\hat{x}$ (in fact χ could even be a hard cutoff function). Writing $f = \chi f + (1-\chi)f$, we have
$$Rf = R(\chi f) + R((1-\chi)f).$$
The Schwartz kernel of an analytic double fibration transform satisfies $\mathrm{WF}_a(R)' = C$ where C is as in Definition 1.5. Now $\mathrm{WF}_a(R((1-\chi)f)) \subset C(\mathrm{WF}_a((1-\chi)f))$ by properties of general linear operators [Hör85, Theorem 8.5.5]. Since $\pi_L^{-1}(\hat{z},\hat{\zeta}) = \{(\hat{z},\hat{\zeta},\hat{x},\hat{\eta})\}$ by the Bolker condition, we always have $(\hat{z},\hat{\zeta}) \notin \mathrm{WF}_a(R((1-\chi)f))$. It is thus enough to show that $(\hat{z},\hat{\zeta}) \notin \mathrm{WF}_a(R(\chi f))$ implies $(\hat{x},\hat{\eta}) \notin \mathrm{WF}_a(f)$.
2. After localizing near $\hat{x}$, we reduce the problem to a model FIO
$$Tf(z) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n''}} e^{i(\phi(z,x')-x'')\cdot\eta} a(z,x) f(x)\, d\eta\, dx \tag{1.5}$$
where the double fibration Z is locally given by $\{x'' = \phi(z,x')\}$ for suitable coordinates $x = (x', x'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, and f is supported near $\hat{x}$. From now on we only require the local Bolker condition that $d\pi_L|_{(\hat{z},\hat{\zeta},\hat{x},\hat{\eta})}$ is injective, or equivalently that the matrix
$$(\phi_z(\hat{z},\hat{x}')^T,\ \partial_{x'}(\phi_z(\hat{z},x')^T\hat{\eta})\,|_{x'=\hat{x}'}) \quad \text{is injective.}$$

This condition implies that π_L is injective in a neighborhood of $(\hat{z}, \hat{\zeta}, \hat{x}, \hat{\eta})$ in C. This, together with structural properties of C, implies that microlocalization on the (z, ζ) side combined with localization on the x side effectively implies microlocalization on the (x, η) side, without the need to introduce additional cutoffs in the η variable (see Lemma 3.4).

3. We need to prove that $(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(Tf) \implies (\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(f)$. We apply the FBI transform characterization of the analytic wave front set [Sjö82, Definition 6.1]. If L_λ denotes the FBI transform with Gaussian wave packets, we have

$$(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(Tf) \iff L_\lambda Tf(z, \zeta) = O(e^{-c\lambda}) \text{ uniformly for } (z, \zeta) \text{ near } (\hat{z}, \hat{\zeta}).$$

We write $L_\lambda T$ in terms of its Schwartz kernel $K_\lambda(z, \zeta, x)$, which can be expressed as an oscillatory integral. Then we have

$$\int K_\lambda(z, \zeta, x) f(x)\, dx = O(e^{-c\lambda}) \text{ for } (z, \zeta) \text{ near } (\hat{z}, \hat{\zeta}). \tag{1.6}$$

4. Next we simplify the oscillatory integral expression K_λ as

$$K_\lambda(z, \zeta, x) = c\lambda^{\frac{3N}{4}} \int_{\mathcal{U}} e^{i\lambda\Psi(\zeta', x; z, \zeta)} \tilde{a}(\zeta', x)\, d\zeta'$$

for some phase function Ψ and amplitude $\tilde{a}$. Here we use that $a(z, x)$ is independent of η, so we can integrate out the η variable to obtain a delta function of Z. We show that the phase has a unique nondegenerate critical point, apply analytic stationary phase [Sjö82, Theorem 2.8] to the formula for K_λ, and substitute the result in (1.6) to obtain

$$\int e^{i\lambda\psi(x, z, \zeta)} \tilde{a}_1(x, z, \zeta; \lambda) f(x)\, dx = O(e^{-c\lambda}). \tag{1.7}$$

for certain ψ and $\tilde{a}_1$.

5. The final step is to verify that the left hand side of (1.7) is a generalized FBI transform applied to f. This involves proving that the phase ψ satisfies the estimates in [Sjö82, Definition 6.1], including quadratic growth for $\mathrm{Im}(\psi)$. Establishing these estimates is the most technical part of the argument. Once this has been done, applying [Sjö82, Definition 6.1] in (1.7) shows that $(\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(f)$ as required.

Remark 1.9. From the FIO point of view, it would be natural to consider operators of the form (1.5) where the analytic function $a(z, x)$ is replaced by a classical analytic symbol $a(z, x, \eta)$ (see e.g. [Trè22]). Such operators can also be handled by the method above. However, the analytic stationary phase argument will include an additional integral over $\eta \in \mathbb{R}^n$ which is not directly covered by [Sjö82, Theorem 2.8]. Since operators of the form (1.5) are already sufficient for our applications to integral geometry transforms, we will only give a sketch of this extension in Remark 5.18.

We discuss some previous work related to recovering analytic singularities. The possibility that the C^∞ argument in [Gui85] could be extended to the analytic case was raised in [BQ93, Qui93]. However, since a detailed proof of the required general FIO composition result in the analytic category was not available, results for particular transforms were proved by alternative means in [SU08, FSU08, Kri09, KS09, Ste17, HZ17] building on the FBI transform method in [KSU07]. Results based on realizing R^*R explicitly as an analytic pseudodifferential operator are given in [BQ87, SU05]. On the FIO theory side, Theorem 4.5 of [Sjö82] gives a construction of a microlocal parametrix for an elliptic analytic FIO associated to a canonical transformation. Recent work of [RSN20] gives a composition calculus under a transversal intersection condition. Neither of these works can be directly applied in our setting. Other related references on analytic microlocal analysis are [SKK73, Trè80, BJ20].

There have also been several studies related to C^∞ singularities in cases where the Bolker condition is violated. Some of these are related to seismic imaging applications, where the Bolker condition corresponds to a travel time injectivity condition [NS97, tKSV98]. In these results the composition R^*R may not be covered by the clean intersection calculus, but one typically interprets R^*R as a generalized Fourier integral operator whose wave front relation can be used to describe artifacts appearing in the imaging process. See e.g. [GU89, GU90, Nol00, FLU03, SU13, FGGN22] and references therein for various results of this type.

We would also like to mention that there is a large literature on double fibration transforms in various symmetric and homogeneous settings [GGG03, Hel11] and in the complex setting in connection with Penrose transforms [BE89].

Organization. This paper is organized as follows. Section 2 collects basic properties of double fibrations needed later, and Section 3 gives several equivalent characterizations of the Bolker condition. In Section 4 we give an alternative approach to double fibration ray transforms based on vector fields on fiber bundles, and discuss geodesic and null bicharacteristic ray transforms in more detail. Section 5 gives the proof of Theorem 1.2 based on analytic microlocal analysis. Finally, in Section 6 we prove Theorems 1.3–1.8 as consequences of Theorem 1.2.

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2. Basic properties of double fibration transforms

We begin by collecting certain properties of double fibration transforms that will be used later. We will state these properties in the smooth category, but the same properties continue to hold if one replaces smooth by analytic throughout.

Let $\mathcal{G}$ and X be oriented smooth manifolds without boundary so that $\dim(\mathcal{G}) = N$ and $\dim(X) = n$. Assume that Z is a smooth oriented embedded submanifold of $\mathcal{G}\times X$ and consider the projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ and $\pi_X : Z \to X$, as illustrated in the diagram (1.1).

We will assume that

$$\pi_{\mathcal{G}} : Z \to \mathcal{G} \text{ is a submersion.}$$

This forces $\dim(Z) \geq N$. If $\dim(Z) = N$ (resp. $\dim(Z) = N + n$) then the sets $G_z = \pi_X(\pi_{\mathcal{G}}^{-1}(z))$ will be 0-dimensional (resp. n-dimensional) and one does not obtain a standard integral geometry transform. For these reasons we will assume in this section that

$$N + n > \dim(Z) > N.$$

We will also write

$$\begin{aligned} \dim(Z) &= N + n', \\ n &= n' + n'', \end{aligned}$$

where $1 \leq n', n'' \leq n-1$.

Since $\pi_{\mathcal{G}}$ is a submersion, for any $z \in \mathcal{G}$ the fiber $\pi_{\mathcal{G}}^{-1}(z)$ is an embedded n'-dimensional submanifold of Z, and correspondingly the set

$$G_z := \pi_X(\pi_{\mathcal{G}}^{-1}(z)) \tag{2.1}$$

is an embedded n'-dimensional submanifold of X. We equip G_z with an orientation form as follows (see [Qui80] for more on the choice of measures).

Lemma 2.1. *Given orientation forms ω_Z on Z and $\omega_{\mathcal{G}}$ on $\mathcal{G}$, there is a natural orientation form ω_{G_z} on each G_z induced by the submersion $\pi_{\mathcal{G}}$.*

Proof. The submersion $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ induces on each fiber $\pi_{\mathcal{G}}^{-1}(z)$ the orientation form

$$\omega_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \dots, v_{n'}) = \frac{\omega_Z(v_1, \dots, v_{n'}, w_1, \dots, w_N)}{\omega_{\mathcal{G}}(d\pi_{\mathcal{G}}(w_1), \dots, d\pi_{\mathcal{G}}(w_N))}$$

where $\{v_1, \dots, v_{n'}\}$ spans $T_{(z,x)}\pi_{\mathcal{G}}^{-1}(z)$ and $\{v_1, \dots, v_{n'}, w_1, \dots, w_N\}$ spans $T_{(z,x)}Z$. Here we used that $\pi_{\mathcal{G}}$ is a submersion and $d\pi_{\mathcal{G}}(v_j) = 0$ to ensure that $\{d\pi_{\mathcal{G}}(w_1), \dots, d\pi_{\mathcal{G}}(w_N)\}$ is a basis of $T_z\mathcal{G}$. Furthermore, a standard change of basis formula for orientation forms ensures that the above definition does not depend on the choice of $\{w_1, \dots, w_N\}$ as long as $\{v_1, \dots, v_{n'}, w_1, \dots, w_N\}$ spans $T_{(z,x)}Z$. It is then enough to note that $\pi_X : \pi_{\mathcal{G}}^{-1}(z) \to G_z$ is a diffeomorphism and to define $\omega_{G_z}(d\pi_X(v_1), \dots, d\pi_X(v_{n'})) = \omega_{\pi_{\mathcal{G}}^{-1}(z)}(v_1, \dots, v_{n'})$. □

Define the weighted double fibration transform $R : C_c^\infty(X) \to C^\infty(\mathcal{G})$ by

$$Rf(z) = \int_{G_z} \kappa(z,x) f(x)\, d\omega_{G_z}(x), \qquad z \in \mathcal{G}, \tag{2.2}$$

for some smooth and nowhere vanishing weight $\kappa(z,x)$. The following result, which implies that R indeed maps into $C^\infty(\mathcal{G})$, characterizes R as an FIO associated with the conormal bundle N^*Z, acting on functions with values in the half density bundle $\Omega^{1/2}$.

Theorem 2.2. *Assume that $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ is a submersion and $\dim(Z) = N + n'$. Then R is a Fourier integral operator of order $\frac{n}{4} - \frac{n'}{2} - \frac{N}{4}$ whose canonical relation is*

$$C = (N^*Z \setminus 0)' = \{(z, A(z,x)\eta, x, -\eta) \mid (z,x) \in Z,\ \eta \in N_x^* G_z,\ \eta \neq 0\}$$

where $A(z,x)$ is is a linear map $N_x^*G_z \to T_z^*\mathcal{G}$ depending smoothly on $(z,x) \in Z$. The operator R has a nowhere vanishing homogeneous principal symbol

$$a_Z \in S^{\frac{n-n'}{2}}(N^*Z, \Omega^{1/2}).$$

This result follows from [GS77, Sections VI.3 and VI.6] or [Gui85]. We will also give a proof below since some of the computations will be used in the proof of Theorem 1.2. As a consequence of Theorem 2.2, R is well defined as a map $\mathcal{E}'(X) \to \mathcal{D}'(\mathcal{G})$. We will later work under the assumption that $N \geq n$ and the left and right projections (π_L and π_R) acting on C have full rank. In this case, the rank of $d\pi_L$ will be $n+N$ while the rank of $d\pi_R$ will be $2n$. Then by Theorem 4.3.2 of [Hör71], $R : H_c^s(X) \to H_{\text{loc}}^{s+n'/2}(\mathcal{G})$.

For many of our arguments, it will be convenient to write Z locally as the graph $\{x'' = \phi(z, x')\}$ for some coordinates $x = (x', x'')$ in X with $x' \in \mathbb{R}^{n'}$ and $x'' \in \mathbb{R}^{n''}$.

Lemma 2.3. *Let $Z \subset \mathbb{R}^N \times \mathbb{R}^n$, $\dim(Z) > N$, be an embedded smooth submanifold such that*

$$\pi_N : Z \to \mathbb{R}^N \text{ is a submersion.}$$

If $(\hat{z}, \hat{x}) \in Z$, we can find open subsets $U \subset\subset \mathbb{R}^{n'+n''}$ containing $\hat{x}$ and $V \subset\subset \mathbb{R}^N$ containing $\hat{z}$ with $U = U' \times U''$ and $U' \subset\subset \mathbb{R}^{n'}$, $U'' \subset\subset \mathbb{R}^{n''}$ open, and a smooth function $\phi : V \times U' \to U''$ such that

$$Z \cap (V \times U) = \{(z, x) \mid x'' = \phi(z, x'), z \in V, x' \in U'\}.$$

Conversely, if $Z = \{x'' = \phi(z, x')\}$ for some smooth ϕ, then π_N is a submersion.

Proof. Let j be the embedding $Z \to \mathbb{R}^N \times \mathbb{R}^n$. Then $dj(w) = (d\pi_N(w), d\pi_n(w))$ for $w \in TZ$, so we may write dj in local coordinates as the $(N + n) \times \dim(Z)$ matrix

$$dj = \begin{pmatrix} d\pi_N \\ d\pi_n \end{pmatrix}.$$

Now dj has $\dim(Z)$ linearly independent columns since j is an immersion. This means that dj has $\dim(Z)$ linearly independent rows. Furthermore, $d\pi_N$ has N linearly independent rows since π_N is a submersion. It follows that we can choose $n' = \dim(Z) - N$ components x' of x, with $x = (x', x'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, such that the differential of the projection $\pi_{N+n'} : Z \ni (z, x) \mapsto (z, x') \in \mathbb{R}^N \times \mathbb{R}^{n'}$ is a bijection at $(\hat{z}, \hat{x})$. Let $\pi_{n''} : Z \to \mathbb{R}^{n''}$ be the projection to the other components.

By the inverse function theorem there is a small neighborhood $V \times U' \subset \mathbb{R}^N \times \mathbb{R}^{n'}$ of $(\hat{z}, \hat{x}')$ such that $\pi_{N+n'}^{-1} : V \times U' \to Z$ is a smooth diffeomorphism from $V \times U'$ to a neighborhood of $(\hat{z}, \hat{x})$ in Z (here we use that Z is an embedded submanifold). So the map $\phi := \pi_{n''} \circ \pi_{N+n'}^{-1} : V \times U' \to \mathbb{R}^{n''}$ is the desired map. The converse follows since $\pi_N(z, x', \phi(z, x')) = z$. □

We now compute N^*Z, considered as a subset of $T^*\mathcal{G} \times T^*X$, and describe the linear map $A(z, x)$ appearing in Theorem 2.2 more precisely.

Lemma 2.4. *Assume that $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ is a submersion. One has*

$$N^*Z = \{(z, \zeta, x, \eta) \mid (z, x) \in Z,\ \eta \in N_x^*G_z,\ \zeta = A(z, x)\eta\}$$

where $A(z, x)$ is a linear map $N_x^*G_z \to T_z^*\mathcal{G}$ depending smoothly on $(z, x) \in Z$. If Z is locally given by $\{x'' = \phi(z, x')\}$, then in these coordinates

$$\begin{aligned} T_xG_z &= \{(\eta', \phi_{x'}(z, x')\eta') \mid \eta \in \mathbb{R}^{n'}\}, \\ N_x^*G_z &= \{(-\phi_{x'}(z, x')^T\eta'', \eta'') \mid \eta'' \in \mathbb{R}^{n''}\}, \\ N_{(z,x)}^*Z &= \{(z, -\phi_z(z, x')^T\eta'', (x', \phi(z, x')), (-\phi_{x'}(z, x')^T\eta'', \eta'')) \mid \eta'' \in \mathbb{R}^{n''}\}, \end{aligned}$$

and

$$A(z, x)(-\phi_{x'}^T\eta'', \eta'') = -\phi_z^T\eta''.$$

Proof. By Lemma 2.3, we can write Z locally as the set $\{x'' = \phi(z, x')\}$. Any curve x_s in G_z satisfies $x_s'' = \phi(z, x_s')$ and therefore

$$\dot{x}_0'' = \phi_{x'}(z, x_0')\dot{x}_0'.$$

This gives the formula for T_xG_z. Then any vector of the form $(-\phi_{x'}(z, x')^T\eta'', \eta'')$ must be in $N_x^*G_z$, and a dimension count shows that all vectors in $N_x^*G_z$ are of this form.

For any curve (z_s, x_s) in Z we have

$$\dot{x}_0'' = \phi_z(z_0, x_0')\dot{z}_0 + \phi_{x'}(z_0, x_0')\dot{x}_0'.$$

It follows that $T_{(z_0,x_0)}Z = \{(v, u', \phi_z(z_0, x_0')v + \phi_{x'}(z_0, x_0')u') \mid v \in \mathbb{R}^N,\ u' \in \mathbb{R}^{n'}\}$. Then $(\zeta, \eta) \in N^*_{(z_0,x_0)}Z$ if and only if

$$\zeta(v) + \eta'(u') + \eta''(\phi_z(z_0, x_0')v + \phi_{x'}(z_0, x_0')u') = 0$$

for all v and u'. Choosing $v = 0$ we have $\eta'(u') + \eta''(\phi_{x'}(z_0, x_0')u') = 0$ for all u', which is equivalent to $\eta \in N^*_{x_0}G_{z_0}$. Then varying v gives $\zeta = -\phi_z(z_0, x_0')^T\eta''$. This proves the formula for $N^*_{(z,x)}Z$.

Finally, the argument above shows that any $(z, \zeta, x, \eta) \in N^*Z$ must satisfy $x \in G_z$, $\eta \in N^*_xG_z$, and $\zeta = A(z, x)\eta$ where $A(z, x)$ is defined on $N^*_xG_z$ by the required formula $A(z, x)(-\phi_{x'}^T\eta'', \eta'') = -\phi_z^T\eta''$. In order to state precisely the smooth dependence of A on (z, x), consider the smooth vector bundles E and F over Z with fibers at (z, x) given by $E_{(z,x)} = N^*_xG_z$ and $F_{(z,x)} = T^*_z\mathcal{G}$. Note that F is a pullback bundle, and E is a smooth vector bundle with local trivializations given by $((z, x', \phi(z, x')), (-\phi_z^T\eta'', \eta'')) \mapsto ((z, x'), \eta'')$. Then A is indeed a smooth section of the bundle $\mathrm{Hom}(E, F)$. □

We proceed to the proof of Theorem 2.2. It follows from [Gui85] that if Z is a double fibration, then R is an FIO whose Schwartz kernel is essentially the delta function $\kappa\delta_Z$. To state this properly we need to consider R acting on half densities. If we fix nonvanishing half densities on $\mathcal{G}$ and X, we obtain identifications between functions and half densities and we can identify R with an operator, still denoted by R, that acts on half density valued functions.

Proof of Theorem 2.2. To simplify notation we assume without loss of generality that $\kappa(z, x) = 1$. We define δ_Z precisely via the formula

$$\langle \delta_Z, \Psi\rangle = \int_{\mathcal{G}} \int_{G_z} \Psi(z, x)\, d\omega_{G_z}(x)\, d\omega_{\mathcal{G}}(z), \qquad \Psi \in C_c^\infty(\mathcal{G} \times X).$$

Then δ_Z is a continuous linear functional on $C_c^\infty(\mathcal{G} \times X)$, i.e. a distribution density on $\mathcal{G} \times X$. It follows from the definitions that

$$\langle \delta_Z, \psi(z)f(x)\rangle = \int_{\mathcal{G}} (Rf(z))\psi(z)\, d\omega_{\mathcal{G}}(z), \qquad f \in C_c^\infty(X),\ \psi \in C_c^\infty(\mathcal{G}).$$

This shows that δ_Z is the (density valued) Schwartz kernel of R.

To see that δ_Z behaves indeed like a delta function of Z, let $(z_0, x_0) \in Z$ and use Lemma 2.3 to find smooth positively oriented local coordinates (z, x', x'') in some neighborhood $V \times U$ of (z_0, x_0) so that $V \subset \mathcal{G}$ and $U \subset X$ are open, $Z \cap (V \times U) = \{(z, x', x'') \in V \times U \mid x'' = \phi(z', x')\}$, and $G_z \cap U = \{(x', x'') \in U \mid x'' = \phi(z, x')\}$ for $z \in V$. In these coordinates $d\omega_{G_z}(x) = a_0(z, x')\, dx'$ and $d\omega_{\mathcal{G}}(z) = b(z)\, dz$ where a_0 and b are smooth positive functions. Then if Ψ has small support near (z_0, x_0) we have

$$\langle \delta_Z, \Psi\rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^n} \delta_{\{x''=\phi(z,x')\}}(z, x', x'')\Psi(z, x', x'')a_0(z, x')b(z)\, dx'\, dz.$$

Thus in these local coordinates δ_Z becomes the conormal distribution $a_0(z, x')b(z)\delta_{\{x''=\phi(z,x')\}}$ in $\mathbb{R}^{N+n}$ whose symbol is positive, smooth and homogeneous of degree zero. We may also write δ_Z, or equivalently the Schwartz kernel $R(z, x)$ of R, as the oscillatory integral

$$R(z, x) = \int_{\mathbb{R}^{n''}} e^{i(\phi(z,x')-x'')\cdot\eta} a(z, x)\, d\eta \tag{2.3}$$

for some positive nonvanishing smooth function a.

After multiplying by nonvanishing smooth half densities we can interpret δ_Z as a half density valued distribution conormal to Z. Since N^*Z has fiber dimension $n - n'$, we see that δ_Z has

principal symbol in $S^{\frac{n-n'}{2}}(N^*Z, \Omega^{1/2})$ according to the convention in [Hör85, Definition 18.2.10]. Then by [Hör85, Theorem 18.2.11] δ_Z is a conormal distribution of order m in $\mathcal{G} \times X$, where

$$m + \frac{N+n}{4} = \frac{n-n'}{2}.$$

It follows that R is a Fourier integral operator whose Schwartz kernel is a conormal to the Lagrangian manifold $N^*Z \setminus 0$, and thus R has canonical relation $C = (N^*Z \setminus 0)'$ (see [Hör85, Section 25.2]). The formula for C follows from Lemma 2.4. □

Up to now we have considered submanifolds $Z \subset \mathcal{G} \times X$ such that $\pi_{\mathcal{G}}$ is a submersion. From now on we will also assume that π_X is a submersion, i.e. that Z is a double fibration. This is locally characterized as follows.

Lemma 2.5. *Let $Z \subset \mathcal{G} \times X$ be locally given by $\{x'' = \phi(z, x')\}$. Then $\pi_X : Z \to M^{\text{int}}$ is a submersion iff the matrix $\phi_z(z, x')$ is surjective iff the map $A(z, x)$ in Lemma 2.4 is injective.*

Proof. We identify $(z, x', \phi(z, x')) \in Z$ with (z, x'). Since $\pi_X(z, x') = (x', \phi(z, x'))$, we have $d\pi_X(\dot{z}, \dot{x}') = (\dot{x}', \phi_z \dot{z} + \phi_{x'} \dot{x}')$. Thus $d\pi_X$ is surjective iff ϕ_z is surjective. This is equivalent with $A(z, x)$ being injective by Lemma 2.4. □

Since π_X is a submersion, the fibers $\pi_X^{-1}(x)$ are smooth manifolds in Z. Correspondingly the sets

$$H_x := \pi_{\mathcal{G}}(\pi_X^{-1}(x))$$

are embedded $(N - n'')$-dimensional submanifolds in $\mathcal{G}$.

For a double fibration, both projections $\pi_{\mathcal{G}}$ and π_X are submersions and thus the roles of the z and x variables are somewhat symmetric. This leads to the following analogue of the results given above.

Lemma 2.6. *Let Z be a double fibration with $N + n > \dim(Z) > n$. Then*

$$N^*Z = \{(z, \zeta, x, \eta) \mid (z, x) \in Z,\ \zeta \in N_z^* H_x,\ \eta = B(z, x)\zeta\}$$

where $B(z, x)$ is a linear map $N_z^* H_x \to T_x^* X$ depending smoothly on $(z, x) \in Z$. If $(z_0, x_0) \in Z$, there are local coordinates $z = (z', z'')$ near z_0 and a smooth $\mathbb{R}^{n''}$-valued function $b(x, z')$ near (x_0, z_0') such that near (z_0, x_0), b_x is surjective and one has

$$Z = \{(z, x) \mid z'' = b(x, z')\},$$

$$N^*_{(z,x)} Z = \{((z', b(x, z')), (-b_{z'}(x, z')^T \zeta'', \zeta''), x, -b_x(x, z')^T \zeta'') \mid \zeta'' \in \mathbb{R}^{n''}\}.$$

In these coordinates one has $B(z, x)\zeta = -b_x(x, z')^T \zeta''$. Conversely, if $b(x, z')$ is a smooth function such that b_x is surjective, then $Z = \{z'' = b(x, z')\}$ is a double fibration.

Proof. This follows by interchanging the roles of z and x above. Since π_X is a submersion, we can use Lemma 2.3 to find coordinates $z = (z', z'')$ with $z'' \in \mathbb{R}^{n''}$ such that locally $Z = \{z'' = b(x, z')\}$ for some smooth function b. Since Z is a double fibration, b_x is surjective by Lemma 2.5. Conversely, if $Z = \{z'' = b(x, z')\}$ where b_x is surjective, then Z is a double fibration by Lemmas 2.3 and 2.5. □

If Z is a double fibration, it follows from the above results that for any $(z, x) \in Z$ the map $A(z, x)$ is a bijection $N_x^* G_z \to N_z^* H_x$ with inverse map given by $B(z, x)$. We can thus write N^*Z in a more symmetric form as

$$N^*Z = \{(z, \zeta, x, \eta) \mid (z, x) \in Z,\ \zeta = A(z, x)\eta,\ \eta = B(z, x)\zeta\},$$

and we can parametrize points in N^*Z over (z,x) either by $\eta \in N_x^*G_z$ or by $\zeta \in N_z^*H_x$.

3. The Bolker condition

In this section we give several equivalent characterizations for the Bolker condition [Gui85] in the setting of double fibration transforms. This condition states that the left projection $\pi_L : C \to T^*\mathcal{G}\backslash 0$ in the microlocal diagram (1.1) is an injective immersion. From this point on we will also assume that $N \geq n$. One reason is that the Bolker condition can only hold when $N \geq n$ (see Lemma 3.3). Another reason is that the double fibration transform maps function in X to functions in $\mathcal{G}$, and the problem of inverting this transform will be formally underdetermined unless $N \geq n$.

3.1. **Injectivity of π_L.** As discussed in Section 1.1, the injectivity of π_L is in some way related to variations and conjugate points. We begin by discussing these notions in the double fibration setting. However, we will use the terminology *Z-conjugate points* instead of conjugate points since the two notions are not always equivalent. Recall the linear map $A(z,x) : N_x^*G_z \to T_z^*\mathcal{G}$ from Lemma 2.4, and denote the adjoint map by $A(z,x)^* : T_z\mathcal{G} \to (N_x^*G_z)^*$. By Lemma 2.5, $A(z,x)$ is injective and $A(z,x)^*$ is surjective. The following notions generalize those in Definition 1.3.

Definition 3.1. Let z_s be a curve through z_0 in $\mathcal{G}$ with $\dot{z}_0 := \partial_s z_s|_{s=0} = w$. The family (G_{z_s}) is called a *variation* of G_{z_0}, and we define the associated *variation field*

$$J_w : G_{z_0} \to (N^*G_{z_0})^*, \quad J_w(x) = A(z,x)^*w.$$

Given $z \in \mathcal{G}$ and $x, y \in G_z$ let $V_z(x,y)$ be the space of all possible directions of variation at x among variations of G_z that keep y fixed, i.e.

$$V_z(x,y) = \{J_w(x) \mid w \in T_z\mathcal{G},\ J_w(y) = 0\}.$$

We say that $x, y \in G_z$ are *Z-conjugate* along G_z if $V_z(x,y)$ is a strict subspace of $(N_x^*G_z)^*$.

To justify the definition of $J_w(x)$, let (G_{z_s}) be a variation of G_{z_0} with $\partial_s z_s \mid_{s=0} = w$, and fix some $x_0 \in G_{z_0}$. If x_s is any smooth curve through x_0 with $x_s \in G_{z_s}$, then $(z_s, x_s) \in Z$ and therefore any element in $N^*_{(z_0,x_0)}Z$ annihilates $(\partial_s z_s, \partial_s x_s) \mid_{s=0}$. Since covectors in $N^*_{(z_0,x_0)}Z$ have the form $(A(z_0,x_0)\eta, \eta)$ with $\eta \in N^*_{x_0}G_{z_0}$, we have

$$\eta(\dot{x}_0) = -A(z_0,x_0)\eta(\dot{z}_0) = -J_w(x_0)(\eta), \qquad \eta \in N^*_{x_0}G_{z_0}.$$

Thus $-J_w(x_0)$ describes the normal component of the variation of x_0 within the manifolds G_{z_s}. We could fix a Riemannian metric on X and identify $(N_x^*G_z)^*$ with the orthocomplement of T_xG_z in this metric, and then J_w would become an actual orthogonal vector field on G_{z_0}.

The following lemma gives several characterizations of Z-conjugate points. By (c) the notion of Z-conjugate points only makes sense when $N \geq 2n''$ (that is, if $N < 2n''$ then all pairs of points on any G_z are Z-conjugate).

Lemma 3.1. *Let $x, y \in G_z$ with $x \neq y$. The following conditions are equivalent.*

(a) *x and y are not Z-conjugate along G_z.*
(b) *The space $V_z(x,y)$ has dimension n''.*
(c) *The space $\{w \in T_z\mathcal{G} \mid J_w(x) = J_w(y) = 0\}$ has dimension $N - 2n''$.*
(d) $\ker(A(z,x)^*) + \ker(A(z,y)^*) = T_z\mathcal{G}$.

Proof. The equivalence of (a) and (b) is clear since $\dim((N_x^* G_z)^*) = n''$. For the equivalence of (b) and (c), note that $\ker(A(z,y)^*)$ has dimension $N - n''$ since $A(z,y)^*$ is surjective, and apply the rank-nullity theorem to $A(z,x)^* : \ker(A(z,y)^*) \to (N_x^* G_z)^*$ whose range is $V_z(x,y)$ to get

$$\dim(\{w \in T_z\mathcal{G} \mid J_w(x) = J_w(y) = 0\}) + \dim(V_z(x,y)) = N - n''.$$

Finally, (c) means $\dim(\ker(A(z,x)^*) \cap \ker(A(z,y)^*)) = N - 2n''$, which is equivalent to (d) by the general formula $\dim(E+F) + \dim(E \cap F) = \dim(E) + \dim(F)$. □

The following result gives a characterization for the global part of the Bolker condition.

Lemma 3.2. *Let $(z,\zeta,x,\eta) \in C$. Then $\pi_L^{-1}(z,\zeta) = \{(z,\zeta,x,\eta)\}$ iff*

$$\textit{for any } y \in G_z \setminus \{x\}, \textit{ the space } V_z(x,y) \textit{ is not annihilated by } \eta.$$

Moreover, $\pi_L : C \to T^*\mathcal{G} \setminus 0$ is injective iff there are no pairs of Z-conjugate points on any G_z.

Proof. By Lemma 2.5, the map $A(z,x)$ is injective. Now if $\pi_L(z,\zeta,x,\eta) = \pi_L(z,\zeta,x,\eta_1)$ we have $\eta = \eta_1$, since $\zeta = A(z,x)\eta = A(z,x)\eta_1$. Therefore, π_L is injective if and only if, for each pair of distinct points $(z,x),(z,y) \in Z$, we have a trivial intersection $\mathrm{im}(A(z,x)) \cap \mathrm{im}(A(z,y)) = \{0\}$. This latter condition is equivalent to Lemma 3.1 part (d).

More specifically, $\pi_L^{-1}(z,\zeta) = \{(z,\zeta,x,\eta)\}$ if and only if

$$A(z,x)\eta \notin \mathrm{im}(A(z,x_1)), \qquad \forall x_1 \in G_z \setminus \{x\}.$$

This latter condition is equivalent to the fact that there exists $w \in T_z\mathcal{G} = (T_z^*\mathcal{G})^*$ such that $w(A(z,x)\eta) \neq 0$ and $w(A(z,x_1)\eta_1) = 0$ for all $\eta_1 \in N_{x_1}^* G_z$, namely $A(z,x_1)^* w = 0$. □

3.2. **Injectivity of $d\pi_L$.** The following result gives equivalent conditions for $d\pi_L$ to be injective. The equivalence of (1) and (2) is a general property of Lagrangian manifolds (see Lemma 4.3 of [SdH02]). Condition (3) gives a geometric interpretation: $d\pi_L|_{(z_0,\zeta_0,x_0,\eta_0)}$ is injective if and only if there are so many manifolds G_z (with z near z_0) through x_0 such that one can obtain any direction in $T_{x_0}^* X$ by varying η_0 within vectors conormal to the manifolds G_z. Condition (4) gives another geometric interpretation in terms of tangents of G_z. Finally, conditions (5) and (6) describe injectivity of $d\pi_L$ in terms of the local representations $Z = \{x'' = \phi(z,x')\} = \{z'' = b(x,z')\}$. Condition (5) will be used frequently in the computations in Section 5.

Lemma 3.3. *Let $Z \subset \mathcal{G} \times X$ be a double fibration. Given any $(z_0,\zeta_0,x_0,\eta_0) \in C$, the following conditions are equivalent.*

(1) *$d\pi_L|_{(z_0,\zeta_0,x_0,\eta_0)}$ is injective.*
(2) *$d\pi_R|_{(z_0,\zeta_0,x_0,\eta_0)}$ is surjective.*
(3) *For any $\xi_0 \in T_{x_0}^* X$ there are smooth curves z_s through z_0 in H_{x_0} and η_s through η_0 in $T_{x_0}^* X$ such that $\eta_s \in N_{x_0}^* G_{z_s}$ and $\dot\eta_0 = \xi_0$.*
(4) *If $F : H_{x_0} \to (T_{x_0}X)^{n'}$ is any smooth map such that $\{F_1(z),\dots,F_{n'}(z)\}$ spans $T_{x_0}G_z$ for z near z_0, then the map $(\eta_0(dF_1(\,\cdot\,)),\dots,\eta_0(dF_{n'}(\,\cdot\,)))$ takes $T_{z_0}H_{x_0}$ onto $\mathbb{R}^{n'}$.*
(5) *If Z is given by $\{x'' = \phi(z,x')\}$ near (z_0,x_0), then*

$$\textit{the } N \times n \textit{ matrix } \Big(\phi_z(z_0,x_0')^T,\ \ \partial_{x'}(\phi_z(z_0,x')^T\eta_0'')|_{x'=x_0'}\Big) \textit{ is injective.}$$

(6) *If Z is given by $\{z'' = b(x,z')\}$ near (z_0,x_0), then*

$$\textit{the } n \times N \textit{ matrix } \Big(b_x(x_0,z_0')^T,\ \ \partial_{z'}(b_x(x_0,z')^T\zeta_0'')|_{z'=z_0'}\Big) \textit{ is surjective.}$$

Proof. By Lemma 2.3, we can express Z as the set $\{x'' = \phi(z, x')\}$ near (z_0, x_0). By the description of C in Lemma 2.4, any curve Γ_s in C with $\Gamma_0 = (z_0, \zeta_0, x_0, \eta_0)$ can be written as

$$\Gamma_s = (z_s, \phi_z(z_s, x'_s)^T \eta''_s, (x'_s, \phi(z_s, x'_s)), (-\phi_{x'}(z_s, x'_s)^T \eta''_s, \eta''_s))$$

where z_s, x'_s and η''_s are curves in $\mathbb{R}^N$, $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$, respectively. We have chosen signs so that $\eta_0 = (-\phi_{x'}(z_0, x'_0)^T \eta''_0, \eta''_0)$.

We first show (1) $\iff$ (5). Now

$$d\pi_L(\dot\Gamma_0) = (\dot z_0, \partial_s(\phi_z(z_s, x'_s)^T \eta''_0)|_{s=0} + \phi_z(z_0, x'_0)^T \dot\eta''_0).$$

It follows that $d\pi_L(\dot\Gamma_0) = 0$ if and only if

$$\dot z_0 = 0, \qquad \phi_z(z_0, x'_0)^T \dot\eta''_0 + \partial_{x'}(\phi_z(z_0, x')^T \eta''_0)|_{x'=x'_0} \dot x'_0 = 0.$$

Thus $d\pi_L|_{(z_0,\zeta_0,x_0,\eta_0)}$ is injective if and only if (5) holds.

Next we show (2) $\iff$ (5). We have

$$d\pi_R(\dot\Gamma_0) = ((\dot x'_0, \partial_s(\phi(z_s, x'_s))|_{s=0}), (-\partial_s(\phi_{x'}(z_s, x'_s)^T \eta''_s)|_{s=0}, \dot\eta''_0)).$$

Thus in matrix form

$$d\pi_R(\dot\Gamma_0) = \begin{pmatrix} I & 0 & 0 \\ * & \phi_z(z_0, x'_0) & 0 \\ * & -\partial_z(\phi_{x'}(z, x'_0)^T \eta''_0)|_{z=z_0} & * \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \dot x'_0 \\ \dot z_0 \\ \dot\eta''_0 \end{pmatrix}.$$

Then $d\pi_R|_{(z_0,\zeta_0,x_0,\eta_0)}$ is surjective if and only if

$$\begin{pmatrix} \phi_z(z_0, x'_0) \\ -\partial_z(\phi_{x'}(z, x'_0)^T \eta''_0)|_{z=z_0} \end{pmatrix} \text{ has full rank.} \tag{3.1}$$

After taking the transpose and noting that $(\partial_z(\phi_{x'}^T \eta''))^T = \partial_{x'}(\phi_z^T \eta'')$, the last condition is equivalent with the matrix $(\phi_z^T, -\partial_{x'}(\phi_z^T \eta''_0))$ being injective, which is equivalent with (5).

The next step is to show (3) $\iff$ (5). Assume that z_s is a curve through z_0 in H_{x_0} and η_s is a curve through η_0 in $T^*_{x_0}X$ such that $\eta_s \in N^*_{x_0} G_{z_s}$. Since $z_s \in H_{x_0}$ we have $\phi(z_s, x'_0) = x''_0$, which implies $\phi_z(z_0, x'_0)\dot z_0 = 0$. We also have

$$\eta_s = (-\phi_{x'}(z_s, x'_0)^T \eta''_s, \eta''_s)$$

where η''_s is a curve in $\mathbb{R}^{n''}$. We have

$$\dot\eta_0 = (-\partial_z(\phi_{x'}(z, x'_0)^T \eta''_0)|_{z=z_0} \dot z_0 - \phi_{x'}(z_0, x'_0)^T \dot\eta''_0, \dot\eta''_0).$$

It follows that the condition in (3) holds if and only if

$$\partial_z(\phi_{x'}(z, x'_0)^T \eta''_0)|_{z=z_0} : \ker(\phi_z(z_0, x'_0)) \to \mathbb{R}^{n'} \text{ is surjective.}$$

Since Z is a double fibration, $\dim(\ker(\phi_z)) = N - n'' \geq n'$ using Lemma 2.5. Thus (3) holds if and only if $\partial_z(\phi_{x'}^T \eta''_0)|_{\ker(\phi_z)}$ has rank n', which is equivalent with its kernel $\ker(\phi_z) \cap \ker(\partial_z(\phi_{x'}^T \eta''_0))$ having dimension $N - n$. This is again equivalent with (3.1), which was seen above to be equivalent with (5).

Next we show (3) $\iff$ (4). Assume that (3) holds, fix $\xi_0 \in T^*_{x_0}X$ and let z_s and η_s be as in (3). If F is as in (4), then $\eta_s(F_j(z_s)) = 0$ for $1 \leq j \leq n'$. Taking derivatives at $s = 0$, we get

$$\xi_0(F_j(z_0)) + \eta_0(dF_j(\dot z_0)) = 0, \qquad 1 \leq j \leq n'. \tag{3.2}$$

Since ξ_0 was arbitrary, this implies the conclusion in (4). Conversely, assume (4), let $\{e_1, \dots, e_n\}$ be a basis of $T^*_{x_0}X$ with $e_j = F_j(z_0)$ for $1 \leq j \leq n'$, and let $\xi_0 \in T^*_{x_0}X$. We use (4) to find a curve z_s

through z_0 in H_{x_0} such that (3.2) holds. By Lemma 2.4, after writing Z locally as $\{x'' = \phi(z, x')\}$, any curve η_s through η_0 in $T^*_{x_0}X$ with $\eta_s \in N^*_{x_0}G_{z_s}$ has the form

$$\eta_s = (-\phi_{x'}(z_s, x'_0)\eta''_s, \eta''_s)$$

where η''_s is a curve through η''_0 in $\mathbb{R}^{n''}$. By choosing $\dot{\eta}''_0$ in a suitable way, we can ensure that η_s satisfies

$$\dot{\eta}_0(e_j) = \xi_0(e_j), \qquad n' + 1 \le j \le n.$$

Since $\eta_s(F_j(z_s)) = 0$, taking the derivative at $s = 0$ and using (3.2) gives

$$\dot{\eta}_0(e_j) = \xi_0(e_j), \qquad 1 \le j \le n'.$$

Thus $\dot{\eta}_0 = \xi_0$, which shows (3).

Finally, we show the equivalence (6) $\Longleftrightarrow$ (2). When Z is given by $\{z'' = b(x, z')\}$ near (z_0, x_0), the projection $d\pi_R$ is given in matrix form by

$$d\pi_R = \begin{pmatrix} \partial_{z'}(b_x(x_0, z')^T\zeta''_0)|_{z'=z'_0} & b_x(x_0, z'_0)^T & * \\ 0 & 0 & I \end{pmatrix}.$$

Therefore, $d\pi_R$ is surjective if and only if the matrix $\big(b_x(x_0, z'_0)^T,\ \partial_{z'}(b_x(x_0, z')^T\zeta''_0)|_{z'=z'_0}\big)$ has full rank. □

If $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective, then π_L is injective in some neighborhood of (z, ζ, x, η) in C. We conclude this section with a lemma stating that π_L is even injective if one restricts to a sufficiently small neighborhood of (z, ζ) and a neighborhood of x. This lemma will not be used explicitly later, but it justifies the claim made in Step 2 of Section 1.4.

Lemma 3.4. *Suppose that $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective. There are neighborhoods $\mathcal{V}$ of (z, ζ) in $T^*\mathcal{G}$ and U of x in X such that π_L is injective in $\tilde{C}$, where*

$$\tilde{C} = \{(\tilde{z}, \tilde{\zeta}, \tilde{x}, \tilde{\eta}) \in C \mid (\tilde{z}, \tilde{\zeta}) \in \mathcal{V},\ \tilde{x} \in U\}.$$

Proof. We argue by contradiction and assume that there are sequences $(z_j, \zeta_j) \to (z, \zeta)$, $x_{j,k} \to x$ as $j \to \infty$, and $\eta_{j,k} \in T^*_{x_{j,k}}X$ for $k = 1, 2$ such that $(z_j, \zeta_j, x_{j,k}, \eta_{j,k}) \in C$ but $(x_{j,1}, \eta_{j,1}) \neq (x_{j,2}, \eta_{j,2})$. By Lemma 2.6 we have

$$\eta_{j,k} = -B(z_j, x_{j,k})\zeta_j.$$

Since B is smooth and $\eta = -B(z, x)\zeta$, we have $\eta_{j,1} \to \eta$ and $\eta_{j,2} \to \eta$. Then $(x_{j,1}, \eta_{j,1}) \neq (x_{j,2}, \eta_{j,2})$ are in a very small neighborhood of (x, η) for j large. This contradicts the fact that π_L is injective near (z, ζ, x, η) in C. □

4. The case of ray transforms

In this section we consider double fibration transforms as in Section 2 in the special case where the manifolds G_z are one-dimensional. We will show that each G_z is an integral curve of some vector field Y on Z. This provides an alternative point of view to double fibration ray transforms. We also discuss the Bolker condition for ray transforms and its relation to conjugate points..

4.1. **Double fibration vector fields.** We show that in a double fibration where the fibers G_z are one-dimensional, the fibers are related to integral curves of a vector field:

Lemma 4.1. *Let M be a manifold with smooth boundary, and let $\mathcal{G}$ be an orientable manifold without boundary. Suppose that Z is an embedded orientable submanifold of $\mathcal{G}\times M^{\mathrm{int}}$ with $\dim(Z) = \dim(\mathcal{G})+1$, let $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ be a surjective submersion with connected fibers, and let $\pi_M : Z \to M^{\mathrm{int}}$ be a proper surjective submersion.*

Then Z is a smooth fiber bundle over M^{int}, and there is a vector field Y on Z with $d\pi_M(Y)$ nonvanishing such that $\pi_{\mathcal{G}}^{-1}(z)$ is an integral curve of Y for any $z \in \mathcal{G}$. The vector field Y is unique up to multiplication by a nonvanishing function on Z. Each integral curve is either periodic, or its projection on M is injective. If we fix orientation forms on Z and $\mathcal{G}$ and let ω_{G_z} be the induced orientation form on G_z, then Y can be chosen so that

$$\int_{G_z} f\,\omega_{G_z} = \int_{-\tau_-(z)}^{\tau_+(z)} f(\pi_M(\gamma_z(t)))\,dt \tag{4.1}$$

for any f such that this is well defined. Here $\gamma_z : (-\tau_-(z), \tau_+(z)) \to Z$ is the maximally extended integral curve of Y through z (if γ_z is periodic then its domain is chosen to be one of its minimal periods). The parametrization of curves is nonredundant in the following sense: given any $(z_0, x_0) \in Z$ there is a neighborhood V of z_0 in $\mathcal{G}$ that can be identified with a submanifold $\tilde{V}$ through (z_0, x_0) in Z such that the integral curves of Y are transverse to $\tilde{V}$.

Proof. Since $\pi_M : Z \to M^{\mathrm{int}}$ is a proper surjective submersion, Ehresmann's lemma ensures that π_M is a locally trivial fibration and hence Z is a fiber bundle over M^{int}. Let $\omega_{\pi_{\mathcal{G}}^{-1}(z)}$ be the orientation form on $\pi_{\mathcal{G}}^{-1}(z)$ as in Lemma 2.1.

We can now define $Y(z, x)$ to be the unique positively oriented vector in the one-dimensional space $T_{(z,x)}\pi_{\mathcal{G}}^{-1}(z)$ that satisfies $\omega_{\pi_{\mathcal{G}}^{-1}(z)}(Y) = 1$. Then Y is a smooth vector field on Z with $d\pi_{\mathcal{G}}(Y) = 0$, and for any $(z, x_0) \in Z$ the maximally extended integral curve of Y through (z, x_0) is a component of $\pi_{\mathcal{G}}^{-1}(z)$, hence equal to $\pi_{\mathcal{G}}^{-1}(z)$ since the fibers are connected. The vector field Y is unique up to multiplication by a nonvanishing function since any such Y must lie in the one-dimensional space $T_{(z,x)}\pi_{\mathcal{G}}^{-1}(z)$. We have $d\pi_M(Y) \neq 0$ since any $v \in TZ$ with $d\pi_{\mathcal{G}}(v) = d\pi_M(v) = 0$ must satisfy $v = 0$. If $\gamma_z(t)$ is an integral curve of Y and if $\pi_M(\gamma_z(t_1)) = \pi_M(\gamma_z(t_2))$, then from $\pi_{\mathcal{G}}(\gamma_z(t_1)) = \pi_{\mathcal{G}}(\gamma_z(t_2))$ we obtain $\gamma_z(t_1) = \gamma_z(t_2)$. Thus each integral curve is either periodic or its projection to M is injective. The formula (4.1) follows since $\omega_{\pi_{\mathcal{G}}^{-1}(z)}(Y) = 1$ and

$$\omega_{G_z}(d\pi_M(v)) = \omega_{\pi_{\mathcal{G}}^{-1}(z)}(v), \qquad v \in T_{(z,x)}\pi_{\mathcal{G}}^{-1}(z).$$

Finally, given $(z_0, x_0) \in Z$ we use Lemma 2.3 to write Z locally in a neighborhood $V \times U'$ of (z_0, x_0') as $\{x'' = \phi(z, x')\}$. Then $\tilde{V} = \{(z, x_0', \phi(z, x_0')) \mid z \in V\}$ is a submanifold of Z that can be identified with V. The tangents of $\tilde{V}$ are of the form $(w, 0', \phi_z(z, x_0')w)$, whereas Y takes the form $(0, \lambda, \phi_{x'}(z, x_0')\lambda)$. □

Remark 4.2. See [Hel11] for classical examples of double fibration ray transforms integrating over periodic curves.

We will next prove a converse to Lemma 4.1, in the setting already discussed in Section 1.1. Assume that (M, g) is an oriented manifold with smooth boundary, and $\pi : \Xi \to M$ is a smooth fiber bundle over M whose fibers Ξ_x are manifolds without boundary (then $\partial\Xi = \pi^{-1}(\partial M)$). Let

Y be a vector field in Ξ with flow Φ_t, and consider its "horizontal projection"

$$Y^h = d\pi \circ Y.$$

Given any $z \in \Xi$ we consider the maximally extended integral curve $\gamma_z : [-\tau_-(z), \tau_+(z)] \to \Xi$, $\gamma_z(t) = \Phi_t(z)$, so that $\dot\gamma_z(t) = Y(\gamma_z(t))$ and $\gamma_z(0) = z$. Note that the functions $\tau_\pm$ are not necessarily continuous and in general $\tau_\pm(z) \in [0, \infty]$, though we will only consider z for which $\tau_\pm(z) < \infty$. For any curve $\gamma_z(t)$, there is a corresponding base space curve $x_z(t) = \pi(\gamma_z(t))$ whose tangent vector is $\dot x_z(t) = Y^h(\Phi_t(z)))$.

Let $\mathcal{G}$ be a submanifold of Ξ such that Y is never tangent to $\mathcal{G}$ (e.g. $\mathcal{G}$ could be an open subset of $\partial_+\Xi$). We consider the set

$$Z = \{(z, x_z(t)) \mid z \in \mathcal{G},\ t \in (-\tau_-(z), \tau_+(z))\} \subset \mathcal{G} \times M.$$

The following converse to Lemma 4.1 shows that under certain conditions Z is a double fibration. Each part below exactly corresponds to the related part in Definition 1.2.

Lemma 4.3. *Let $D = \{(z,t) \mid z \in \mathcal{G},\ t \in (-\tau_-(z), \tau_+(z))\}$ and consider the map*

$$F : D \to \mathcal{G} \times M, \quad F(z,t) := (z, x_z(t)).$$

(i) *F maps into $\mathcal{G} \times M^{\mathrm{int}}$ iff $x_z(t) \in M^{\mathrm{int}}$ for $z \in \mathcal{G}$ and $t \in (-\tau_-(z), \tau_+(z))$.*
(ii) *F is injective iff the curves $x_z(t)$ do not self-intersect for $z \in \mathcal{G}$.*
(iii) *F is an immersion iff $\dot x_z(t) \neq 0$ for $(z,t) \in D$.*
(iv) *If the conditions in* (i)–(iii) *hold and additionally $\tau_\pm(z) < \infty$ for all $z \in \mathcal{G}$, then F is an embedding of D into $\mathcal{G} \times M^{\mathrm{int}}$.*
(v) *If the conditions in* (i)–(iv) *hold and additionally $V_z(t) \oplus \mathbb{R}Y^h = T_{x_z(t)}M$ for all $z \in \mathcal{G}$ and all t, then $\pi_M : Z \to M^{\mathrm{int}}$ is a submersion and $Z = F(D) \subset \mathcal{G} \times M^{\mathrm{int}}$ is a double fibration. In this case,*

$$N^*Z = \{(z, (\pi \circ \Phi_t)^*\eta, x_z(t), -\eta) \mid z \in \mathcal{G},\ t \in (-\tau_-(z), \tau_+(z)),\ \eta(\dot x_z(t)) = 0\}. \tag{4.2}$$

If $\mathcal{G}$ is an open subset of $\partial\Xi$, or more generally if $\dim(\mathcal{G}) = \dim(\Xi) - 1$, the condition $V_z(t) \oplus \mathbb{R}Y^h = T_{x_z(t)}M$ holds automatically.

Proof. (i) is clear. For (ii), one has $F(z_1,t_1) = F(z_2,t_2)$ iff $z_1 = z_2 = z$ and $x_z(t_1) = x_z(t_2)$. This shows (ii).

(iii) Let (z_s, t_s) be a curve in D. Since $x_z(t) = \pi(\Phi_t(z))$, we have

$$dF(\dot z_0, \dot t_0) = (\dot z_0, \dot x_{z_0}(t_0)\dot t_0 + d\pi(d\Phi_{t_0}(\dot z_0))). \tag{4.3}$$

We see that $dF|_{(z_0,t_0)}$ is injective iff $\dot x_{z_0}(t_0) \neq 0$.

(iv) Suppose that $\tau_+(z) < \infty$ for all $z \in \mathcal{G}$, and let (z_j, t_j) be a sequence in D with $(z_j, x_{z_j}(t_j)) \to (z,x)$ in $\mathcal{G} \times M^{\mathrm{int}}$. Since τ_+ is upper semicontinuous by Lemma 4.4, we have $\limsup \tau_+(z_j) \le \tau_+(z) < \infty$. Thus $t_j \le \tau_+(z)$, and similarly $t_j \ge -\tau_-(z)$. In particular (t_j) is a bounded sequence, and hence after replacing (t_j) by a subsequence we have $t_j \to t_0 \in [-\tau_-(z), \tau_+(z)]$. This implies that $x = x_z(t_0)$, and $t_0 \in (-\tau_-(z), \tau_+(z))$ since $x \in M^{\mathrm{int}}$. It follows that $F(D)$ is closed in $\mathcal{G} \times M^{\mathrm{int}}$. Moreover, if the conditions in (i)–(iii) also hold, then F is an embedding.

(v) Since F is an embedding, any smooth curve in Z is of the form $\rho_s = F(z_s, t_s) = (z_s, x_{z_s}(t_s))$ where z_s and t_s are smooth curves in $\mathcal{G}$ and $\mathbb{R}$, respectively. Then

$$d\pi_{\mathcal{G}}(\dot\rho_0) = \dot z_0,$$

which shows that $d\pi_{\mathcal{G}}$ is surjective. Similarly,

$$d\pi_M(\dot\rho_0) = \partial_s(\pi(\Phi_{t_s}(z_s))|_{s=0} = d\pi(Y(\Phi_{t_0}(z_0))\dot t_0 + d\Phi_{t_0}(\dot z_0)). \tag{4.4}$$

According to Definition 1.1, $d\pi(d\Phi_t(\dot z_0)) = J_{\dot z_0}(t)$. We also note for later purposes that

$$Y(\Phi_t(z)) = \partial_t\Phi_t(z) = \partial_s(\Phi_t(\Phi_s(z)))|_{s=0} = d\Phi_t(Y(z)). \tag{4.5}$$

This shows that

$$d\pi_M(\dot\rho_0) = J_{\dot z_0}(t_0) + \dot t_0 Y^h(\Phi_{t_0}(z_0)).$$

The assumption $V_z(t) \oplus \mathbb{R}Y^h = T_{x_z(t)}M$ for all z, t ensures that π_M is a submersion. Thus Z is a double fibration. The formula (4.2) for N^*Z follows since vectors in $T_{(z,x)}Z$ take on the form (4.3) (see also Lemma 2.4). Finally, if $\dim(\mathcal{G}) = \dim(\Xi) - 1$, then $T_z\mathcal{G} + \mathbb{R}Y = T_z\Xi$ by the assumption that Y is never tangent to $\mathcal{G}$. Since Φ_t is a flow, $d\Phi_t$ is an isomorphism. By (4.4) and (4.5) we have $d\pi_M(\dot\rho_0) = d\pi(d\Phi_{t_0}(Y(z_0)\dot t_0 + \dot z_0))$ and thus $V_z(t) \oplus \mathbb{R}Y^h = T_{x_z(t)}M$. □

Lemma 4.4. *The map $z \mapsto \tau_+(z)$ is upper semicontinuous.*

Proof. Let $z_j \to z$ and suppose by contradiction that $\limsup \tau_+(z_j) > \tau_+(z)$. Without loss of generality this means that there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ we have that $\tau_+(z_j) > \tau_+(z) + \epsilon$ for all $j \in \mathbb{N}$. This means that there exists a bounded sequence $t_j \in (\tau_+(z) + \epsilon, \tau_+(z_j))$ with $t_j \to \tau_+(z) + \epsilon$ such that $\Phi_{t_j}(z_j) \in \Xi$. By the joint continuity of $\Phi_\cdot(\cdot)$ we get that $\Phi_{\tau_+(z)+\epsilon}(z) \in \Xi$ since Ξ is closed. This is true for all $\epsilon \in (0, \epsilon_0)$, which contradicts the definition of $\tau_+(z)$. □

4.2. **Ray transforms satisfying the Bolker condition.** We will next look at the Bolker condition in the special case of ray transforms. Let z_s be a smooth curve in $\mathcal{G}$ with $z_0 = z$ and $\partial_s z_s|_{s=0} = w$. In this subsection we slightly change notation and denote by $\tilde J_w(x)$ the variation field along G_z as in Definition 3.1. We can relate $\tilde J_w$ with the vector field $J_w : (-\tau_-(z), \tau_+(z)) \to TM$ along x_z from Definition 1.1, given by

$$J_w(t) = \partial_s x_{z_s}(t)|_{s=0} = d\pi(d\Phi_t(w)).$$

For all $\eta \in N^*_{x_z(t)}G_z$, since $\eta(J_w(t)) = \eta((\pi \circ \Phi_t|_{\mathcal{G}})_* w) = (\pi \circ \Phi_t|_{\mathcal{G}})^*\eta(w) = (A(z, x_z(t))\eta)(w)$, we have

$$\eta(J_w(t)) = \tilde J_w(x_z(t))(\eta).$$

Thus $\tilde J_w(x_z(t)) = 0$ iff $J_w(t)$ is tangential, i.e. $J_w(t) \parallel \dot x_z(t)$. This means that $\tilde J_w(x_z(t))$ can be identified with $J_w(t)$, when the latter is considered as an element of $T_{x_z(t)}M/T_{x_z(t)}G_z$.

Recall from Definition 1.3 the space

$$V_z(t, s) = \{J_w(t) \mid w \in T_z\mathcal{G},\ J_w(s) \parallel \dot x_z(s)\},$$

If $t \neq s$, we say that $x_z(t)$ and $x_z(s)$ are *Z-conjugate* along x_z if $V_z(t, s) + \mathbb{R}Y^h$ is a strict subspace of $T_{x_z(t)}M$. Now $V_z(t, s) + \mathbb{R}Y^h = T_{x_z(t)}M$ iff $V_z(x_z(t), x_z(s))$ is all of $(N^*_{x_z(t)}G_z)^*$, which implies that this notion of Z-conjugate points is equivalent to the one in Definition 3.1. From Lemma 3.1, we see that the following are equivalent:

(a) $x_z(t)$ and $x_z(s)$ are not Z-conjugate along x_z.
(b) $V_z(t, s) + \mathbb{R}Y^h = T_{x_z(t)}M$.
(c) The space $\{w \in T_z\mathcal{G} \mid J_w(t), J_w(s) \text{ are tangential}\}$ has dimension $N - (2n - 2)$.

Remark 4.5. It follows that when $N < 2n - 2$, all pairs of points on any x_z are Z-conjugate. Moreover, when $N = 2n-2$ the notion of two points being Z-conjugate is equivalent to the existence of a nontrivial variation field that is tangential at the two points.

In particular, if $\Xi = SM$ and $Y = X_g$ is the geodesic vector field, then we have the geodesic X-ray transform. Variation fields are standard Jacobi fields $J_w(t) = d\pi(d\Phi_t(w))$, but w is now restricted to lie in $T_z\mathcal{G}$ instead of $T_z(TM)$. Note that the tangential Jacobi field $\dot{x}_z(t)$ formally corresponds to taking $w = Y$, and $t\dot{x}_z(t)$ corresponds to the radial direction in $T_z(TM)$. In the usual case where $\dim(\mathcal{G}) = 2n-2$ and $\mathcal{G}$ is never tangent to Y, Z-conjugate points for the geodesic X-ray transform in the sense defined above correspond precisely to conjugate points for geodesics in the standard sense. Indeed, (c) above fails iff there is $w \in T_z\mathcal{G} \setminus 0$ such that $J_w(t)$ and $J_w(s)$ are tangential to x_z, and after subtracting a linear combination of $\dot{x}_z(r)$ and $r\dot{x}_z(r)$ from $J_w(r)$ this is equivalent with having a nontrivial Jacobi field such that $J(t) = J(s) = 0$.

The following result shows that the Bolker condition can indeed be characterized as in Definition 1.4.

Lemma 4.6. *Let R be a double fibration ray transform and let $(z, \zeta, x, \eta) \in C$ with $x = x_z(t)$.*

(a) $\pi_L^{-1}(z, \zeta) = \{(z, \zeta, x, \eta)\}$ *iff* $V_z(t, s)$ *is not annihilated by* η *for any* $s \neq t$.
(b) π_L *is injective iff there are no pairs of* Z*-conjugate points on any* x_z.
(c) $d\pi_L|_{(z,\zeta,x,\eta)}$ *is injective iff* $d(Y^h(x, \xi(\,\cdot\,)))(T_zH_x)$ *is not annihilated by* η*, where* $(x, \xi(\tilde{z}))$ *is the unique point over* x *on the integral curve through* $\tilde{z} \in H_x$.
(d) *Let* $\dim(\mathcal{G}) = \dim(\Xi) - 1$. *Then* $d\pi_L|_{(z,\zeta,x,\eta)}$ *is injective iff* $d(Y^h(x, \,\cdot\,))(T_{(x,\xi(z))}\Xi_x)$ *is not annihilated by* η.

Proof. Part (a) follows from Lemma 3.2 and the identification of $J_w(t)$, modulo tangential vectors, with $\tilde{J}_w(x_z(t))$. Part (b) follows from part (a).

Part (c) is a consequence of Lemma 3.3 part (4) with the choice $F : H_x \to T_xM$, $F(\tilde{z}) = Y^h(\Phi_{t(\tilde{z})}(\tilde{z}))$ where $t(\tilde{z})$ is the unique time such that $x_{\tilde{z}}(t(\tilde{z})) = x$. The map $\tilde{z} \mapsto t(\tilde{z})$ is smooth by the implicit function theorem, and since $\Phi_{t(\tilde{z})}(\tilde{z}) = (x, \xi(\tilde{z}))$ it follows that also $\xi : H_x \to \Xi_x$ is smooth.

For part (d), note that when $\dim(\mathcal{G}) = \dim(\Xi) - 1$, one has $\dim(H_x) = \dim(\Xi_x)$ and thus ξ maps between spaces of the same dimension. If z_s is a curve in H_x with $z_0 = z$, we use (4.5) to obtain

$$\begin{aligned} d\xi(\dot{z}_0) &= \partial_s(\Phi_{t(z_s)}(z_s))|_{s=0} = d\Phi_{t(z_0)}(\dot{z}_0) + Y(\Phi_{t(z_0)}(z_0))\partial_s(t(z_s))|_{s=0} \\ &= d\Phi_{t(z_0)}(\dot{z}_0 + Y(z_0)\partial_s(t(z_s))|_{s=0}). \end{aligned}$$

Now $d\Phi_t$ is an isomorphism and Y is never tangent to H_x. Thus $d\xi$ is injective and hence invertible. We obtain

$$d(Y^h(x, \xi(\,\cdot\,)))(T_zH_x) = d(Y^h(x, \,\cdot\,))(T_{(x,\xi(z))}\Xi_x).$$

The result follows from (c). □

4.3. **Example 1: Geodesic X-ray transform.** Let us consider the geodesic X-ray transform (see [PSU23] for more details). Let (M, g) be an oriented manifold with smooth boundary, let $SM = \{(x, v) \in TM \mid |v|_g = 1\}$ be the unit sphere bundle, and let $\partial_+SM = \{\ (x, v) \in SM \mid x \in \partial M,\ g(v, \nu) < 0\}$ where ν is the unit outer normal to ∂M. We take $\Xi = SM$ and $Y = X_g$, where X_g is the geodesic vector field. Let $\mathcal{G}$ be a submanifold of ∂_+SM, so Y is never tangent to $\mathcal{G}$. Let also $\kappa \in C^\infty(\mathcal{G} \times M^{\text{int}})$ be nowhere vanishing. Then the curves $x_z(t)$ are unit speed geodesics, and the weighted geodesic X-ray transform is given by

$$Rf(z) = \int_0^{\tau_+(z)} \kappa(z, x_z(t)) f(x_z(t))\, dt, \qquad z \in \mathcal{G}. \tag{4.6}$$

Let us assume the following conditions for all $z \in \mathcal{G}$:

(i) (No tangential intersections) $x_z(t) \in M^{\text{int}}$ for $t \in (0, \tau_+(z))$;
(ii) (No self-intersections) $t \mapsto x_z(t)$ is injective;
(iv) (Nontrapping) $\tau_+(z) < \infty$;
(v) (Enough variations) $V_z(t) + \mathbb{R}\dot{x}_z(t) = T_{x_z(t)}M$ for $t \in (0, \tau_+(z))$.

Under these conditions, Lemma 4.3 ensures that R is a ray transform coming from a double fibration. The no singular points condition (iii) always holds since $|\dot{x}_z(t)| = 1$. Condition (i) holds e.g. if ∂M is strictly convex. Conditions (ii) and (iv) hold e.g. if (M, g) is a simple manifold, but they may fail for trapping manifolds such as the catenoid. As noted above, (v) is automatically satisfied if $\mathcal{G}$ is an open subset of $\partial_+ SM$, i.e. $\dim(\mathcal{G}) = 2n-2$. Condition (v) holds also in many other cases, e.g. if $M \subset \mathbb{R}^n$, g is the Euclidean metric, and $\mathcal{G}$ consists of all lines whose direction vector is orthogonal to e_n.

Let us next study the Bolker condition at $(z, \zeta, x, \eta) \in C$ where $x = x_z(t)$. In the usual case where $\dim(\mathcal{G}) = 2n-2$, the map $d\pi_L$ is always injective by Lemma 4.6 part (d) since $Y^h(x, \cdot)$ is the identity map. Moreover, if the geodesics through $z \in \mathcal{G}$ have no conjugate points in the usual sense, then π_L is injective by Lemma 4.6 part (b) and Remark 4.5. Even if there are conjugate points or $\dim(\mathcal{G}) < 2n-2$, the Bolker condition might still hold at some points $(z, \zeta, x, \eta) \in C$, but then one has to verify the conditions in Lemma 4.6 parts (a) and (c) instead.

We collect some of the above results in the following proposition, formulated in terms of a fixed geodesic x_{z_0} that is never tangent to ∂M (hence also geodesics x_z for z near z_0 are never tangent). This result is already contained e.g. in [SU08].

Proposition 4.7. *Let (M, g) be a manifold with smooth boundary, let $z_0 \in \partial_+ SM$ satisfy $\tau_+(z_0) < \infty$, and assume that the geodesic $x_{z_0} : [0, \tau_+(z_0)] \to M$ does not self-intersect, meets ∂M transversally at the endpoints, and otherwise stays in M^{int}. Let also $\kappa \in C^\infty(\mathcal{G} \times M^{\text{int}})$ be nowhere vanishing. If $\mathcal{G}$ is a sufficiently small neighborhood of z_0 in $\partial_+ SM$, then the geodesic ray transform given by (4.6) is a double fibration ray transform. If there are no conjugate points along x_{z_0}, then the Bolker condition is satisfied at every $(z_0, \zeta, x, \eta) \in C$ where $x = x_{z_0}(t)$ for some t.*

Proof. The only thing to check is that if x_{z_0} does not self-intersect, then x_z does not self-intersect for z close to z_0. We argue by contradiction and suppose that $z_j \to z_0$ but $x_{z_j}(t_j) = x_{z_j}(s_j)$ for some $t_j, s_j \in [0, \tau_+(z_j)]$ with $t_j \neq s_j$. Since $x_{z_0}(t)$ meets ∂M transversally at the endpoints, τ_+ is smooth near z_0, and hence by compactness, after passing to a subsequence, we have $t_j \to t_0$ and $s_j \to s_0$. Since x_{z_0} is injective, we must have $t_0 = s_0$. In local coordinates near $x_{z_0}(t_0)$ we write

$$x_z(t) - x_z(s) = \dot{x}_z(s)(t-s) + O((t-s)^2)$$

where the implied constant depends on second derivatives of $(z, t) \mapsto x_z(t)$. Since these second derivatives are bounded, for j large one has

$$0 = |x_{z_j}(t_j) - x_{z_j}(s_j)| \geq \frac{1}{2}|\dot{x}_{z_j}(s_j)||t_j - s_j|.$$

This is a contradiction since $t_j \neq s_j$ and since $|\dot{x}_{z_j}(s_j)|$ has a uniform lower bound for j large. □

4.4. **Example 2: Null bicharacteristic ray transform.** Let M be an n-manifold with smooth boundary, let $\pi : T^*M \setminus 0 \to M$ be the natural projection, and let $p \in C^\infty(T^*M \setminus 0)$. Let H_p be the Hamilton vector field, and note that the fiberwise derivative $\nabla_\xi p$ is related to the Hamilton vector field H_p by

$$\nabla_\xi p = d\pi(H_p).$$

Then $\nabla_\xi p(x,\xi) \in T_xM$, and in local coordinates $\nabla_\xi p(x,\xi) = \partial_{\xi_j} p(x,\xi)\partial_{x_j}$. The Hessian of $p(x,\xi)$ with respect to ξ is defined as the derivative

$$\nabla_\xi^2 p(x,\xi) = d(\nabla_\xi p(x,\,\cdot\,))|_\xi.$$

It follows that $\nabla_\xi^2 p(x,\xi) : T_x^*M \to T_xM$. In local coordinates

$$\nabla_\xi^2 p(x,\xi)\theta = \partial_{\xi_j\xi_k} p(x,\xi)\theta_k \partial_{x_j}.$$

Define $\Xi = p^{-1}(0) = \{(x,\xi) \in T^*M \setminus 0 \mid p(x,\xi) = 0\}$. We make the standing assumption that

$$\nabla_\xi p \neq 0 \text{ everywhere on } \Xi.$$

Then Ξ is a smooth fiber bundle over M, and each fiber $\Xi_x = \{\xi \in T_x^*M \setminus 0 \mid p(x,\xi) = 0\}$ is a smooth $(n-1)$-dimensional submanifold of T_x^*M. For any $\xi \in \Xi_x$, we have the identification

$$T_\xi \Xi_x = \{\eta \in T_x^*M \mid \eta(\nabla_\xi p(x,\xi)) = 0\} = (\nabla_\xi p(x,\xi))^\perp. \tag{4.7}$$

Let $Y = H_p|_\Xi$ be the Hamilton vector field, which is tangent to Ξ. As before, let Φ_t be the flow of Y with $\gamma_z(t) = \Phi_t(z)$ and $x_z(t) = \pi(\gamma_z(t))$. Let $\mathcal{G}$ be an open subset of $\partial_+\Xi$. Let also $\kappa \in C^\infty(\mathcal{G} \times M^{\mathrm{int}})$ be a nowhere vanishing function. This setting gives rise to the weighted null bicharacteristic ray transform

$$Rf(z) = \int_{-\tau_-(z)}^{\tau_+(z)} \kappa(z, x_z(t)) f(x_z(t))\, dt, \qquad z \in \mathcal{G}, \tag{4.8}$$

whenever this expression is well defined. Since $\dot{x}_z = \nabla_\xi p$ is nonvanishing on Ξ, the curves x_z have no singular points. Moreover, since $\dim(\mathcal{G}) = 2n-2$, the variation condition (v) in Lemma 4.3 is satisfied. Then R will be a double fibration ray transform if we assume the following:

(i) (No tangential intersections) $x_z(t) \in M^{\mathrm{int}}$ for $z \in \mathcal{G}$ and $t \in (-\tau_-(z), \tau_+(z))$;
(ii) (No self-intersections) $t \mapsto x_z(t)$ is injective for all $z \in \mathcal{G}$;
(iv) (Nontrapping) $\tau_\pm(z) < \infty$ for all $z \in \mathcal{G}$.

Next we consider the Bolker condition. From now on we assume that p is homogeneous of degree m in ξ, which will make the statements below cleaner. The following lemma shows that for the null bicharacteristic ray transform, there is always one normal direction missing from the possible directions of variation fields (but on the other hand tangential directions are always there).

Lemma 4.8. *Let p be homogeneous in ξ. If $z \in \mathcal{G}$ and $\Phi_t(z) = (x(t), \xi(t))$, then $\xi(t) \perp V_z(t,0)$ and $\dot{x}_z(t) \in V_z(t,0)$ for $t \neq 0$.*

Proof. Let $w \in T_z\mathcal{G}$ and $J_w(t) = d\pi(d\Phi_t(w))$, and suppose that $J_w(0) \parallel \dot{x}_z(0)$, i.e. $J_w(0) \parallel \nabla_\xi p$. Let $\lambda|_{(x,\xi)} = \xi_j\, dx^j$ be the canonical 1-form on T^*M, and note that

$$f(t) := \xi(t)(J_w(t)) = \lambda(d\Phi_t(w)) = (\Phi_t^*\lambda)(w).$$

One has $f(0) = 0$ since $\xi \perp \nabla_\xi p$ on Ξ by the homogeneity relation $\xi_j \partial_{\xi_j} p = mp$. We use Cartan's formula and the homogeneity relation again to obtain that

$$\partial_t(\Phi_t^*\lambda)|_{t=0} = \mathcal{L}_{H_p}\lambda = (di_{H_p} + i_{H_p}d)\lambda = d(mp) + d\lambda(H_p,\,\cdot\,) = (m-1)dp.$$

Thus we have

$$f'(t) = \partial_s((\Phi_{t+s}^*\lambda)(w))|_{s=0} = (m-1)dp(d\Phi_t(w)).$$

Since $d\Phi_t(w)$ is tangential to Ξ, it follows that $f'(t) = 0$. We have shown that $f(t) = \xi(t)(J_w(t)) = 0$ for all $w \in T_z\mathcal{G}$, which implies that $\xi(t) \perp V_z(t,0)$.

Let now $z = (x, \xi_0)$ and let $w = \dot{z}_0 \in T_z\mathcal{G}$ be a radial vector where $z_s = (x, (1+s)\xi_0) \in \mathcal{G}$ by homogeneity. Since the curve $x_{z_s}(t)$ is obtained from $x_z(t)$ by reparametrization, we see that $J_w(t) = \partial_s(x_{z_s}(t))|_{s=0}$ is always parallel to $\dot{x}_z(t)$. □

Motivated by Lemma 4.8, we say that two points $x_z(t)$ and $x_z(s)$ are *not conjugate* along x_z if $V_z(t,s) = \ker(\xi(t))$, i.e. all directions of variation in the kernel of $\xi(t)$ are possible. This is equivalent to surjectivity of the map $d\pi \circ d\Phi_t : \{w \in T_z\mathcal{G} \mid J_w(s) \parallel \dot{x}_z(s)\} \to \ker(\xi(t))$. Since this is a map between $(n-1)$-dimensional spaces and since one can add a radial vector to w, we see that

$$x_z(t) \text{ and } x_z(s) \text{ are conjugate iff there is } w \in T_z\mathcal{G},\ w \neq 0, \text{ with } J_w(t) = J_w(s) = 0.$$

See [LOSU20] for an analysis of conjugate points in the case of the light ray transform.

We can now show that if there are no conjugate points and a nondegeneracy condition holds, then the Bolker condition at $(z, \zeta, x, \eta) \in C$ with $\Phi_t(z) = (x, \xi)$ holds precisely when $\eta \nparallel \xi$.

Lemma 4.9. *Let p be homogeneous in ξ, and assume that*

$$\nabla_\xi^2 p \text{ is nondegenerate on } \Xi. \tag{4.9}$$

Let $(z, \zeta, x, \eta) \in C$ where $\Phi_t(z) = (x, \xi)$. Then $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective iff $\eta \nparallel \xi$. If $\eta \nparallel \xi$ and there are no conjugate points on x_z, then $\pi_L^{-1}(z, \zeta) = \{(z, \zeta, x, \eta)\}$.

Proof. Since $Y^h = \nabla_\xi p$, we have $d(Y^h(x, \cdot))|_\xi(\theta) = \nabla_\xi^2 p(x, \xi)\theta$. Using Lemma 4.6 part (d) and (4.7), we obtain that

$$d\pi_L|_{(z,\zeta,x,\eta)} \text{ is injective iff } \nabla_\xi^2 p(x,\xi)\eta \nparallel \nabla_\xi p(x,\xi).$$

Since p is homogeneous, we have $\partial_{\xi_j\xi_k} p \xi_k = (m-1)\partial_{\xi_j} p$, which gives $\nabla_\xi^2 p(x,\xi)\xi = (m-1)\nabla_\xi p$. Since $\nabla_\xi^2 p(x,\xi)$ is assumed to be nondegenerate, we must have $m \neq 1$ and $(\nabla_\xi^2 p)^{-1}\nabla_\xi p \parallel \xi$. Consequently $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective iff $\eta \nparallel \xi$. If x_z has no conjugate points and $\eta \nparallel \xi$, then $V_z(t,s) = \ker(\xi(t))$ for $t \neq s$ and the second statement follows from Lemma 4.6. □

We also show that (4.9) ensures the absence of conjugate points on short curves.

Lemma 4.10. *Assume that (4.9) holds. For any $z \in \Xi$, there is $\varepsilon > 0$ so that $x_z|_{[-\varepsilon,\varepsilon]}$ has no conjugate points.*

Proof. Let $w \in T_z\Xi \setminus 0$ satisfy $J_w(0) = 0$, i.e. $d\pi(w) = 0$. Then $w = \dot{z}_0$ where $z_s = (x, \xi_s) \in \Xi_x$, so we can identify w with an element of T_x^*M. If we fix some Riemannian metric on M and denote by D_t the covariant derivative, we have

$$D_t J_w(0) = D_t \partial_s x_{z_s}(t)|_{s=t=0} = D_s(\nabla_\xi p(\Phi_t(z_s)))|_{s=t=0} = D_s(\nabla_\xi p(x, \xi_s))|_{s=0} = \nabla_\xi^2 p(x,\xi)w.$$

By (4.9) we have $D_t J_w(0) \neq 0$. It follows that for some $\varepsilon > 0$, $J_w(t) \neq 0$ when $t \in [-\varepsilon, \varepsilon] \setminus 0$. This shows that $x_z|_{[-\varepsilon,\varepsilon]}$ has no points conjugate to $x_z(0)$. We can repeat this argument starting at $x_z(\rho)$ instead of $x_z(0)$ using a related variation field $J_{w_\rho}(t;\rho)$ with $w_\rho \in T_{\Phi_\rho(z)}\Xi$ with $|w_\rho| = 1$ in some fixed Riemannian metric. This implies by continuity that for some $\varepsilon > 0$, if $\rho, t \in [-\varepsilon, \varepsilon]$ and $t \neq \rho$, one cannot have $J_w(t) = J_w(\rho) = 0$ for some $w \in T_z\Xi$. □

The following result collects some of the facts given above (again, the only thing to prove is that the curves x_z do not self-intersect for z close to z_0, and this proceeds as in Proposition 4.7).

Proposition 4.11. *Let M be a manifold with smooth boundary, let $p \in C^\infty(T^*M \setminus 0)$, and let $\nabla_\xi p$ be nonvanishing on $\Xi = p^{-1}(0)$. Let also $\kappa \in C^\infty(\mathcal{G} \times M^{\mathrm{int}})$ be nowhere vanishing. Suppose that $z_0 \in \partial_+\Xi$ satisfies $\tau_+(z_0) < \infty$ and that the curve x_{z_0} meets ∂M transversally at the endpoints, otherwise stays in M^{int}, and does not self-intersect. If $\mathcal{G}$ is a sufficiently small neighborhood of z_0 in $\partial_+\Xi$, then the null bicharacteristic ray transform given by (4.8) is a double fibration ray transform. If p is homogeneous in ξ, (4.9) holds and there are no conjugate points along x_{z_0}, then the Bolker condition is satisfied at every $(z_0, \zeta, x, \eta) \in C$ where $\Phi_t(z_0) = (x, \xi)$ for some t and $\eta \nparallel \xi$.*

5. Recovering the analytic wave front set

In this section we prove Theorem 1.2, which we restate here for the reader's convenience.

Theorem 5.1. *Let R be an analytic double fibration transform as in Definition 1.5, and assume that the Bolker condition holds at $(\hat{z}, \hat{\zeta}, \hat{x}, \hat{\eta}) \in C$. Then for any $f \in \mathcal{E}'(X)$, we have*

$$(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(Rf) \implies (\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(f).$$

5.1. **The model operator.** We introduce here a model operator relevant for Theorem 5.1, related to the local representation $Z = \{x'' = \phi(z, x')\}$ of a double fibration given in Lemma 2.3. Write $x = (x', x'') \in \mathbb{R}^{n'+n''}$ and $z \in \mathbb{R}^N$ with $N \geq n' + n''$. We recall that $n = n' + n'' = \dim(X)$, $N = \dim(\mathcal{G})$, and $N + n' = \dim(Z)$. We assume that the map

$$\phi : \mathbb{R}^N \times \mathbb{R}^{n'} \to \mathbb{R}^{n''}$$

is analytic in a neighborhood $V \times U'$ of a fixed point $(\hat{z}, \hat{x}')$, where $V \subset \mathbb{R}^N$ is a neighborhood of $\hat{z}$ and $U' \subset \mathbb{R}^{n'}$ is a neighborhood of $\hat{x}'$. Define

$$Z := \{(z, x', x'') \in V \times U' \times \mathbb{R}^{n''} \mid x'' = \phi(z, x')\}. \tag{5.1}$$

We choose a neighborhood U'' of $\hat{x}'' := \phi(\hat{z}, \hat{x}')$ in $\mathbb{R}^{n''}$ such that if we set $U := U' \times U''$, then $Z \subset V \times U$.

We define the model operator T as the FIO

$$Tf(z) = \int_U T(z, x) f(x)\, dx \in \mathcal{D}'(V)$$

for $f \in \mathcal{E}'(U)$. The kernel $T(z, x) \in \mathcal{D}'(V \times U)$ is given by the oscillatory integral

$$T(z, x) := \int_{\mathbb{R}^{n''}} e^{i(\phi(z,x') - x'')\cdot\eta''} a(z, x)\, d\eta'' \tag{5.2}$$

where $a(z, x)$ is real valued and analytic on $V \times U$. That is, the Schwartz kernel of T is a conormal distribution associated with the conormal bundle of Z, and $a(z, x)$ is a nonvanishing amplitude defined on $V \times U$ that is independent of η''.

Let $C := (N^*Z \setminus 0)' \subset T^*\mathbb{R}^N \times T^*\mathbb{R}^{n'+n''}$ be the canonical relation associated to the phase function

$$\Phi : \mathbb{R}^{N+n'+n''} \times \mathbb{R}^{n''} \to \mathbb{R}, \quad \Phi(z, x, \eta'') := (\phi(z, x') - x'') \cdot \eta''.$$

The conormal bundle N^*Z is given by

$$\begin{aligned} N^*Z &= \Big\{ \big((z, -\partial_z\Phi(z, x)), (x, \partial_x\Phi(z, x))\big) \;\Big|\; \partial_{\eta''}\Phi(z, x) = 0, z \in V, x \in U' \times \mathbb{R}^{n''} \Big\} \\ &= \Big\{ \big(z, \phi_z(z, x')^T\eta'', x', \phi(z, x'), \phi_{x'}(z, x')^T\eta'', -\eta''\big) \;\Big|\; \eta'' \in \mathbb{R}^{n''}, z \in V, x' \in U' \Big\}. \end{aligned} \tag{5.3}$$

We will assume a local version of the Bolker condition (i.e. injectivity of $d\pi_L$, see Lemma 3.3): we assume that at $(\hat{z}, \hat{x}', \hat{\eta}'') \in V \times U' \times \mathbb{R}^{n''}$ the $N \times (n' + n'')$ matrix

$$(\phi_z(\hat{z}, \hat{x}')^T,\ \partial_{x'}(\phi_z(\hat{z}, x')^T \hat{\eta}'') \mid_{x'=\hat{x}'}) \text{ has linearly independent columns.} \tag{5.4}$$

For elliptic FIOs T as above, we will prove the following analytic regularity statement for distributions supported in a sufficiently small neighborhood of $\hat{x}$.

Theorem 5.2. *Assume* (5.4). *There exists a neighborhood $\hat{U} \subset\subset U$ containing $\hat{x}$ such that for all $f \in \mathcal{E}'(\hat{U})$,*

$$(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(Tf) \implies (\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(f).$$

5.2. **Proof of Theorem 5.1.** We will prove that Theorem 5.1 is a consequence of Theorem 5.2, by showing that the operators considered in Theorem 5.2 are local representations of operators in Theorem 5.1.

First we give a simple characterization of a double fibration that will ensure that when one uses [Hör85, Theorem 8.5.5] below, there will no extra elements appearing in the wave front set. One direction would already follow from Lemmas 2.4 and 2.6.

Lemma 5.3. *Let Z be an embedded submanifold in $\mathcal{G} \times X$. The projections $\pi_{\mathcal{G}} : Z \to \mathcal{G}$ and $\pi_X : Z \to X$ are submersions if and only if N^*Z satisfies*

$$\{(z, \zeta, x, \eta) \in N^*Z \setminus 0 \mid \zeta = 0\} = \{(z, \zeta, x, \eta) \in N^*Z \setminus 0 \mid \eta = 0\} = \emptyset. \tag{5.5}$$

Proof. The map $d\pi_{\mathcal{G}}|_{(z,x)} : T_{(z,x)}Z \to T_z\mathcal{G}$ is surjective if and only if for any $v \in T_z\mathcal{G}$ there is some $(z, v, x, u) \in TZ$. The last condition implies that any $(z, \zeta, x, 0) \in N^*Z$ must satisfy $\zeta = 0$. So

$$\{(z, \zeta, x, \eta) \in N^*Z \setminus 0 \mid \zeta = 0\} = \emptyset.$$

Conversely, suppose

$$\{(z, \zeta, x, \eta) \in N^*Z \setminus 0 \mid \zeta = 0\} = \emptyset$$

but $d\pi_{\mathcal{G}}|_{(z,x)}$ is not surjective. Then its range has codimension ≥ 1 and hence there is $\zeta \in T_z^*\mathcal{G} \setminus 0$ with $\zeta(d\pi_{\mathcal{G}}(T_{z,x}Z)) = \{0\}$. It follows that $(z, \zeta, x, 0) \in N^*Z$ with $\zeta \neq 0$ which is a contradiction. The argument for $d\pi_X$ is analogous. □

Proof of Theorem 5.1. We first write $f = \psi f + (1 - \psi)f$, where $\psi \in C_c^\infty(X)$ satisfies $\psi = 1$ near $\hat{x}$ and $\mathrm{supp}(\psi)$ is contained in an analytic coordinate chart of X. Since we have (5.5) and since $\pi_L^{-1}(\hat{z}, \hat{\zeta}) = \{(\hat{z}, \hat{\zeta}, \hat{x}, \hat{\eta})\}$ by the Bolker condition, we can apply [Hör85, Theorem 8.5.5] to conclude that $(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(R((1 - \psi)f))$. So we have that $(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(R(\psi f))$.

Working near $\hat{x}$ and $\hat{z}$, we may identify X with part of $\mathbb{R}^n$ and $\mathcal{G}$ with part of $\mathbb{R}^N$. By Lemma 2.3, there are open sets $V \subset \mathbb{R}^N$ containing $\hat{z}$ and $U' \subset \mathbb{R}^{n'}$ containing $\hat{x}'$ such that the submanifold $Z \cap (V \times U) \subset \mathbb{R}^N \times \mathbb{R}^{n'+n''}$ can be described in analytic coordinates by $x'' = \phi(z, x')$ for some analytic function $\phi : V \times U' \to U'' \subset \mathbb{R}^{n''}$. Here $U := U' \times U''$ for some small set $U'' \subset \mathbb{R}^{n''}$. It was proved in (2.3) that in such local coordinates, one has $R(\psi f) = T(\psi f)$ where T is the model operator as in (5.2).

The Bolker condition and Lemma 3.3 ensure that (5.4) holds. So Theorem 5.2 then applies to show that there is an open set $\hat{U} \subset\subset U$ such that for all $F \in \mathcal{E}'(\hat{U})$,

$$(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(TF) \implies (\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(F). \tag{5.6}$$

We proved above that $(\hat{z}, \hat{\zeta}) \notin \mathrm{WF}_a(T(\psi f))$, where ψ can be chosen to be supported in $\hat{U}$. It follows that $(\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(\psi f)$, which proves that $(\hat{x}, \hat{\eta}) \notin \mathrm{WF}_a(f)$ as required. □

5.3. **Wave packet decomposition and representation of Tf.** To prove Theorem 5.2, instead of oscillatory integral methods, we will appeal to wave packet decompositions and FBI transforms. We will use conventions as in [Mar02] (with h replaced by λ^{-1} and ξ replaced $-\xi$).

From this point on, we will often denote elements of T^*V by (v_1, v_2) instead of (z, ζ) and elements of T^*U by (u_1, u_2) instead of (x, η). We also set $\hat{v}_1 := \hat{z}$, $\hat{u}_1 := \hat{x}$ etc. for consistency in notation.

Let T be given by (5.2), let $m(y) = c_n e^{-\frac{1}{2}|y|^2}$ and $M(z) = c_N e^{-\frac{1}{2}|z|^2}$ be Gaussians centred at the origin on $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, with $n = n' + n''$ and $c_n = 2^{-n/2}\pi^{-3n/4}$. For $\lambda > 0$, $v = (v_1, v_2) \in T^*V \cong V \times \mathbb{R}^N$ and $u = (u_1, u_2) \in T^*U \cong U \times \mathbb{R}^n$, we consider Gaussian wave packets defined by $M_v^\lambda(z) := \lambda^{3N/4} e^{i\lambda z\cdot v_2} M(\lambda^{1/2}(z - v_1))$ and $m_u^\lambda(y) := \lambda^{3n/4} e^{i\lambda y\cdot u_2} m(\lambda^{1/2}(y - u_1))$.

For $\hat{U} \subset\subset U$ and for any compactly supported distribution $f \in \mathcal{E}'(\hat{U})$ we can define the FBI transform $(L_m^\lambda f)(u)$ formally by

$$(L_m^\lambda f)(u) := \int f(y)\overline{m_u^\lambda(y)}\, dy = \langle f, m_u^\lambda\rangle_{L^2}.$$

The FBI transform $(L_M^\lambda g)(v)$ for compactly supported distributions $g \in \mathcal{E}'(V)$ can be defined analogously. The FBI transform satisfies the inversion formula

$$f(x) = \int (L_m^\lambda f)(u) m_u^\lambda(x) du = \int \langle f, m_u^\lambda\rangle m_u^\lambda(x)\, du \tag{5.7}$$

and the same of course holds for the transform L_M^λ. Let T be an FIO defined by (5.2) and $f \in \mathcal{E}'(\hat{U})$. We can use (5.7) to deduce the formula

$$(L_M^\lambda Tf)(v) = \int f(x) \left[\int \langle Tm_u^\lambda, M_v^\lambda\rangle \overline{m_u^\lambda(x)}\, du\right] dx := \int f(x) K_\lambda(x, v)\, dx \tag{5.8}$$

where

$$K_\lambda(x, v) = \int \langle Tm_u^\lambda, M_v^\lambda\rangle \overline{m_u^\lambda(x)}\, du. \tag{5.9}$$

In Section 5.5 we will prove that the kernel $K_\lambda(x, v)$ can be expressed as follows.

Proposition 5.4. *There exists a relatively compact open set $\mathcal{U} \subset \mathbb{R}^{N-n''}$ and an analytic diffeomorphism*

$$z(\cdot, \cdot) : \mathcal{U} \times U \to Z$$

such that

$$K_\lambda(x, v) = c_{n,N} \lambda^{\frac{3N}{4}} \int_{\zeta' \in \mathcal{U}} e^{i\lambda\Psi(\zeta', x, v)} \tilde{a}(\zeta', x)\, d\zeta' \tag{5.10}$$

where $\Psi(\zeta', x, v) := -z(\zeta', x) \cdot v_2 + i\frac{(z(\zeta', x) - v_1)^2}{2}$ and $\tilde{a}(\zeta', x)$ is analytic and nonvanishing.

Before proceeding further we introduce a notation to streamline our presentation. If $\Omega \subset \mathbb{R}^k$ is an open subset, we denote by $\Omega_\mathbb{C}$ an open subset of $\mathbb{C}^k$ with $\Omega_\mathbb{C} \cap \mathbb{R}^k = \Omega$. Here we use the natural embedding $\mathbb{R}^k \subset \mathbb{C}^k$. Similarly we have the extension of the set Z defined in (5.1), given by

$$Z_\mathbb{C} := \{(z, x) \in V_\mathbb{C} \times U_\mathbb{C} \mid \phi(z, x') - x'' = 0\} \tag{5.11}$$

where ϕ is now the unique holomorphic extension of the original ϕ, defined on a complex neighborhood of $V \times U'$.

We elaborate a bit more on some elementary properties of the holomorphic extension ϕ as they will be useful for us later. Writing $z \in \mathbb{C}^N$ as $z = z^{\mathbb{R}} + iz^{i\mathbb{R}}$ with $z^{\mathbb{R}}$ and $z^{i\mathbb{R}}$ real vectors, we write ϕ as

$$\phi(z, x') = \phi^{\mathbb{R}}(z^{\mathbb{R}}, z^{i\mathbb{R}}, x') + i\phi^{i\mathbb{R}}(z^{\mathbb{R}}, z^{i\mathbb{R}}, x'). \tag{5.12}$$

Since ϕ takes on real values for real z, we have that

$$\phi^{i\mathbb{R}}(z^{\mathbb{R}}, 0, x') = 0, \qquad z^{\mathbb{R}} \in \mathbb{R}^N,\ x' \in \mathbb{R}^{n'}. \tag{5.13}$$

Note that the same holds for (real) derivatives in $z^{\mathbb{R}}$ and x.

The phase appearing in (5.10) will be the key component in our analysis of this kernel. In particular we are interested in its critical points as a map $\zeta' \mapsto \Psi(\zeta', x, v)$ and in the dependence of the critical point on $(x, v) \in U_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$ where $\mathcal{V}$ is given below. In order to state the required properties of the phase function, we introduce a map χ as follows.

Lemma 5.5. *Assume (5.4). There is a neighborhood $\tilde{C}$ of $(\hat{v}_1, \hat{v}_2, \hat{u}_1, \hat{u}_2)$ in C, an analytic $(N + n)$-dimensional submanifold $\mathcal{V} = \pi_L(\tilde{C})$ of $T^*\mathbb{R}^N$ containing $(\hat{v}_1, \hat{v}_2)$, and an analytic surjective submersion*

$$\chi := \pi_R \circ \pi_L^{-1} : \mathcal{V} \to \pi_R(\tilde{C}) \tag{5.14}$$

where $\pi_R(\tilde{C})$ is an open set in T^*U.

Proof. By (5.4) and Lemma 3.3, we know that π_L is an immersion and π_R is a submersion near $(\hat{v}_1, \hat{v}_2, \hat{u}_1, \hat{u}_2)$. Hence there is a neighborhood $\tilde{C}$ of $(\hat{v}_1, \hat{v}_2, \hat{u}_1, \hat{u}_2)$ in C such that $\pi_L|_{\tilde{C}} : \tilde{C} \to T^*\mathbb{R}^N$ is an embedding and $\mathcal{V} = \pi_L(\tilde{C})$ is an analytic submanifold. Then χ is a well defined analytic map and it is a submersion if $\tilde{C}$ is chosen small enough. Hence also $\pi_R(\tilde{C})$ is open. □

The graph of χ can be characterized in the following way (see (5.3), here we write $u_j = (u_j', u_j'')$):

$$\begin{aligned} (u_1, u_2) = \chi(v_1, v_2) \quad &\iff \quad (v_1, v_2, u_1, u_2) \in \tilde{C} \\ &\iff \quad v_2 = \phi_z(v_1, u_1')^T u_2'',\ u_1'' = \phi(v_1, u_1'),\ u_2' = -\phi_{x'}(v_1, u_1')^T u_2'' \end{aligned} \tag{5.15}$$

The proof of the following proposition will be given in Section 5.6 and this will be the most technical part of the argument.

Proposition 5.6. *There exist open subsets $\hat{U} \subset\subset U$ containing $\hat{x}$ and $\hat{\mathcal{V}} \subset\subset \mathcal{V}$ containing $\hat{v} = (\hat{v}_1, \hat{v}_2)$ such that for all $(x, v) \in \hat{U}_{\mathbb{C}} \times \hat{\mathcal{V}}_{\mathbb{C}}$, the map $\zeta' \mapsto \Psi(\zeta', x, v)$ defined on $\mathcal{U}$ has a unique nondegenerate critical point $\zeta_c'(x, v)$ which depends holomorphically on (x, v). The phase function $\psi(x, v) := \Psi(\zeta_c'(x, v), x, v)$ satisfies the following conditions:*

1. *If we write $v = (v_1, v_2)$, then $\psi(\pi(\chi(v)), v) + v_1 \cdot v_2 = 0$ where χ is the map given by (5.14).*
2. $(x, d_x\psi(x, v))\,|_{x=\pi(\chi(v))} = \chi(v)$.
3. $\mathrm{Im}(\psi(x, v)) \geq c|x - \pi(\chi(v))|^2$ *if x and v are real valued.*

5.4. **Stationary phase and proof of Theorem 5.2.** Let $\hat{U} \subset \mathbb{R}^{n'+n''}$ be the open subset containing $\hat{x}$ appearing in the statement of Proposition 5.6. For all $f \in \mathcal{E}'(\hat{U})$ we need to show that if $(\hat{v}_1, \hat{v}_2) \notin \mathrm{WF}_a(Tf)$ then $(\hat{x}, \hat{u}_2) \notin \mathrm{WF}_a(f)$.

By the assumption that $(\hat{v}_1, \hat{v}_2)$ is not in the analytic wave front set, by (5.8) and by [Sjö82, Definition 6.1], there is $\epsilon > 0$ such that for any $v = (v_1, v_2)$ in some neighborhood of $(\hat{v}_1, \hat{v}_2)$ one

has

$$\int f(x)K_\lambda(x,v_1,v_2)\,dx = O(e^{-\epsilon\lambda}) \tag{5.16}$$

as $\lambda \to \infty$. The kernel $K_\lambda(x,v_1,v_2)$ is given by (5.10). Proposition 5.6 asserts that for complex (x,v_1,v_2) near $(\hat{x},\hat{v}_1,\hat{v}_2)$, the phase $\zeta' \mapsto \Psi(\zeta',x,v)$ defined on $\mathcal{U}$ has a unique nondegenerate critical point $\zeta_c(x,v_1,v_2)$. So by stationary phase (Theorem 2.8 and Remark 2.10 of [Sjö82]) applied to the expression (5.10) for $K_\lambda(x,v_1,v_2)$ we get that

$$\int f(x)e^{i\lambda\psi(x,v_1,v_2)}A(x,v_1,v_2;\lambda)\,dx = O(e^{-\epsilon\lambda})$$

for (v_1,v_2) complex near $(\hat{v}_1,\hat{v}_2)$. By the stationary phase expansion formula (Theorem 2.8 and Remark 2.10 of [Sjö82]), the new symbol $A(x,v_1,v_2;\lambda)$ is a nonvanishing classical analytic symbol in the sense of [Sjö82, Section 1]. We get then that

$$\int f(x)e^{i\lambda(\psi(x,v_1,v_2)+v_1\cdot v_2)}A(x,v_1,v_2;\lambda)\,dx = O(e^{-\epsilon\lambda}).$$

We now make a change of variable in the (v_1,v_2) variable. By Lemma 5.5, $\chi : \mathcal{V} \to \pi_R(\tilde{C}) \subset T^*U$ is an analytic surjective submersion onto the open set $\pi_R(\tilde{C})$. By the local submersion theorem (which follows from the implicit function theorem and hence works in the analytic category), there exists an analytic map $\chi^+ : \pi_R(\tilde{C}) \to \mathcal{V}$ satisfying

$$\chi \circ \chi^+ = \mathrm{Id}.$$

Therefore if $\hat{U}$ is chosen small enough, we can define $\tilde{\psi} : \hat{U} \times \pi_R(\tilde{C}) \to \mathbb{C}$ by

$$\tilde{\psi}(x,u_1,u_2) = (\psi(x,v_1,v_2) + v_1\cdot v_2)|_{(v_1,v_2)=\chi+(u_1,u_2)}.$$

Note that the variable x is untouched in this change of parameters. The above integral becomes

$$\int f(x)e^{i\lambda\tilde{\psi}(x,u_1,u_2)}\tilde{A}(x,u_1,u_2;\lambda)\,dx = O(e^{-\epsilon\lambda}) \tag{5.17}$$

for all (u_1,u_2) near $(\hat{x},\hat{u}_2)$. By Proposition 5.6 part (1) the phase function $\tilde{\psi}(x,u_1,u_2)$ vanishes when $x = u_1$, and by part (2) it satisfies

$$d_x(\tilde{\psi}(x,u_1,u_2))|_{x=u_1} = u_2.$$

Furthermore, by property (3) of Proposition 5.6

$$\mathrm{Im}(\tilde{\psi}(x,u_1,u_2)) \geq c|x-u_1|^2$$

for real (u_1,u_2). By [Sjö82, Definition 6.1] we have that $(\hat{x},\hat{u}_2) \notin \mathrm{WF}_a(f)$.

5.5. $K_\lambda(x,v)$ **as an oscillatory integral and proof of Proposition 5.4.** We observe that the integral kernel $K_\lambda(x,v)$ given by (5.9) involves oscillatory wave packets which concentrate simultaneously in space and frequency. Writing out the integrals explicitly using (5.2) for the operator T and integrating out the u_2 and y variables we have that for $(x,v) \in U \times \mathcal{V}$,

$$K_\lambda(x,v) = c_{n,N}\lambda^{\frac{3N}{4}+n''}\int_{z\in V}\int_{\eta\in\mathbb{R}^{n''}} e^{i\lambda(\phi(z,x')-x'')\cdot\eta}e^{-i\lambda z\cdot v_2}e^{-\frac{\lambda|z-v_1|^2}{2}}a(z,x)\,d\eta\,dz. \tag{5.18}$$

The η integral in (5.18) results in $\lambda^{-n''}\delta_0(\phi(z,x')-x'')$ which is a distribution supported on the submanifold $Z \subset V \times U$.

The following lemma gives another coordinate representation for Z.

Lemma 5.7. *The set Z has the following properties:*

(1) *The set Z is an analytic submanifold of $V \times U$ of dimension $N + n'$.*
(2) *We can find $\Omega \subset V \times U$ containing $(\hat{v}_1, \hat{x})$, an open set $\tilde{\Omega} \subset \mathbb{R}^N \times \mathbb{R}^{n'+n''}$, and an analytic diffeomorphism of the form $(\zeta, x) \mapsto (z(\zeta, x), x)$ from $\tilde{\Omega}$ to Ω such that*

$$Z \cap \Omega = \{(z(\zeta, x), x) \mid \zeta = (\zeta'', \zeta') = (0, \zeta'), \zeta' \in \mathcal{U}, x \in U\}$$

for some neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^{N-n''}$. Also, the matrix $z_\zeta(\zeta, x)$ is invertible.
(3) *The coordinate system constructed above extends holomorphically to a coordinate system (possibly after shrinking both Ω and $\tilde{\Omega}$) from $\tilde{\Omega}_\mathbb{C}$ to $\Omega_\mathbb{C}$ such that $Z_\mathbb{C} \cap \Omega_\mathbb{C}$ is again given by $\zeta'' = 0$.*

Proof. The statement (1) is a consequence of (2) so we proceed with (2). By (5.4), we must have that the $n'' \times N$ matrix $\partial_z \phi$ has full rank. Choose $\Omega \subset V \times U$ small containing $(\hat{v}_1, \hat{x})$ and without loss of generality we may assume that $\Omega = V \times U$. Since $N > n''$, we may assume that at $(\hat{v}_1, \hat{x}) \in Z \cap \Omega$, the first n'' vectors

$$\{\partial_{z_1} \phi, \dots, \partial_{z_{n''}} \phi\}$$

are linearly independent. For any $\zeta'' = (\zeta_1, \dots, \zeta_{n''})$ close to the origin and $x \in U$, use implicit function theorem to find the unique solution $z'' := (z_1, \dots, z_{n''})$ to the equation

$$\phi(z, x') - x'' = \zeta''$$

so that

$$(z_1, \dots, z_{n''}, \underbrace{z_{n''+1}, \dots, z_N}_{z'}) \in V.$$

The dependence of $(z_1, \dots, z_{n''})$ on the variables

$$(\zeta'', z', x) \in \mathbb{R}^{n''} \times \mathbb{R}^{N-n''} \times \mathbb{R}^{n'+n''}$$

is analytic.

Setting $\zeta' = z'$ and $\zeta = (\zeta'', \zeta')$ gives us an analytic map $z(\zeta, x)$ solving $\phi(z(\zeta, x), x') - x'' = \zeta''$. By direct computation we see that the map $(\zeta, x) \mapsto (z(\zeta, x), x)$ has nondegenerate Jacobian and is therefore locally an analytic diffeomorphism. Consequently, the matrix $z_\zeta(\zeta, x)$ is invertible.

To see (3), extend $(\zeta, x) \mapsto (z(\zeta, x), x)$ holomorphically to a complex neighborhood $\tilde{\Omega}_\mathbb{C}$ containing $\tilde{\Omega}$. By the fact that the real Jacobian of this holomorphic map is nonvanishing, the complex Jacobian is also nonvanishing. After possibly shrinking $\tilde{\Omega}_\mathbb{C}$ and $\Omega_\mathbb{C}$ we can conclude that $\zeta(\cdot, \cdot)$ is therefore a diffeomorphism between $\tilde{\Omega}_\mathbb{C}$ and $\Omega_\mathbb{C}$.

By construction of $z(\zeta'', \zeta', x)$, $\phi(z(\zeta'', \zeta'), x') - x'' = \zeta''$ for $(\zeta, x) \in \mathcal{U} \times U$. This identity therefore holds by unique continuation for the holomorphic extension to the complex domain. So $Z_\mathbb{C} \cap \Omega_\mathbb{C}$ is again given by $\{\zeta'' = 0\}$. □

In what follows we assume without loss of generality that $Z = \Omega \cap Z$.

We make a remark on the notation which will follow. We are often interested in coordinate systems for Z or $Z_\mathbb{C}$ which are given by $(z(0, \zeta', x), x)$. Therefore, to simplify notation we omit the 0 in the first argument and write $z(\zeta', x)$ in place of $z(0, \zeta', x)$ in this case.

Proof of Proposition 5.4. Integrating in η, (5.18) becomes

$$K_\lambda(x,v) = c_{n,N}\lambda^{\frac{3N}{4}} \int\limits_{Z^x} e^{-i\lambda z\cdot v_2} e^{-\frac{\lambda|z-v_1|^2}{2}} a(z,x)\,dz$$

where for each $x \in U$,

$$Z^x := \{z \in V \mid (z,x) \in Z\} = \{z(\zeta',x) \mid \zeta' \in \mathcal{U}\}$$

by Lemma 5.7. After a change of coordinate into the ζ' variable this integral becomes

$$K_\lambda(x,v) = c_{n,N}\lambda^{\frac{3N}{4}} \int\limits_{\zeta'\in\mathcal{U}} e^{i\lambda\Psi(\zeta',x,v)}\tilde{a}(\zeta',x)\,d\zeta'$$

for some nonvanishing real analytic function $\tilde{a}(\zeta',x)$ defined on $\mathcal{U}\times U$. □

5.6. **Critical points of Ψ and proof of Proposition 5.6.** We now examine the critical points of the phase function

$$\Psi(\zeta';x,v) := -z(\zeta',x)\cdot v_2 + i\frac{(z(\zeta',x)-v_1)^2}{2}$$

appearing in (5.10). This function has a holomorphic extension when the variables are allowed to be complex which we still denote by Ψ. For what follows we treat ζ' as a variable for the function Ψ and x and $v = (v_1, v_2)$ as parameters.

Let $\hat{x} \in U$ and $\hat{v} = (\hat{v}_1, \hat{v}_2) \in \mathcal{V}$ satisfy $\hat{x} = \pi(\chi(\hat{v}))$ which automatically guarantees that $(\hat{v}_1, \hat{x}) \in Z$ by (5.15). Choose Ω as in statement (2) of Lemma 5.7. Then Lemma 5.7 gives a $\hat{\zeta}' \in \mathcal{U}$ such that $z(\hat{\zeta}', \hat{x}) = \hat{v}_1$. Statement (3) of Lemma 5.7 provides a complex neighborhood $\Omega_{\mathbb{C}}$ of $\mathbb{C}^N \times \mathbb{C}^{n'+n''}$ containing $(\hat{v}_1, \hat{x})$ which is diffeomorphic to a complex neighborhood $\tilde{\Omega}_{\mathbb{C}}$ containing $((0,\hat{\zeta}'), \hat{x})$. The diffeomorphism is given by

$$(\zeta, x) \in \tilde{\Omega}_{\mathbb{C}} \mapsto (z(\zeta,x), x) \in \Omega_{\mathbb{C}}$$

near $((0,\hat{\zeta}'), \hat{x})$. The submanifold $Z_{\mathbb{C}} \cap \Omega_{\mathbb{C}}$ is given by $\zeta = (0, \zeta')$ where $\zeta' \in \mathcal{U}_{\mathbb{C}}$ and $\mathcal{U}$ is given as in Lemma 5.7.

We first look at the possibility of critical points when $x = \pi(\chi(v))$:

Lemma 5.8. *For each real $(x, v = (v_1, v_2)) \in U \times \mathcal{V}$, writing $x(v) = \pi(\chi(v))$, the following holds:*

(1) *$\Psi(\cdot; x(v), v)$ has a real critical point $\zeta'_c(x(v), v) \in \mathcal{U}$ satisfying $z(\zeta'_c(x(v), v), x(v)) = v_1$.*
(2) *The matrix of complex derivatives $\left(\frac{\partial^2\Psi}{\partial\zeta'^2}\right)$ at $\zeta'_c(x(v), v)$ is nondegenerate.*
(3) *If $\mathcal{U}$ is chosen small enough, the critical point $\zeta'_c(x(v), v)$ is the unique critical point of $\zeta' \mapsto \Psi(\cdot; x(v), v)$ in $\mathcal{U}$.*

Proof. Let $(x, u_2) = \chi(v_1, v_2)$. To see (1) we use (5.15) to get

$$v_2 = \phi_z(v_1, x')^T u''_2, \quad \phi(v_1, x') = x''. \tag{5.19}$$

We see from the second equality that $(v_1, x) \in Z$ and by Lemma 5.7 we can find $\zeta'_c \in \mathcal{U}$ real such that $z(\zeta'_c, x) = v_1$. We claim that ζ'_c is a critical point of $\Psi(\cdot; x, v)$.

Indeed using the first equality in (5.19) for v_2 we get:

$$-\partial_{\zeta'}(z(\zeta',x)\cdot v_2)\,|_{\zeta'=\zeta'_c} = -\partial_{\zeta'} z^T v_2 = -(\partial_{\zeta'} z)^T(\phi_z(v_1,x'))^T u''_2 = (\partial_{\zeta'}(\phi(z(\zeta',x),x')))^T u''_2\,|_{\zeta'=\zeta'_c} = 0.$$

The second last equality comes by the chain rule. The last equality comes from the fact that $\phi(z(\zeta', x), x') = x''$ is independent of ζ'. The fact that $\partial_{\zeta'}(z(\zeta', x) - v_1)^2 \mid_{\zeta'=\zeta'_c} = 0$ is trivial using $z(\zeta'_c, x) = v_1$.

To see (2), we observe that since Ψ extends holomorphically in the ζ variable to a complex neighborhood of ζ'_c, the complex Hessian coincides with the real Hessian by the Cauchy-Riemann equations. Hence it is enough to consider the real Hessian (with respect to ζ') of $\Psi(\zeta'; x, v) = -z(\zeta', x) \cdot v_2 + i\frac{(z(\zeta',x)-v_1)^2}{2}$ at ζ'_c. Direct computation using $z(\zeta'_c, x) = v_1$ yields that this Hessian is of the form $B + iA^TA$ where $A = z_{\zeta'}(\zeta'_c, x)$ is an injective real matrix (since z_ζ is invertible by Lemma 5.7 (2)) while $B = -\partial^2_{\zeta'}(z \cdot v_2)|_{\zeta'=\zeta'_c}$ is a symmetric real matrix.

It is easy to see that such a matrix acting on complex vectors has trivial kernel. Indeed, let $(B + iA^TA)(a + ib) = 0$ for real vectors $a, b \in \mathbb{R}^{N-n''}$. We then have that $Ba = A^TAb$ and $Bb = -A^TAa$. Taking the inner product of the first equation with b and the second equation with a we get

$$b \cdot Ba = |Ab|^2, \ -a \cdot Bb = |Aa|^2.$$

Since B is symmetric the two equations combine to yield $-|Aa|^2 = |Ab|^2$, so $Ab = Aa = 0$. Now injectivity of A gives that $a = b = 0$. So (2) is verified.

The statement (3) follows directly from (2) and the holomorphic implicit function theorem. □

The holomorphic implicit function theorem together with the non-degeneracy of the Hessian of $\zeta' \mapsto \Psi(\zeta'; x(v), v)$ at $\zeta' = \zeta'_c(x(v), v)$ provides us complex critical points even when x is not necessarily equal to $\pi(\chi(v))$:

Corollary 5.9. *Given any $U \times \mathcal{V}$ small enough containing $(\hat{x}, \hat{v})$ with $\hat{x} = \pi(\chi(\hat{v}))$ and $\mathcal{U}$ small enough containing $\zeta'_c(\hat{x}, \hat{v})$, for all $(x, v) \in U_\mathbb{C} \times \mathcal{V}_\mathbb{C}$ the map $\zeta' \mapsto \Psi(\zeta'; x, v)$ has a unique complex critical point $\zeta'_c(x, v) \in \mathcal{U}_\mathbb{C}$ which depends holomorphically on (x, v). Moreover, if $(x, v) \in U \times \mathcal{V}$ and $x = \pi(\chi(v))$, then $\zeta'_c(x, v)$ is real.*

At this point it is convenient to introduce the following auxiliary phase function:

$$\Phi(z, \eta; x, v) := (\phi(z, x') - x'') \cdot \eta - z \cdot v_2 + \frac{i(z - v_1)^2}{2} \tag{5.20}$$

which has a holomorphic extension that we still denote by Φ. Clearly, $\Phi(z(\zeta', x), \eta; x, v) = \Psi(\zeta'; x, v)$.

We treat $(z, \eta) \in V_\mathbb{C} \times \mathbb{C}^{n''}$ as variables and $(x, v) \in U \times \mathcal{V}$ as parameters where $(x, v) \in U \times \mathcal{V}$ (in most cases we will consider real (x, v) though (z, η) can still be complex). By the characterization of χ in (5.15), when $\chi(v) = (x, u_2)$, we see using direct calculation that the point $(v_1, u''_2) \in \mathbb{R}^N \times \mathbb{R}^{n''}$ is a real critical point of $(z, \eta) \mapsto \Phi(z, \eta; x, v)$. For arbitrary (x, v), we will try to find complex critical points (z_c, η_c) for $\Phi(\cdot; x, v)$. If they exist, critical points (z_c, η_c) are given by the equation:

$$\phi(z_c, x') - x'' = 0, \qquad \phi_z(z_c, x')^T\eta_c - v_2 = -i(z_c - v_1). \tag{5.21}$$

The critical points of Ψ are related to that of Φ in the following way:

Lemma 5.10. *Suppose $U \times \mathcal{V}$ is sufficiently small containing $(\hat{x}, \hat{v})$ and let $(x, v) \in U_\mathbb{C} \times \mathcal{V}_\mathbb{C}$. Suppose (z_c, η_c) is a complex critical point of $\Phi(\cdot; x, v)$ so that $(z_c, x) \in V_\mathbb{C} \times U_\mathbb{C}$, then we can find $\zeta'_0 \in \mathcal{U}_\mathbb{C}$ such that $z(\zeta'_0, x) = z_c$ and ζ'_0 is a critical point of $\Psi(\cdot; x, v)$.*

Proof. The first equation of (5.21) indicates that $(z_c, x) \in Z_\mathbb{C}$ (see (5.11) for definition of $Z_\mathbb{C}$). So we may find $\zeta'_0 \in \mathcal{U}_\mathbb{C}$ such that $z_c = z(\zeta'_0, x)$. We now need to show that ζ'_0 is a critical point of $\Psi(\cdot; x, v)$.

Observe that $\phi(z(\zeta', x), x') - x'' = 0$ for all $\zeta' \in \mathcal{U}_{\mathbb{C}}$ so differentiating in ζ' we get

$$(\partial_{\zeta'} z(\zeta_0', x))^T \phi_z(z(\zeta_0', x), x')^T = 0.$$

Multiply the second equation of (5.21) by $(\partial_{\zeta'} z(\zeta_0', x))^T$ and use the above identity we have that

$$\partial_{\zeta'} z(\zeta_0', x)^T v_2 = i\partial_{\zeta'} z(\zeta_0', x)^T (z(\zeta_0', x) - v_1).$$

This is precisely what it means for ζ_0' to solve $\Psi_{\zeta'}(\zeta_0'; x, v) = 0$. □

Lemma 5.11. *Let (x, v) be real and close to $(\hat{x}, \hat{v})$ with $x = \pi(\chi(v))$, and write $\chi(v) = (x, u_2)$. The point $(z, \eta) := (v_1, u_2'')$ is a critical point of Φ in $V \times \mathbb{R}^{n''}$. Furthermore, at this point the complex Hessian of Φ is nondegenerate with positive semidefinite imaginary part.*

Proof. By the characterization of χ in (5.15), when $\chi(v) = (x, u_2)$, the point $(z, \eta) = (v_1, u_2'')$ solves (5.21) and hence it is a critical point of Φ.

We use again that the complex Hessian is equal to real Hessian by holomorphicity, so it is enough to consider the real Hessian of Φ at (v_1, u_2''). By (5.4) $\phi_z(z, x')$ has full rank when (x, v) is close to $(\hat{x}, \hat{v})$. Direct calculation gives the following block form for the Hessian of Φ:

$$\begin{pmatrix} \partial_z^2 (\phi(z, x') \cdot u_2'') + iI & \phi_z(z, x')^T \\ \phi_z(z, x') & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_z^2 (\phi(z, x') \cdot u_2'') & \phi_z(z, x')^T \\ \phi_z(z, x') & 0 \end{pmatrix}}_{A} + i \underbrace{\begin{pmatrix} I_{N\times N} & 0 \\ 0 & 0 \end{pmatrix}}_{B}$$

(5.22)

Note that both A and B are real valued symmetric matrices. To show that $A+iB$ is nondegenerate, we suppose $(A+iB)(a+ib) = 0$ for some $a, b \in \mathbb{R}^{N+n''}$. Then $Aa = Bb$ and $Ab = -Ba$, which gives that $Bb \cdot b = -Ba \cdot a$ since A is symmetric. Using the fact that $B = B^T B$ yields $Ba = Bb = 0$. This means that $a = (0, a'')$ and $b = (0, b'')$ where $a'', b'' \in \mathbb{R}^{n''}$. We then have that

$$\phi_z(z, x')^T (a'' + ib'') = 0.$$

Using the fact that $\phi_z(z, x')$ is real and has full rank shows $a'' = b'' = 0$. So we have shown that the complex Hessian of Φ at (z, η) is nondegenerate. □

The next result gives a correspondence between the complex critical points of Φ and Ψ.

Lemma 5.12. *For all $(x, v) \in U_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$, let $\zeta_c'(x, v)$ be the critical point of Ψ deduced in Corollary 5.9. Then for all (x, v) near $(\hat{x}, \hat{v})$, there exists a unique critical point $(\mathcal{Z}_c(x, v), \eta_c(x, v)) \in V_{\mathbb{C}} \times \mathbb{C}^{n''}$ of Φ and it satisfies*

$$\mathcal{Z}_c(x, v) = z(\zeta_c'(x, v), x). \tag{5.23}$$

In particular, if x and v are real and $x = \pi(\chi(v))$, then writing $(x, u_2) = \chi(v)$, we have that $(\mathcal{Z}_c(x, v), \eta_c(x, v)) = (v_1, u_2'')$ and thus the critical point is real.

Proof. By Lemma 5.11, when $\chi(\hat{v}) = (\hat{x}, \hat{u}_2)$, the point $(\hat{v}_1, \hat{u}_2'')$ is a real critical point of Φ. Furthermore, at this point the complex Hessian of Φ is nondegenerate. The holomorphic implicit function theorem allows us to find critical points $(\mathcal{Z}_c(x, v), \eta_c(x, v)) \in V_{\mathbb{C}} \times \mathbb{C}^{n''}$ of $\Phi(\cdot; x, v)$ as (x, v) vary in $U_{\mathbb{C}} \times V_{\mathbb{C}}$ with $\mathcal{Z}_c(\hat{x}, \hat{v}) = \hat{v}_1$. The dependence of $\mathcal{Z}_c(x, v)$ on (x, v) is holomorphic.

For each $(\mathcal{Z}_c(x, v), \eta_c(x, v))$ use Lemma 5.10 to find $\zeta_0'(x, v)$ so that $\zeta_0'(x, v)$ is a critical point of $\Psi(\cdot; x, v)$ and $z(\zeta_0'(x, v), x) = \mathcal{Z}_c(x, v)$. By the uniqueness statement of Corollary 5.9, $\zeta_0'(x, v) = \zeta_c'(x, v)$. This means that $\mathcal{Z}_c(x, v) = z(\zeta_c'(x, v), x)$.

To see that $(\mathcal{Z}_c(x,v),\eta_c(x,v))$ is the unique critical point in $V_{\mathbb{C}}\times\mathbb{C}^{n''}$, suppose $(\tilde{\mathcal{Z}}_c(x,v),\tilde{\eta}_c(x,v))\in V_{\mathbb{C}}\times\mathbb{C}^{n''}$ is another critical point of $\Phi(\cdot;x,v)$. Then by Lemma 5.10, there is $\tilde{\zeta}_0'(x,v)\in\mathcal{U}_{\mathbb{C}}$ which is a critical point of $\Psi(\cdot;x,v)$ such that $\tilde{\mathcal{Z}}_c(x,v)=z(\tilde{\zeta}_0'(x,v),x)$. Uniqueness of critical points stated in Corollary 5.9 of $\Psi(\cdot;x,v)$ then forces $\tilde{\zeta}_0'(x,v)=\zeta_c'(x,v)$. This means $\tilde{\mathcal{Z}}_c(x,v)=\mathcal{Z}_c(x,v)$. The second equation of (5.21) and injectivity of the matrix $\phi_z(\mathcal{Z}_c(x,v),x')^T$ then forces $\tilde{\eta}_c(x,v)=\eta_c(x,v)$.

Finally, when (x,v) is real and $x=\pi(\chi(v))$, the last statement in the lemma follows from Lemma 5.11 and the uniqueness part above. □

The identity in Lemma 5.12 gives us a convenient way to prove the following quantitative estimate:

Lemma 5.13. *There is a constant $c_0>0$ such that for all $(x,v)\in U\times\mathcal{V}$, we have the estimate*

$$|z(\zeta_c'(x,v),x)-v_1|\geq c_0|x-\pi(\chi(v))|. \tag{5.24}$$

Proof. By (5.23), it is equivalent to show that

$$|\mathcal{Z}_c(x,v)-v_1|\geq c|x-\pi(\chi(v))|. \tag{5.25}$$

To this end, for all $v\in\mathcal{V}$, let $(u_1,u_2)=\chi(v)$. By the characterization of χ in (5.15) we have that

$$\phi(v_1,u_1')=u_1'',\quad \phi_z(v_1,u_1')^Tu_2''-v_2=0,\quad u_2'=-\phi_{x'}(v_1,u_1')^Tu_2''. \tag{5.26}$$

Meanwhile, by (5.21) $\mathcal{Z}_c(x,v)$ satisfies

$$\phi(\mathcal{Z}_c(x,v),x')=x'',\quad \phi_z(\mathcal{Z}_c(x,v),x')^T\eta_c(x,v)-v_2=i(\mathcal{Z}_c(x,v)-v_1). \tag{5.27}$$

Subtracting the first equation of (5.26) and the first equation of (5.27), Lipschitz continuity of ϕ gives that

$$|x''-u_1''|\leq c(|v_1-\mathcal{Z}_c(x,v)|+|x'-u_1'|).$$

So to show (5.25), it in fact suffices to show

$$|x'-u_1'|\leq c|v_1-\mathcal{Z}_c(x,v)|. \tag{5.28}$$

To this end, we look at the second equation of (5.27) and subtract from it the second equation of (5.26):

$$\phi_z(\mathcal{Z}_c(x,v),x')^T\eta_c(x,v)-\phi_z(v_1,u_1')^Tu_2''=O(\mathcal{Z}_c(x,v)-v_1).$$

Using standard continuity estimates allows us to replace the $\phi_z(\mathcal{Z}_c(x,v),x')^T\eta_c(x,v)$ in the first term by $\phi_z(v_1,x')^T\eta_c(x,v)$ to get, for (x,v) in $U_{\mathbb{C}}\times\mathcal{V}_{\mathbb{C}}$ (chosen small enough),

$$\phi_z(v_1,x')^T\eta_c(x,v)-\phi_z(v_1,u_1')^Tu_2''=O(\mathcal{Z}_c(x,v)-v_1).$$

Using the above expression, the estimate (5.28) would be established if we could show that for all (x',η) near (u_1',u_2'') we have that

$$|x'-u_1'|+|\eta-u_2''|\leq C|\phi_z(v_1,x')^T\eta-\phi_z(v_1,u_1')^Tu_2''|. \tag{5.29}$$

To establish (5.29) we observe that by (5.4) the Jacobian matrix

$$A:=\left(\partial_{x'}\left(\phi_z(v_1,x')^T\eta\right)|_{x'=u_1'},\quad \phi_z(v_1,u_1')^T\right)$$

is an $N\times(n'+n'')$ matrix of full rank when $(v_1,u_1')\in V\times U'$ and η is near $\hat{\eta}$. As $N\geq n'+n''$, the matrix A^TA is invertible.

We now expand the function $(x',\eta)\mapsto\phi_z(v_1,x')^T\eta$ around the point $(x',\eta)=(u_1',u_2'')$ to get

$$\phi_z(v_1,x')^T\eta-\phi_z(v_1,u_1')^Tu_2''=A\begin{pmatrix}x'-u_1'\\ \eta-u_2''\end{pmatrix}+Q\left(\begin{pmatrix}x'-u_1'\\ \eta-u_2''\end{pmatrix}\right)$$

for some quadratic form Q. The estimate (5.29) comes now directly from the invertibility of $A^T A$. $\square$

Fix $(\hat{x}, \hat{v})$ and $U \times \mathcal{V}$ as in Corollary 5.9. We now define the holomorphic phase on $U_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$ by

$$\begin{aligned} \psi(x,v) &:= \Psi(\zeta_c'(x,v);x,v) \\ &= -z(\zeta_c'(x,v),x)\cdot v_2 + i\frac{(z(\zeta_c'(x,v),x)-v_1)^2}{2}. \end{aligned} \tag{5.30}$$

We showed in Lemma 5.8 that when $x = \pi(\chi(v))$, the critical point $\zeta_c'(x,v)$ is real valued and consequently $z(\zeta_c'(x,v),x) = v_1$ which implies $\psi(x,v)$ is real valued. It turns out that when $x - \pi(\chi(v))$ is small, the imaginary part of $\psi(x,v)$ is bounded below by the imaginary part of $z(\zeta_c'(x,v),x)$:

Lemma 5.14. *Writing $z(\zeta_c'(x,v),x) = z^{\mathbb{R}}(\zeta_c'(x,v),x) + iz^{i\mathbb{R}}(\zeta_c'(x,v),x)$ where $z^{\mathbb{R}}$ and $z^{i\mathbb{R}}$ are real, we have for real $x \in U$ and $v \in \mathcal{V}$ the estimate*

$$\mathrm{Im}(\psi(x,v)) \geq C|z^{i\mathbb{R}}(\zeta_c'(x,v),x)|^2 \tag{5.31}$$

when $|x - \pi(\chi(v))|$ is sufficiently small.

Proof. Setting

$$\mathcal{Z}_c(x,v) = \mathcal{Z}_c^{\mathbb{R}}(x,v) + i\mathcal{Z}_c^{i\mathbb{R}}(x,v),$$

for real valued $\mathcal{Z}_c^{\mathbb{R}}(x,v)$ and $\mathcal{Z}_c^{i\mathbb{R}}(x,v)$, it suffices by (5.23) to show that

$$\mathrm{Im}(\psi(x,v)) \geq C|\mathcal{Z}_c^{i\mathbb{R}}(x,v)|^2. \tag{5.32}$$

To this end we follow the presentation of Lemma 7.7.8 in [Hör85, vol. I]. We simplify notation by $(z,\eta) = w$ and $(x,v) = y$, where both w and y are real. We have that

$$\Phi(z,\eta;x,v) = \Phi(w;y) = \psi(y) + \sum_{|\alpha|=2} \frac{H_\alpha(y)}{\alpha!}(w-w(y))^\alpha + O((w-w(y))^3).$$

Here $w(y)$ denotes the critical point $w(y) = (\mathcal{Z}_c(x,v), \eta_c(x,v))$ (recall that the variable y denotes (x,v)). We remark that while w is real, $w(y)$ can be complex and that

$$\lim_{y\to(\pi(\chi(v)),v)} \mathrm{Im}(w(y)) = 0 \tag{5.33}$$

for any $v \in \mathcal{V}$ by Lemma 5.12. Note the identity

$$H_\alpha(y) = \partial_w^\alpha \Phi(w(y);y), \qquad |\alpha| = 2. \tag{5.34}$$

For real valued $w \in \mathbb{R}^N \times \mathbb{R}^{n''}$, the definition of Φ gives that $\mathrm{Im}(\Phi(w;y)) \geq 0$, so

$$0 \leq \mathrm{Im}(\psi(y)) + \mathrm{Im}\left(\sum_{|\alpha|=2} \frac{H_\alpha(y)}{\alpha!}(w-w(y))^\alpha + O((w-w(y))^3)\right).$$

Now choose $w = \mathrm{Re}(w(y)) - t|\mathrm{Im}(w(y))|$ with $t \in \mathbb{R}^{N+n''}$ and $|t| \leq 1$. We get

$$|\mathrm{Im}(w(y))|^2\left(-\mathrm{Im}\left(\sum_{|\alpha|=2}\frac{H_\alpha(y)}{\alpha!}\left(t + i\frac{\mathrm{Im}(w(y))}{|\mathrm{Im}(w(y))|}\right)^\alpha + O(|\mathrm{Im}(w(y))|)\right)\right) \leq \mathrm{Im}(\psi(y)).$$

Since the inequality holds for all $|t| \leq 1$ we can take the supremum to obtain that

$$|\mathrm{Im}(w(y))|^2 \sup_{|t|\leq 1}\left(-\mathrm{Im}\left(\sum_{|\alpha|=2}\frac{H_\alpha(y)}{\alpha!}\left(t + i\frac{\mathrm{Im}(w(y))}{|\mathrm{Im}(w(y))|}\right)^\alpha + O(|\mathrm{Im}(w(y))|)\right)\right) \leq \mathrm{Im}(\psi(y)).$$

Thus, by (5.33), proving (5.32) amounts to showing that

$$\sup_{|t|\leq 1}\left(-\mathrm{Im}\left(\sum_{|\alpha|=2}\frac{H_\alpha(y)}{\alpha!}(t+i\omega)^\alpha\right)\right)\geq c>0 \tag{5.35}$$

uniformly for $y\in U\times\mathcal{V}$ and $\omega\in\mathbb{R}^N\times\mathbb{R}^{n''}$ with $|\omega|=1$.

Using continuity and compactness, it suffices to prove (5.35) when $y=\hat{y}=(\hat{x},\hat{v})$. Using (5.34), we see that this can be written as

$$\sup_{|t|\leq 1}-\mathrm{Im}\left((t+i\omega)\cdot H(\hat{y})(t+i\omega)\right)\geq c>0$$

where $H(\hat{y})$ is the Hessian of the map $w=(z,\eta)\mapsto\Phi(z,\eta;x,v)$ evaluated at

$$w(\hat{y})=(\mathcal{Z}_c(\hat{x},\hat{v}),\eta_c(\hat{x},\hat{v})))=(\hat{v}_1,\hat{u}_2'')$$

(recall that the variable y is shorthand for the variables (x,v) and $w(y)$ denotes $(\mathcal{Z}_c(x,v),\eta_c(x,v))$). Lemma 5.11 states that $H(\hat{y})$ is a symmetric nondegenerate matrix whose imaginary part is positive semi-definite. Lemma 7.7.9 in [Hör85, Vol. I] then gives the desired uniform lower bound. □

Corollary 5.15. *There are constants $c_1,c_2>0$ such that for all $(x,v)\in U\times\mathcal{V}$, the following estimate holds:*

$$\mathrm{Re}\left[(z(\zeta_c'(x,v),x)-v_1)^2\right]\geq c_1|x-\pi(\chi(v))|^2-c_2\mathrm{Im}(\psi(x,v)).$$

Proof. Since v_1 is real, this is a direct consequence of Lemma 5.13 and Lemma 5.14. □

Lemma 5.16. *For all $(x,v)\in U\times\mathcal{V}$ and for all $\epsilon>0$,*

$$-\mathrm{Im}(z(\zeta_c'(x,v),x)\cdot v_2)\geq -C\epsilon|x-\pi(\chi(v))|^2-C\epsilon^{-1}\mathrm{Im}(\psi(x,v))-C\epsilon^{-1}|x-\pi(\chi(v))|^3.$$

The constant C is uniform over $(x,v)\in U\times\mathcal{V}$ and independent of ϵ.

Proof. We again use (5.23) to deduce that the estimate in the Lemma is equivalent to the same estimate for $-\mathcal{Z}_c(x,v)\cdot v_2$. To this end, we write $\mathcal{Z}_c(x,v)=\mathcal{Z}_c^{\mathbb{R}}(x,v)+i\mathcal{Z}_c^{i\mathbb{R}}(x,v)$. Observe that for $v\in\mathcal{V}$, by Lemma 5.8 part (1) and (5.23),

$$\mathcal{Z}_c(\pi(\chi(v)),v)=\mathcal{Z}_c^{\mathbb{R}}(\pi(\chi(v)),v)+i\mathcal{Z}_c^{i\mathbb{R}}(\pi(\chi(v)),v)=v_1. \tag{5.36}$$

Now write ϕ in terms of its real and imaginary part as in (5.12) to obtain from (5.21)

$$\phi^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v),\mathcal{Z}_c^{i\mathbb{R}}(x,v),x')=0.$$

Differentiating in x we get

$$\begin{aligned}0=&\left(\mathcal{Z}_c^{\mathbb{R}}\right)_x^T\phi_{z^{\mathbb{R}}}^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v),\mathcal{Z}_c^{i\mathbb{R}}(x,v),x')^T\\&+\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T\phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v),\mathcal{Z}_c^{i\mathbb{R}}(x,v),x')^T+\phi_x^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v),\mathcal{Z}_c^{i\mathbb{R}}(x,v),x')^T.\end{aligned} \tag{5.37}$$

If we denote $\chi(v)=(u_1(v),u_2(v))$ and write $u_2(v)=(u_2(v)',u_2(v)'')\in\mathbb{R}^{n'+n''}$, then by (5.15) we have that

$$v_2=\phi_z(v_1,u_1(v)')^Tu_2(v)'' \tag{5.38}$$

We multiply (5.37) by $u_2(v)''$ to get

$$(5.39)\quad 0 = \left(\mathcal{Z}_c^{\mathbb{R}}\right)_x^T \phi_{z^{\mathbb{R}}}^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v), \mathcal{Z}_c^{i\mathbb{R}}(x,v), x')^T u_2(v)'' + \left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T \phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v), \mathcal{Z}_c^{i\mathbb{R}}(x,v), x')^T u_2(v)'' + \phi_x^{i\mathbb{R}}(\mathcal{Z}_c^{\mathbb{R}}(x,v), \mathcal{Z}_c^{i\mathbb{R}}(x,v), x')^T u_2(v)''.$$

In the above expression both $\left(\mathcal{Z}_c^{\mathbb{R}}\right)_x$ and $\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x$ still depend on x. Now we set $x = \pi(\chi(v)) = u_1(v)$ and use (5.36). Equation (5.39) becomes

$$(5.40)\qquad 0 = \left(\mathcal{Z}_c^{\mathbb{R}}\right)_x^T \phi_{z^{\mathbb{R}}}^{i\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)'' + \left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T \phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)'' + \phi_x^{i\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)''.$$

Now use (5.13) with $x = u_1(v)$ to obtain that

$$(5.41)\qquad 0 = \left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T \phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)''.$$

Since ϕ is holomorphic, the Cauchy-Riemann equations give $\phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(v_1, 0, u_1(v)') = \phi_{z^{\mathbb{R}}}^{\mathbb{R}}(v_1, 0, u_1(v)')$. Thus we get from (5.13) and (5.38) that when $x = u_1(v)$,

$$(5.42)\quad \phi_{z^{i\mathbb{R}}}^{i\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)'' = \phi_{z^{\mathbb{R}}}^{\mathbb{R}}(v_1, 0, u_1(v)')^T u_2(v)'' = \phi_z(v_1, u_1(v)')^T u_2(v)'' = v_2.$$

Combine (5.41) and (5.42) to get

$$(5.43)\qquad 0 = \left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T |_{x=u_1(v)}\, v_2 = \partial_x \left(\mathcal{Z}_c^{i\mathbb{R}} \cdot v_2\right) |_{x=u_1(v)}\,.$$

We now differentiate (5.39) with respect to x and set $x = u_1(v)$ to get

$$(5.44)\qquad 0 = \left(\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T A + B\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x + \partial_x^2(\mathcal{Z}_c^{i\mathbb{R}}(x,v)\cdot v_2)\right) |_{x=u_1(v)}$$

for some matrices A and B depending on v. Indeed, differentiating the first term of (5.39) with respect to x yields an expression of the form $B\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x$ by using (5.13) multiple times. Differentiating the second term of (5.39) with respect to x and setting $x = u_1(v)$ yields an expression of the form $\left(\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T A + \partial_x^2(\mathcal{Z}_c^{i\mathbb{R}}(x,v)\cdot v_2)\right) |_{x=u_1(v)}$ due to (5.42) and Lemma 5.8 part (1). Finally, differentiate the third term of (5.39) with respect to x and setting $x = u_1(v)$ yields an expression of the form $B\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x$ by (5.13).

We now Taylor expand using (5.43), (5.44), and the fact that $\mathcal{Z}_c^{i\mathbb{R}}(u_1(v), v)) = 0$ to get

$$\begin{aligned}
|\mathrm{Im}(\mathcal{Z}_c(x,v)\cdot v_2)| &= |\mathcal{Z}_c^{i\mathbb{R}}(x,v)\cdot v_2| \\
&\le |\langle (x-u_1(v)), \left(\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x^T A + B\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x\right)(x-u_1(v))\rangle| + C|x-u_1(v)|^3 \\
&\le \epsilon^{-1}\left|\left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x (x-u_1(v))\right|^2 + \epsilon|x-u_1(v)|^2 + C|x-u_1(v)|^3 \\
&\le \epsilon^{-1}\left|\mathcal{Z}_c^{i\mathbb{R}}(x,v)\right|^2 + \epsilon|x-u_1(v)|^2 + C\epsilon^{-1}|x-u_1(v)|^3. \qquad (5.45)
\end{aligned}$$

In the last inequality we used $\mathcal{Z}_c^{i\mathbb{R}}(x,v) = \left(\mathcal{Z}_c^{i\mathbb{R}}\right)_x (u_1(v), v)\,(x-u_1(v)) + O((x-u_1(v))^2)$. Recall

$$\mathcal{Z}_c^{i\mathbb{R}}(x,v) = \mathrm{Im}(\mathcal{Z}_c(x,v)) = \mathrm{Im}(z(\zeta_c'(x,v), x))$$

by (5.23). Finally, applying the estimate (5.31) to the first term in the last line of (5.45) gives

$$|\mathrm{Im}(\mathcal{Z}_c(x,v)\cdot v_2)| \le C\epsilon^{-1}\mathrm{Im}(\psi(x,v)) + \epsilon|x-u_1(v)|^2 + C\epsilon^{-1}|x-u_1(v)|^3$$

and the proof is complete. □

Corollary 5.17. *Let $\hat{x} = \pi(\chi(\hat{v}))$. We may choose $\hat{U} \subset\subset U$ containing $\hat{x}$ and $\hat{\mathcal{V}} \subset\subset \mathcal{V}$ containing $\hat{v}$ small enough such that the phase function $\psi(x, v)$ given by (5.30) satisfies*

$$\mathrm{Im}(\psi(x, v)) \geq c|x - \pi(\chi(v))|^2$$

for all $(x, v) \in \hat{U} \times \hat{\mathcal{V}}$.

Proof. By Corollary 5.15 we have

$$\frac{1}{2}\mathrm{Re}\left[(z(\zeta_c'(x, v), x) - v_1)^2\right] \geq c|x - \pi(\chi(v))|^2 - C\mathrm{Im}(\psi(x, v)).$$

Now choose $\epsilon > 0$ sufficiently small in the estimate of Lemma 5.16 and add it to the above inequality we get

$$\begin{aligned}\mathrm{Im}(\psi(x, v)) &= -\mathrm{Im}(z(\zeta_c'(x, v), x) \cdot v_2) + \frac{1}{2}\mathrm{Re}\left[(z(\zeta_c'(x, v), x) - v_1)^2\right] \\ &\geq c|x - \pi(\chi(v))|^2 - C\epsilon^{-1}\mathrm{Im}(\psi(x, v)) - C\epsilon^{-1}|x - \pi(\chi(v))|^3.\end{aligned}$$

For this choice of ϵ, we now choose $U \times \mathcal{V}$ sufficiently small so that

$$C\epsilon^{-1}|x - \pi(\chi(v))| \leq \frac{c}{2}.$$

for all $(x, v) \in U \times \mathcal{V}$. We then have the desired estimate. □

By Lemma 5.12 the function ψ defined in (5.30) satisfies

$$\psi(x, v) = \Phi(\mathcal{Z}_c(x, v), \eta_c(x, v); x, v) = -\mathcal{Z}_c(x, v) \cdot v_2 + \frac{i(\mathcal{Z}_c(x, v) - v_1)^2}{2}. \tag{5.46}$$

We can now give the proof of the main technical result.

Proof of Proposition 5.6. The existence and uniqueness of the critical point $\zeta_c'(x, v)$ is stated in Corollary 5.9.

Property (1) comes from the definition of ψ for real valued $(x, v) \in U \times \mathcal{V}$ and extends by analyticity to $(x, v) \in U_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$.

For property (2), we first check it for $v \in \mathcal{V}$ real valued and then use analyticity to extend the identity to complex valued $v \in \mathcal{V}_{\mathbb{C}}$. First, write $\chi(v) = (u_1(v), u_2(v))$. Use (5.21) to get

$$\partial_x \left(\phi(\mathcal{Z}_c(x, v), x') - x''\right) = 0.$$

Expanding the derivatives and setting $x = u_1(v)$, so that $\mathcal{Z}_c(u_1(v), v) = v_1$, we get

$$\partial_x \mathcal{Z}_c(x, v)^T|_{x=u_1(v)} \phi_z^T(v_1, u_1(v)') = \begin{pmatrix} -\phi_{x'}^T(v_1, u_1(v)') \\ I_{n''\times n''} \end{pmatrix}.$$

Now multiply both sides of the above equality by $-u_2(v)''$. The identity in (5.15) then gives

$$\partial_x \left(\mathcal{Z}_c(x, v) \cdot v_2\right)|_{x=\pi(\chi(v))} = -u_2(v).$$

Differentiate the expression (5.46) and use the above identity to obtain $d_x\psi(\pi(\chi(v)), v) = u_2(v)$. This is property (2). Property (3) is Corollary 5.17 if we choose $\hat{U}$ and $\hat{\mathcal{V}}$ as in Corollary 5.17. □

We conclude this section with the following remark outlining how one can generalize the argument above to amplitudes $a(z, x, \eta)$ which also depend on η.

Remark 5.18. Let T be an FIO with Schwartz kernel given by

$$T(z,x) := \int_{\mathbb{R}^{n''}} e^{i(\phi(z,x')-x'')\cdot\eta} a(z,x,\eta)\, d\eta$$

where $a(z,x,\eta)$ is a classical analytic symbol. In this case, instead of first performing the η integral as in the proof of Proposition 5.4, we consider the η and z integral jointly in the expression (5.18). We split this integral into two parts:

$$\begin{aligned} K_\lambda(x,v) &= c_{n,N}\lambda^{\frac{3N}{4}+n''} \int\limits_{z\in V}\int\limits_{|\eta|\le R} e^{i\lambda\Phi(z,\eta;x,v)} a(z,x,\eta)\, d\eta\, dz \\ &\quad + c_{n,N}\lambda^{\frac{3N}{4}+n''} \int\limits_{z\in V}\int\limits_{|\eta|\ge R} e^{i\lambda\Phi(z,\eta;x,v)} a(z,x,\eta)\, d\eta\, dz \end{aligned} \tag{5.47}$$

where $\Phi(z,\eta;x,v) = (\phi(z,x')-x'')\cdot\eta - z\cdot v_2 + i\frac{|z-v_1|^2}{2}$. Here R is chosen so that for all (x,v) near $(\hat{x},\hat{v})$,

$$\text{there is no } (z,\eta)\in V\times\mathbb{R}^{n''}\setminus B_R(0) \text{ s.t. } \nabla_{z,\eta}\Phi(z,\eta;x,v)=0. \tag{5.48}$$

One applies stationary phase (see Remark 2.9 and 2.10 of [Sjö82]) to get that

$$c_{n,N}\lambda^{\frac{3N}{4}+n''} \int\limits_{z\in V}\int\limits_{|\eta|\le R} e^{i\lambda\Phi(z,\eta;x,v)} a(z,x,\eta)\, d\eta\, dz \sim \lambda^\ell e^{i\lambda\psi(x,v_1,v_2)} A(x,v_1,v_2;\lambda) \tag{5.49}$$

for some $\ell\in\mathbb{N}$.

Since no critical points of the phase reside in the non-compact part of (5.47) we can write for any $k\in\mathbb{N}$

$$e^{i\lambda\Phi} = \lambda^{-k}\left(\frac{\nabla\bar{\Phi}\cdot\nabla}{|\nabla\Phi|^2}\right)^k e^{i\lambda\Phi} := \lambda^{-k}(\mathcal{L}_\Phi)^k e^{i\lambda\Phi}$$

Applying integration by parts to the non-compact part of (5.47) gives that

$$\begin{aligned} &\int\limits_{z\in V}\int\limits_{|\eta|\ge R} e^{i\lambda\Phi(z,\eta;x,v)} a(z,x,\eta)\, d\eta\, dz \\ &= \lambda^{-k}\int\limits_{z\in V}\int\limits_{|\eta|\ge R} e^{i\lambda\Phi(z,\eta;x,v)} (\mathcal{L}^*)^k a(z,x,\eta)\, d\eta\, dz \\ &+ \sum_{j=1}^{k}\lambda^{-j}\int\limits_{z\in\partial V}\int\limits_{|\eta|\ge R} e^{i\lambda\Phi(z,\eta;x,v)} \frac{\nabla_z\bar{\Phi}\cdot\nu_z}{|\nabla\Phi|^2}(\mathcal{L}^*)^{j-1} a(z,x,\eta) d\eta dz \\ &+ \sum_{j=1}^{k}\lambda^{-j}\int\limits_{z\in V}\int\limits_{|\eta|=R} e^{i\lambda\Phi(z,\eta;x,v)} \frac{\nabla_\eta\bar{\Phi}\cdot\eta}{R|\nabla\Phi|^2}(\mathcal{L}^*)^{j-1} a(z,x,\eta) d\eta dz \end{aligned} \tag{5.50}$$

We now need to verify that each of the terms in (5.50) are exponentially decaying as $\lambda\to\infty$. The first term involving the interior integration can be shown to have exponential decay by using standard pseudoanalytic estimates on the symbol $a(z,x,\eta)$ and by choosing k appropriately as a function of λ.

For the first boundary term, we observe that for v_1 bounded away from the boundary ∂V, the integral decays exponentially as $\lambda\to\infty$.

The second boundary term requires more care. When $x = \pi(\chi(v))$ we can see from relation (5.15) and condition (5.48),that $\iota^*_{V\times\partial B_R(0)}\Phi$ has no critical points. This means that we can integrate by parts in the last term of (5.50) and estimate this term in the same way as we did for the first term of (5.50).

We have argued that all three terms in (5.50) decay exponentially fast. This shows that the second term of (5.47) decays exponentially quickly. This combined with (5.49) shows that

$$K_\lambda(x,v) \sim \lambda^\ell e^{i\lambda\psi(x,v_1,v_2)} A(x,v_1,v_2;\lambda)$$

modulo exponentially decaying terms.

6. Applications to uniqueness theorems in integral geometry

We will now discuss the proofs of Theorems 1.3–1.8 in the introduction. These follow more or less directly from the analytic regularity result in Theorem 1.2, the analysis of the Bolker condition in Sections 3 and 4, and the microlocal analytic continuation result from [Hör85, Theorem 8.5.6'] which is rephrased as follows.

Theorem 6.1. *Let X be an analytic manifold, and let Σ be a C^2 hypersurface through $x_0 \in X$ with conormal ν_0 at x_0. If $f \in \mathcal{D}'(X)$ satisfies $f = 0$ on one side of Σ near x_0 and if $(x_0,\nu_0) \notin \mathrm{WF}_a(f)$ or $(x_0,-\nu_0) \notin \mathrm{WF}_a(f)$, then $f = 0$ near x_0.*

Theorem 1.3 in the introduction is an immediate consequence of Theorem 1.2 and Theorem 6.1.

We proceed to the weighted geodesic X-ray transform R as defined in Section 4.3. In the analytic case, Theorem 6.1 allows us to upgrade Proposition 4.7 to the following uniqueness result that is already contained in [SU08]. As mentioned in the introduction, it might be possible to remove the assumptions that the geodesic has no self-intersections or tangential intersections by using suitable extension procedures.

Proposition 6.2. *Let (M,g) be an analytic manifold with smooth boundary such that g is analytic up to ∂M. Let $z_0 \in \partial_+SM$ satisfy $\tau_+(z_0) < \infty$, and assume that the geodesic $x_{z_0} : [0,\tau_+(z_0)] \to M$ does not self-intersect, has no conjugate points, meets ∂M transversally at the endpoints, and otherwise stays in M^{int}. Let κ be analytic and nowhere vanishing in $\mathcal{G} \times M^{\mathrm{int}}$, and assume that Σ is a C^2 hypersurface such that x_{z_0} is tangent to Σ at $x_0 \in M^{\mathrm{int}}$. If $Rf(z) = 0$ for z near z_0 and if f vanishes on one side of Σ near x_0, then $f = 0$ near x_0.*

We can now prove that the geodesic X-ray transform, possibly with analytic weight, is injective on compact nontrapping strictly convex two-dimensional manifolds in the analytic case.

Proof of Theorem 1.4. Suppose that $f \in C(M)$ and $Rf = 0$. Embed M in a slightly larger analytic nontrapping manifold M_1 with strictly convex boundary, extend g analytically to M_1, and extend f by zero to M_1. Note that since the ray transform of f vanishes and ∂M is strictly convex, we have $f|_{\partial M} = 0$ by looking at short geodesics, so the zero extension is continuous in M_1. Then we have $f \in \mathcal{E}'(M_1^{\mathrm{int}})$ and $\mathrm{supp}(f) \subset M$.

Since ∂M is strictly convex, we can apply Proposition 6.2 near tangential geodesics to ∂M to conclude that f must vanish near ∂M. We now use a layer stripping argument: by [BGL02], the manifold $M \setminus \{p\}$ for some $p \in M^{\mathrm{int}}$ is foliated by strictly convex hypersurfaces Γ_s for $s \in (0,1]$ so that $\Gamma_1 = \partial M$ and $\Gamma_s \to \{p\}$ as $s \to 0$. Define

$$I = \{t \in (0,1] \,:\, f = 0 \text{ in } \cup_{s\in[t,1]} \Gamma_s\}.$$

By the argument above we have $1 \in I$. Moreover, if $t \in I$, then applying Theorem 6.2 for geodesics tangent to Γ_t shows that some neighborhood of t is in I. Thus I is open and closed, so $I = (0, 1]$ by connectedness. This shows that $f = 0$ in $M \setminus \{p\}$ and that $f \equiv 0$ by continuity. □

Next we consider the setting in Section 4.4 and let R be the null bicharacteristic ray transform with nowhere vanishing weight. When all the objects involved are analytic, Theorem 1.5 follows immediately from Proposition 4.11, Theorem 1.2 and Theorem 6.1. We now use this to prove the uniqueness result involving a strictly pseudoconvex foliation. See also [PSUZ19] for further analysis of foliation conditions.

Proof of Theorem 1.6. Define

$$I = \{t \in (0, 1] \mid f = 0 \text{ in an open set containing } \cup_{s\in[t,1]} \Gamma_s\}.$$

By assumption (a) one has $1 \in I$. By definition I is open. To show that I is closed, we suppose that $t_j \in I$ and $t_j \to t \in (0, 1]$. Then $f = 0$ in $\cup_{s\in(t,1)}\Gamma_s$ and in particular f vanishes on one side of Γ_t. Let $x \in \Gamma_t$ and let $\nu = dF(x)$ be conormal to Γ_t at x. By assumption (b), there is $\xi \in \Xi_x$ such that $\nu \perp Y^h(x, \xi)$ and $\nu \nparallel \xi$. Then $z = (x, \xi)$ is such that x_z is tangent to Γ_t at x. Since $p(x, \xi) = 0$ and $\{p, F\}(x, \xi) = 0$, assumption (c) can be rephrased as

$$\partial_s^2[F(x_z(s))]\big|_{s=0} = H_p^2 F(x, \xi) = \{p, \{p, F\}\}(x, \xi) > 0. \tag{6.1}$$

Let $\tilde{M} = \cup_{s\in(0,t+\delta]}\Gamma_s$. If $\delta > 0$ is sufficiently small, the segment of the curve x_z that lies in $\tilde{M}$ is very short by (6.1) and does not have conjugate points by Lemma 4.10. It also meets $\partial\tilde{M}$ transversally at the endpoints and does not self-intersect. We can now invoke Theorem 1.5 in $\tilde{M}$ to conclude that $f = 0$ near x. Repeating this argument for all $x \in \Gamma_t$ shows that $t \in I$. Consequently I is closed, and by connectedness $I = (0, 1]$. □

Finally we consider the transform introduced in Section 1.3.3, repeating some of the statements given there. Let X be a manifold without boundary. We will study a transform that integrates a function f over submanifolds given by

$$G_{\theta,s} = \{x \in X \mid b(x, \theta) = s\}.$$

Here $b : X \times V' \to \mathbb{R}^k$ is a smooth function, $\theta \in V'$ and $s \in V''$, where V' and V'' are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Here $m \geq 1$ and $1 \leq k \leq n - 1$. Comparing with Lemma 2.6, we have relabeled variables so that $z = (z', z'') = (\theta, s)$ and $m = N - n''$, $k = n''$. We also assume that we are considering integrals over $G_{\theta,s}$ with (θ, s) close to a fixed (θ_0, s_0), so we can indeed work in local coordinates and take $\mathcal{G} = V' \times V''$. Moreover, if we let $U \subset X$ be a neighborhood of G_{θ_0,s_0} such that $G_{\theta,s} \subset U$ for $(\theta, s) \in \mathcal{G}$, it is enough to have everything defined for $x \in U$.

We assume that $b_x(x, \theta)$ is surjective for $x \in U$ and $\theta \in V'$. Then Lemma 2.6 ensures that $Z = \{(\theta, s, x) \mid b(x, \theta) = s\}$ is a double fibration. If we fix orientation forms on Z and $\mathcal{G}$ (we can use the Lebesgue measure on $\mathcal{G} \subset \mathbb{R}^N$), we obtain orientation forms on each $G_{\theta,s}$ and consider the weighted double fibration transform

$$Rf(\theta, s) = \int_{G_{\theta,s}} \kappa(\theta, s, x) f(x)\, d\omega_{G_{\theta,s}}(x), \qquad (\theta, s) \in \mathcal{G},$$

where κ is smooth and nowhere vanishing.

We next study the Bolker condition. By Lemma 2.6, the canonical relation of R may be written as

$$C = \{(z, \zeta, x, -B(z, x)\zeta) \mid b(x, \theta) = s,\ \zeta \in N_z^* H_x \setminus 0\} \tag{6.2}$$

where $z = (\theta, s) = (\theta, b(x,\theta))$, and $\zeta \in N_z^* H_x \subset \mathbb{R}^N$ has the form $\zeta = (-b_\theta(x,\theta)^T \zeta'', \zeta'')$. From Lemma 3.3 we see that $d\pi_L$ is injective if for any $x \in U$, $\theta \in V'$ and $\zeta'' \in \mathbb{R}^{n''} \setminus 0$, the linear map

$$\big(b_x(x,\theta)^T, \quad \partial_\theta(b_x(x,\theta)^T \zeta'')\big) \text{ is surjective.}$$

We also see from (6.2) that π_L is injective if for any $\theta \in V'$ and $\zeta'' \in \mathbb{R}^{n''} \setminus 0$, the map

$$x \mapsto (b(x,\theta), b_\theta(x,\theta)^T \zeta'') \text{ is injective on } U.$$

Thus the Bolker condition indeed amounts to (1.3) and (1.4).

Using these facts, the combination of Theorem 1.2 and Theorem 6.1 yields Theorem 1.8 in the introduction.

References

[AZ17] Y. M. Assylbekov and H. Zhou, *Invariant distributions and tensor tomography for Gaussian thermostats*, Comm. Anal. Geom. **25** (2017), no. 5, 895–926, DOI 10.4310/CAG.2017.v25.n5.a1.

[BE89] R. J. Baston and M. G. Eastwood, *The Penrose transform*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1989. Its interaction with representation theory; Oxford Science Publications.

[BGL02] S. Betelú, R. Gulliver, and W. Littman, *Boundary control of PDEs via curvature flows: the view from the boundary. II*, Appl. Math. Optim. **46** (2002), no. 2-3, 167–178, DOI 10.1007/s00245-002-0742-6. Special issue dedicated to the memory of Jacques-Louis Lions.

[Bey84] G. Beylkin, *The inversion problem and applications of the generalized Radon transform*, Comm. Pure Appl. Math. **37** (1984), no. 5, 579–599, DOI 10.1002/cpa.3160370503.

[BQ87] J. Boman and E. T. Quinto, *Support theorems for real-analytic Radon transforms*, Duke Math. J. **55** (1987), no. 4, 943–948, DOI 10.1215/S0012-7094-87-05547-5.

[BQ93] J. Boman and E. T. Quinto, *Support theorems for Radon transforms on real analytic line complexes in three-space*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 877–890, DOI 10.2307/2154410.

[BJ20] Y. G. Bonthonneau and M. Jézéquel, *FBI transform in Gevrey classes and Anosov flows* (2020). arXiv:2001.03610.

[Dai06] N. S. Dairbekov, *Integral geometry problem for nontrapping manifolds*, Inverse Problems **22** (2006), no. 2, 431–445, DOI 10.1088/0266-5611/22/2/003.

[DPSU07] N. S. Dairbekov, G. P. Paternain, P. Stefanov, and G. Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*, Adv. Math. **216** (2007), no. 2, 535–609, DOI 10.1016/j.aim.2007.05.014.

[dHILS23] M. de Hoop, J. Ilmavirta, M. Lassas, and T. Saksala, *Determination of a compact Finsler manifold from its boundary distance map and an inverse problem in elasticity*, Comm. Anal. Geom. (2023), available at arXiv:1901.03902.

[FIO21] A. Feizmohammadi, J. Ilmavirta, and L. Oksanen, *The light ray transform in stationary and static Lorentzian geometries*, J. Geom. Anal. **31** (2021), no. 4, 3656–3682, DOI 10.1007/s12220-020-00409-y.

[FGGN22] R. Felea, R. Gaburro, A. Greenleaf, and C. Nolan, *Microlocal analysis of borehole seismic data*, Inverse Probl. Imaging **16** (2022), no. 6, 1543–1570, DOI 10.3934/ipi.2022026.

[FLU03] D. Finch, I.-R. Lan, and G. Uhlmann, *Microlocal analysis of the x-ray transform with sources on a curve*, Inside out: inverse problems and applications, Math. Sci. Res. Inst. Publ., vol. 47, Cambridge Univ. Press, Cambridge, 2003, pp. 193–218.

[FSU08] B. Frigyik, P. Stefanov, and G. Uhlmann, *The X-ray transform for a generic family of curves and weights*, J. Geom. Anal. **18** (2008), no. 1, 89–108, DOI 10.1007/s12220-007-9007-6.

[GGG03] I. M. Gelfand, S. G. Gindikin, and M. I. Graev, *Selected topics in integral geometry*, Translations of Mathematical Monographs, vol. 220, American Mathematical Society, Providence, RI, 2003. Translated from the 2000 Russian original by A. Shtern.

[GGŠ69] I. M. Gel′fand, M. I. Graev, and Z. Ja. Šapiro, *Differential forms and integral geometry*, Funkcional. Anal. i Priložen. **3** (1969), no. 2, 24–40 (Russian).

[GU89] A. Greenleaf and G. Uhlmann, *Nonlocal inversion formulas for the X-ray transform*, Duke Math. J. **58** (1989), no. 1, 205–240, DOI 10.1215/S0012-7094-89-05811-0.

[GU90] A. Greenleaf and G. Uhlmann, *Microlocal techniques in integral geometry*, Integral geometry and tomography (Arcata, CA, 1989), Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 121–135, DOI 10.1090/conm/113/1108649.

[Gui17] C. Guillarmou, *Lens rigidity for manifolds with hyperbolic trapped sets*, J. Amer. Math. Soc. **30** (2017), no. 2, 561–599, DOI 10.1090/jams/865.

[GS77] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Mathematical Surveys, No. 14, American Mathematical Society, Providence, R.I., 1977.

[Gui85] V. Guillemin, *On some results of Gel'fand in integral geometry*, Pseudodifferential operators and applications (Notre Dame, Ind., 1984), Proc. Sympos. Pure Math., vol. 43, Amer. Math. Soc., Providence, RI, 1985, pp. 149–155, DOI 10.1090/pspum/043/812288.

[Hel66] S. Helgason, *A duality in integral geometry on symmetric spaces*, Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), Nippon Hyoronsha, Tokyo, 1966, pp. 37–56.

[Hel11] S. Helgason, *Integral geometry and Radon transforms*, Springer, New York, 2011.

[HU18] S. Holman and G. Uhlmann, *On the microlocal analysis of the geodesic X-ray transform with conjugate points*, J. Differential Geom. **108** (2018), no. 3, 459–494, DOI 10.4310/jdg/1519959623.

[HZ17] A. Homan and H. Zhou, *Injectivity and stability for a generic class of generalized Radon transforms*, J. Geom. Anal. **27** (2017), no. 2, 1515–1529, DOI 10.1007/s12220-016-9729-4.

[Hör85] L. Hörmander, *The analysis of linear partial differential operators. I-IV*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1985.

[Hör71] L. Hörmander, *Fourier Integral Operators*, Acta Math. **127** (1971), 79–183.

[Ilm18] J. Ilmavirta, *X-ray transforms in pseudo-Riemannian geometry*, J. Geom. Anal. **28** (2018), no. 1, 606–626, DOI 10.1007/s12220-017-9834-z.

[KSU07] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. (2) **165** (2007), no. 2, 567–591, DOI 10.4007/annals.2007.165.567.

[Kri09] V. P. Krishnan, *A support theorem for the geodesic ray transform on functions*, J. Fourier Anal. Appl. **15** (2009), no. 4, 515–520, DOI 10.1007/s00041-009-9061-5.

[KS09] V. P. Krishnan and P. Stefanov, *A support theorem for the geodesic ray transform of symmetric tensor fields*, Inverse Probl. Imaging **3** (2009), no. 3, 453–464, DOI 10.3934/ipi.2009.3.453.

[tKSV98] A. P. E. ten Kroode, D.-J. Smit, and A. R. Verdel, *A microlocal analysis of migration*, Wave Motion **28** (1998), no. 2, 149–172, DOI 10.1016/S0165-2125(98)00004-3.

[LOSU20] M. Lassas, L. Oksanen, P. Stefanov, and G. Uhlmann, *The light ray transform on Lorentzian manifolds*, Comm. Math. Phys. **377** (2020), no. 2, 1349–1379, DOI 10.1007/s00220-020-03703-6.

[Mar02] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Universitext, Springer-Verlag, New York, 2002.

[MSU15] F. Monard, P. Stefanov, and G. Uhlmann, *The geodesic ray transform on Riemannian surfaces with conjugate points*, Comm. Math. Phys. **337** (2015), no. 3, 1491–1513, DOI 10.1007/s00220-015-2328-6.

[Muh77] R. G. Muhometov, *The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry*, Dokl. Akad. Nauk SSSR **232** (1977), no. 1, 32–35 (Russian).

[NS97] C. J. Nolan and W. W. Symes, *Global solution of a linearized inverse problem for the wave equation*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 919–952, DOI 10.1080/03605309708821289.

[Nol00] C. J. Nolan, *Scattering in the presence of fold caustics*, SIAM J. Appl. Math. **61** (2000), no. 2, 659–672, DOI 10.1137/S0036139999356107.

[OSSU20] L. Oksanen, M. Salo, P. Stefanov, and G. Uhlmann, *Inverse problems for real principal type operators*, Amer. J. Math. (2020). arXiv:2001.07599.

[PSU13] G. P. Paternain, M. Salo, and G. Uhlmann, *Tensor tomography on surfaces*, Invent. Math. **193** (2013), no. 1, 229–247, DOI 10.1007/s00222-012-0432-1.

[PSU23] G. P. Paternain, M. Salo, and G. Uhlmann, *Geometric inverse problems—with emphasis on two dimensions*, Cambridge Studies in Advanced Mathematics, vol. 204, Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.

[PSUZ19] G. P. Paternain, M. Salo, G. Uhlmann, and H. Zhou, *The geodesic X-ray transform with matrix weights*, Amer. J. Math. **141** (2019), no. 6, 1707–1750, DOI 10.1353/ajm.2019.0045.

[Qui80] E. T. Quinto, *The dependence of the generalized Radon transform on defining measures*, Trans. Amer. Math. Soc. **257** (1980), no. 2, 331–346, DOI 10.2307/1998299.

[Qui93] E. T. Quinto, *Real analytic Radon transforms on rank one symmetric spaces*, Proc. Amer. Math. Soc. **117** (1993), no. 1, 179–186, DOI 10.2307/2159714.

[RSN20] O. Rouby, J. Sjöstrand, and S. V. Ngọc, *Analytic Bergman operators in the semiclassical limit*, Duke Math. J. **169** (2020), no. 16, 3033–3097, DOI 10.1215/00127094-2020-0022.

[SKK73] M. Sato, T. Kawai, and M. Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Lecture Notes in Math., Vol. 287, Springer, Berlin, 1973, pp. 265–529.

[Sjö82] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque, 95, Astérisque, vol. 95, Soc. Math. France, Paris, 1982, pp. 1–166 (French).

[Ste17] P. Stefanov, *Support theorems for the light ray transform on analytic Lorentzian manifolds*, Proc. Amer. Math. Soc. **145** (2017), no. 3, 1259–1274, DOI 10.1090/proc/13117.

[SU05] P. Stefanov and G. Uhlmann, *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc. **18** (2005), no. 4, 975–1003, DOI 10.1090/S0894-0347-05-00494-7.

[SU08] P. Stefanov and G. Uhlmann, *Integral geometry on tensor fields on a class of non-simple Riemannian manifolds*, Amer. J. Math. **130** (2008), no. 1, 239–268, DOI 10.1353/ajm.2008.0003.

[SU13] P. Stefanov and G. Uhlmann, *Is a curved flight path in SAR better than a straight one?*, SIAM J. Appl. Math. **73** (2013), no. 4, 1596–1612, DOI 10.1137/120882639.

[SdH02] C. C. Stolk and M. V. de Hoop, *Microlocal analysis of seismic inverse scattering in anisotropic elastic media*, Comm. Pure Appl. Math. **55** (2002), no. 3, 261–301, DOI 10.1002/cpa.10019.

[Trè80] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators. Vol. 2*, University Series in Mathematics, Plenum Press, New York-London, 1980. Fourier integral operators.

[Trè22] F. Trèves, *Analytic partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 359, Springer, Cham, 2022.

[UV16] G. Uhlmann and A. Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math. **205** (2016), no. 1, 83–120, DOI 10.1007/s00222-015-0631-7.

[VW21] A. Vasy and Y. Wang, *On the light ray transform of wave equation solutions*, Comm. Math. Phys. **384** (2021), no. 1, 503–532, DOI 10.1007/s00220-021-04045-7.

[Wan18] Y. Wang, *Parametrices for the light ray transform on Minkowski spacetime*, Inverse Probl. Imaging **12** (2018), no. 1, 229–237, DOI 10.3934/ipi.2018009.

[Wan21] Y. Wang, *Microlocal analysis of the light ray transform on globally hyperbolic Lorentzian manifolds*, Amer. J. Math. (2021). arXiv:2104.08576.

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