# INVARIANT SETS AND HYPERBOLIC CLOSED REEB ORBITS

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Abstract. We investigate the effect of a hyperbolic (or, more generally, isolated as an invariant set) closed Reeb orbit on the dynamics of a Reeb flow on the $(2n-1)$-dimensional standard contact sphere, extending two results previously known for Hamiltonian diffeomorphisms to the Reeb setting. In particular, we show that under very mild dynamical convexity type assumptions, the presence of one hyperbolic closed orbit implies the existence of infinitely many simple closed Reeb orbits. The second main result of the paper is a higher-dimensional Reeb analogue of the Le Calvez–Yoccoz theorem, asserting that no closed orbit of a non-degenerate dynamically convex Reeb pseudo-rotation is locally maximal, i.e., isolated as an invariant set. The key new ingredient of the proofs is a Reeb variant of the crossing energy theorem.

Contents



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## 1. Introduction and main results

1.1. **Introduction.** We investigate the impact of hyperbolic or, more generally, locally maximal as an invariant set closed Reeb orbits on the dynamics of Reeb flows on the standard contact sphere $S^{2n-1}$. We prove that, under a very mild dynamical convexity type assumption, the presence of one hyperbolic closed orbit implies the existence of infinitely many simple closed Reeb orbits. In a related theorem, we show that for non-degenerate dynamically convex Reeb flows on the sphere, the same is true when there is a locally maximal closed Reeb orbit. These results hold for all dimensions $2n-1 \geq 3$, but they are primarily of interest when $2n-1 \geq 5$: in dimension three the results can be derived from other known facts in low-dimensional dynamics and three-dimensional contact topology.

In symplectic dynamics, even a minimal input of localized hyperbolicity, such as the presence of one or several hyperbolic periodic orbits, can have a strong impact on non-local dynamics of the system. Moreover, in some instances hyperbolicity can be replaced by the weaker condition that the orbit is locally maximal, i.e., isolated as an invariant set. This phenomenon manifests itself in a variety of disparate ways. We illustrate it by the next two examples, only tangentially related to the main theme of the paper.

The first one is provided by results from, e.g., [Ha, Xi] asserting that $C^1$-generically the stable and unstable manifolds of a hyperbolic periodic point of a Hamiltonian diffeomorphism $\varphi$ of a closed symplectic manifold $M$ have transverse non-empty intersections. As a consequence, $\varphi$ has a horseshoe and positive topological entropy. (This is a construction somewhat similar to the $C^1$-closing lemma.) In dimension two, this is also true $C^\infty$-generically, [LCS22, LCS23]. We also note that for many manifolds $M$, a Hamiltonian diffeomorphism has infinitely many hyperbolic periodic points $C^\infty$-generically; see, e.g., [ÇGG22b, ÇGG23] and references therein.

The second example, more of a symplectic geometric nature, is that for any Hamiltonian diffeomorphism $\varphi\colon M \to M$ with sufficiently many hyperbolic periodic points the spectral norm $\gamma(\varphi^k)$ of the iterates is bounded away from zero, [ÇGG22b]. The lower bound on the required number of hyperbolic points depends only on the topology of $M$; for instance, for $S^2$ just one such a point is sufficient.

In this paper we extend to dynamically convex Reeb flows on $S^{2n-1}$ two results about the effect of hyperbolic or locally maximal periodic points on global dynamics of Hamiltonian diffeomorphisms of $\mathbb{CP}^n$.

The first of these results is that a Hamiltonian diffeomorphism of $\mathbb{CP}^n$ with a hyperbolic periodic point necessarily has infinitely many periodic points; see [GG14, Thm. 1.1] and also [Al22a]. (In fact, the theorem holds for a broader class of closed symplectic manifolds.) In dimension two, i.e., for $S^2 = \mathbb{CP}^1$, this theorem readily follows from the celebrated theorem of Franks from [Fr92, Fr96] asserting that an area preserving diffeomorphism of $S^2$ with more than two periodic points must have infinitely many periodic points; see also [LeC]. Furthermore, when the stable and unstable manifolds of the hyperbolic periodic point intersect transversely and non-trivially, the resulting horseshoe immediately provides infinitely many periodic points. Moreover, as we have pointed out above, such intersections exist $C^1$-generically in all dimensions and $C^\infty$-generically in dimension two. Hence, it is essential that no intersection condition is imposed in [GG14, Thm. 1.1].

Recently, Franks' theorem has been (partially) generalized to a class of symplectic manifolds of any dimension including $\mathbb{CP}^n$ under a minor non-degeneracy requirement; see [Sh22] and also [Al22b, Al23, ÇGG22a]. This higher-dimensional variant of Franks theorem, originally conjectured by Hofer and Zehnder in [HZ], implies [GG14, Thm. 1.1] in all dimensions. A different conjecture inspired by that theorem goes beyond the orbit count and asserts that a Hamiltonian dynamical system (in a very broad sense) must have infinitely many simple periodic orbits whenever it has a periodic orbit which is homologically or geometrically unnecessary, e.g., a non-contractible or degenerate orbit for Hamiltonian diffeomorphisms. We refer the reader to, for instance, [Ba15, Ba17, Ba18, GG16, Gü13, Or17, Or20, Su21] for some sample results in this direction.

In the spirit of [GG14, Thm. 1.1], the first main result of this paper (Theorem A) is an extension of that theorem to dynamically convex, in a very loose sense, Reeb flows on $S^{2n-1}$. Thus this result can be viewed as the first step towards the contact Franks' theorem in all dimensions.

Our second main result (Theorem B) is a Reeb analogue of the higher-dimensional Le Calvez–Yoccoz theorem, [GG18, Thm. 4.1], on invariant sets of pseudo-rotations. Roughly speaking, a Hamiltonian pseudo-rotation is a Hamiltonian diffeomorphism with the minimal possible number of periodic points, where the lower bound is usually interpreted in terms of Arnold's conjecture. Pseudo-rotations in dimension two have been extensively studied by dynamical systems methods (see, e.g., [A-Z, LCY] and references therein) and also by holomorphic curves techniques, [Br15a, Br15b]. Recently, Floer theoretic methods have been used to study Hamiltonian pseudo-rotations in all dimensions; see [ÇS, GG18, JS]. While official definitions of a pseudo-rotation vary (see, e.g., [ÇGG20a, ÇGG20b, Sh20, Sh21]), for $\mathbb{CP}^n$ they all amount to requiring a Hamiltonian diffeomorphism to have exactly $n+1$ periodic points which are then necessarily the fixed points. Pseudo-rotations can have quite complicated dynamics. For instance, the Anosov–Katok conjugation method, originally developed in [AK] (see also [FK]) yields area-preserving diffeomorphisms of $S^2$ with exactly three ergodic measures: the two fixed points and the area form. The conjugation method was extended to higher dimensions in the Hamiltonian setting in [LRS], leading in particular to a construction of Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ with exactly $n+1$ ergodic measures: the fixed points and the volume.

Such pseudo-rotations are uniquely ergodic outside the fixed point set, and one would expect every orbit to be either periodic or dense. This is, however, not true. The celebrated Le Calvez–Yoccoz theorem asserts that for an area-preserving pseudo-rotation of $S^2$ no fixed point is locally maximal, i.e., isolated as an invariant set; see [LCY] and also [Fr99, FM, Sal]. In [GG18, Thm. 4.1] this result is generalized to Hamiltonian pseudo-rotations of $\mathbb{CP}^n$ in all dimensions. We note that here a pseudo-rotation is not required to be non-degenerate, although this is the case in all known examples.

The contact analogue of a pseudo-rotation is a Reeb flow with finitely many periodic orbits. (For the sake of simplicity we are leaving aside the requirement that the number of closed Reeb orbits is minimal; see Section 1.2 and Remark 1.2.) Such flows can also have very involved dynamics. For instance, ergodic pseudo-rotations on $S^{2n-1}$ were constructed in [Ka]; these are Reeb flows on certain $C^\infty$-small perturbations of irrational ellipsoids. In [AGZ], pseudo-rotations of $S^3$ with exactly three invariant measures are constructed by applying the "contact suspension" to Anosov–Katok pseudo-rotations of the disk. In Theorem B we show that no closed Reeb orbit of a dynamically convex, non-degenerate pseudo-rotation of $S^{2n-1}$ is locally maximal.

In the next section we will precisely state our main results, discuss them in more detail and also touch upon the key ingredients of the proofs. Here we only mention that the central new component of the proofs is a Reeb analogue of the Crossing Energy Theorem from [ÇGG21, GG14, GG18].

1.2. **Main results.** Both of our main results concern the Reeb flow of a contact form $\alpha$ on $S^{2n-1}$ supporting the standard contact structure. To state the theorems, recall that a contact form or a Reeb flow is said to be dynamically convex when $\mu_-(x) \geq n+1$ for all closed contractible Reeb orbits $x$, where $\mu_-$ is the lower semi-continuous extension of the Conley–Zehnder index $\mu$. The Reeb flow on a convex hypersurface in $\mathbb{R}^{2n}$ is dynamically convex, [HWZ]. Our first result requires a condition similar to dynamical convexity but notably less restrictive. Namely, denote by $\hat{\mu}(x)$ the mean index of a closed Reeb orbit $x$ and by $2\nu_{alg}(x)$ the algebraic multiplicity of the eigenvalue 1 of the Poincaré return map of $x$. We refer the reader to Section 2.2 for a more detailed discussion of the Conley–Zehnder and mean indices and of dynamical convexity type conditions and for further references.

**Theorem A.** *Assume that $(S^{2n-1}, \alpha)$ has a hyperbolic (simple) closed Reeb orbit $z$ with $\hat{\mu}(z) > 0$ and*

$$\mu_-(x) \geq \max\{3,\, 2 + \nu_{alg}(x)\}$$

*for all, not necessarily simple, periodic orbits $x$ with $\hat{\mu}(x) > 0$. Then the Reeb flow of $\alpha$ has infinitely many simple periodic orbits.*

The condition of the theorem is met when $\alpha$ is dynamically convex and $2n-1 \geq 3$. Indeed, then for all closed Reeb orbits $\mu_-(x) \geq n+1 \geq 3$ and $\mu_-(x) - \nu_{alg}(x) \geq (n+1) - (n-1) \geq 2$ since $\nu_{alg}(x) \leq n-1$; see Section 2.2. Furthermore, note that $\hat{\mu}(x) \geq (n+1) - (n-1) \geq 2$ by (2.6). We emphasize that we do not impose any non-degeneracy requirements on the Reeb flow in Theorem A.

In contrast, the non-degeneracy and essentially full dynamical convexity conditions are essential for the proof of our second main result:

**Theorem B.** *Assume that $(S^{2n-1}, \alpha)$ with $2n-1 \geq 3$ is dynamically convex, non-degenerate and its Reeb flow has only finitely many simple closed orbits, i.e., the flow is a Reeb pseudo-rotation. Then no closed orbit of the flow is locally maximal, i.e., isolated as an invariant set.*

Since hyperbolic orbits are obviously locally maximal, this theorem would be a stronger statement than Theorem A if not for the more restrictive conditions on the Reeb flow.

*Remark* 1.1. In fact, as is easy to see from the proof, we prove a slightly stronger result than Theorem B. Namely, assume that all closed Reeb orbits $x$ of the Reeb flow on $(S^{2n-1\geq 3}, \alpha)$ with $\hat{\mu}(x) > 0$ are non-degenerate and $\mu(x) \geq n+1$ and that one of such orbits is locally maximal. Then the flow has infinitely many simple closed orbits with $\hat{\mu} > 0$.

Both of these results are primarily of interest when $2n-1 \geq 5$. In dimension three, Theorem A readily follows from the existence of a global surface of section, [HWZ], and Franks' theorem, [Fr92, Fr96]. Furthermore the non-degenerate version of Franks' theorem is known to hold in dimension three: every non-degenerate Reeb flow on a closed contact 3-manifold has either exactly two closed orbits, which are then elliptic, or infinitely many; see [CGHP, CDR]. Moreover, for the standard contact sphere $S^3$ this is true without the non-degeneracy requirement, [CGH, GH²M].

Theorem B holds for any non-degenerate Reeb flow with finitely many periodic orbits on a closed 3-manifold. The reason is that the Le Calvez–Yoccoz theorem is in fact local. To be more precise, an irrationally elliptic fixed point (or equivalently an elliptic fixed point which is non-degenerate along with all iterates) of an area preserving diffeomorphism of a surface is never locally maximal. This is an immediate consequence of the topological proof of the Le Calvez–Yoccoz theorem by Franks; see [Fr99, Prop. 3.1] and also [FM]. For the sake of completeness, we have included a proof in the Appendix (Section 7) – see Theorem 7.1 and Corollary 7.2, closely following Franks' argument. However, to the best of our knowledge, nothing like this local result is known in higher dimensions. In other words, it is not known if a non-degenerate (with all iterates) elliptic fixed point of a Hamiltonian diffeomorphism is necessarily locally maximal.

Theorem A and its proof are closely related to the multiplicity problem for simple closed Reeb orbits on $S^{2n-1\geq 5}$, and this is where the dynamical convexity–type condition becomes essential. This problem is an analogue of the Arnold conjecture for Reeb flows on the standard contact sphere and concerns with the minimal number of such orbits. Hypothetically, this number is $n$. The question has been extensively studied and we refer the reader to, e.g., [DL²W, GG20, GGMa, GK, Lo, LZ] for some relevant results and further references. However, all these results require the Reeb flow to meet some additional requirements. Without a dynamical convexity–type condition (or symmetry), it is not even known if in general a Reeb flow on $S^{2n-1\geq 5}$ must have more than one simple closed Reeb orbit, or if there are more than two simple closed Reeb orbits when the flow is non-degenerate; see [Gü15, Rmk. 3.3] and also [AGKM].

Likewise, it is tempting to conjecture that a variant of Franks theorem holds for Reeb flows on $S^{2n-1\geq 5}$: a flow with more than $n$ simple closed Reeb orbits must have infinitely many such orbits. Theorem A and also [Gü15, Thm. 1.7] are the only results known to us supporting this conjecture when $2n-1 \geq 5$.

1.3. **About the proofs.** The proofs of the two main theorems are quite similar and hinge on three key results. These are the Crossing Energy Theorem (Theorem 4.1), the Floer Homology Vanishing Theorem (Theorem 4.3) and the Index Recurrence Theorem (Theorem 4.7).

In the Hamiltonian setting, the Crossing Energy Theorem asserts that whenever a 1-periodic orbit $z$ of a Hamiltonian diffeomorphism $\varphi_H$ is locally maximal (e.g., hyperbolic), every Floer cylinder $u$ for $\varphi_H^k$ asymptotic to the iterates $z^k$ at either end has energy $E(u)$ bounded from below by a constant $c_\infty > 0$ independent of $k$; see [ÇGG21, GG14, GG18]. For our purposes independence of $k$ is crucial;

furthermore, for a fixed $k$ the lower bound readily follows from a suitable variant of Gromov compactness.

A simple proof of the Crossing Energy Theorem in this case is based on the fact that every loop $t \mapsto u(s,t)$, $t \in S^1_k = \mathbb{R}/k\mathbb{Z}$, is an $\epsilon$-pseudo-orbit of the Hamiltonian flow $\varphi_H^t$, i.e., it deviates from the flow by no more than $\epsilon$ in time-one, where $\epsilon$ is small when $e = E(u)$ is small. In fact, we can take $\epsilon = O\big(e^{1/4}\big)$ uniformly in $k$; see [Sa99, Sec. 1.5] or Remark 6.4. Then, arguing by contradiction, we assume that $e \to 0$ for some sequences $k = k_i \to \infty$ and $u = u_{k_i}$. Let $V$ be a compact isolating neighborhood of $z$. For each $u$, pick $s \in \mathbb{R}$ such that $u(s,\cdot)$ is tangent to $\partial V$ and contained in $V$. (Strictly speaking, here we have to work with $V \times \mathbb{R}/\mathbb{Z}$ unless $H$ is autonomous.) Passing to the limit as $k \to \infty$ and hence $\epsilon \to 0$, we obtain an integral curve of $\varphi_H^t$ entirely contained in $V$ and different from $z$, which is impossible since $z$ is locally maximal.

Generalizing this argument to the Reeb and symplectic homology setting encounters several difficulties. First of all, it is not entirely clear how to state the Crossing Energy Theorem on the level of Reeb flows and/or symplectic homology. This forces us to work with admissible Hamiltonians $H$ on the symplectic completion $\widehat{W}$ of a Liouville domain $W$, which are constant on $W$. But then the 1-periodic orbit $\tilde{z}$ of $H$ corresponding to a locally maximal closed orbit $z$ of the Reeb flow on $\partial W$ is no longer maximal and, in addition, a Floer cylinder $u$ for $kH$ asymptotic $\tilde{z}^k$ can hypothetically get arbitrarily close to $W$ where the Hamiltonian vector field $X_H$ is close to zero. As a consequence, the above argument breaks down. We show however that, under certain extra conditions, this does not happen: $u$ remains some distance from $W$; see Theorem 6.1. This is sufficient to prove a variant of the Crossing Energy Theorem suitable for our purposes.

We should mention that recently a few other variants of the Crossing Energy Theorem have been established: for $\mathbb{CP}^n$ by employing generating functions in [Al22a]; for geodesic flows via finite-dimensional approximations in [GGMz]; and finally in [Pr] for certain holomorphic curves in the symplectization by using the machinery of feral holomorphic curves developed in [FH]. Let us, however, emphasize that none of the other variants of the Crossing Energy Theorem is currently applicable in the setting of our main theorems, and hence Theorem 4.1 is indispensable for this work.

The second key ingredient of the proof concerns with the vanishing of the (non-equivariant) symplectic homology $\mathrm{SH}(W)$. To be more precise, denote by $\mathrm{SH}^I(\alpha)$ the filtered symplectic homology of the contact form $\alpha$ on $\partial W$. Then, whenever $\mathrm{SH}(W) = 0$, there exists a constant $C \geq 0$ such that the natural map $\mathrm{SH}^I(\alpha) \to \mathrm{SH}^{I+C}(\alpha)$ is zero. In particular, when $I = \mathbb{R} = I + C$, we have $\mathrm{SH}(\alpha)^I = \mathrm{SH}(W) = \mathrm{SH}^{I+C}(\alpha)$, the map in question is the identity and we get back the assumption that $\mathrm{SH}(W) = 0$. In other words, the condition that $\mathrm{SH}(W) = 0$ implies that the every bar of the persistence module $\mathrm{SH}^{(-\infty,a)}(\alpha)$ has length at most $C$. (This observation is originally due to Kei Irie; we refer the reader to [GS] or Section 4.2 for a proof.) This is the case, for instance, when $W$ is a star-shaped domain in $\mathbb{R}^{2n}$ with smooth boundary, i.e., $\alpha$ is a contact form on the standard contact sphere $S^{2n-1}$.

In general, the statement is no longer literally true if we replace $\mathrm{SH}^I(\alpha)$ by the filtered Floer homology $\mathrm{HF}^I(H)$ for an admissible Hamiltonian $H$ on $\widehat{W}$. For instance, $\mathrm{HF}(H) \neq 0$ in general. However, we show in Theorem 4.3 that an analogue of this vanishing result holds for the family of the filtered Floer homology groups

$\mathrm{HF}^I(kH)$, $k \in \mathbb{N}$, with $C$ independent of $k$, as long as the right end-point of $I$ is within a certain range which grows linearly with $k$.

The final key ingredient of the proof is the Index Recurrence Theorem (Theorem 4.7). This is a symplectic linear algebra or number theory result roughly asserting that for a finite collection $\Phi_i \in \widetilde{\mathrm{Sp}}(2m)$, the sequences of Conley–Zehnder indices of the iterates $\phi_i^k$ have a certain recurrence property. As stated and used here, the theorem was proved in [GG20], but it can also be derived from the common jump theorem from [Lo, LZ]. The two theorems are closely related and make a central component of the proofs of many multiplicity results. For instance, combined with dynamical convexity, the Index Recurrence Theorem allows one in the non-degenerate case to construct infinitely many index intervals of length $2m$ such that each sequence $\mu\big(\Phi_i^k\big)$ enters each interval at most once.

Theorems A and B are proved by contradiction. Assuming that the flow has only finitely many simple periodic orbits we use the Index Recurrence and Crossing Energy Theorems to find arbitrarily long action intervals $I$ such that for a suitable Hamiltonian $H$ and all large $k \in \mathbb{N}$, a locally maximal closed Reeb orbit $z$ gives rise to a non-zero class in $\mathrm{HF}^I(kH)$ with action at the center of the interval. Then the map $\mathrm{HF}^I(kH) \to \mathrm{HF}^{I+C}(kH)$ is non-zero. This is impossible by Theorem 4.3. Theorem A requires much weaker dynamical convexity conditions and no non-degeneracy for other orbits because $\mu(z^k) = k\mu(z)$ for a hyperbolic orbit $z$. This enables us to use the Index Recurrence Theorem in a more precise way tying the index sequences of other closed orbits to $\mu(z^k)$.

*Remark* 1.2. In connection with the discussion in Section 1.2 let us point out an interesting discrepancy between Theorem B and the higher-dimensional Le Calvez–Yoccoz theorem, [GG18, Thm. 4.1], for Hamiltonian pseudo-rotations $\varphi$ of $\mathbb{CP}^n$. The former theorem only requires the Reeb flow to have finitely many simple closed orbits, while in the latter the number of periodic points must be exactly $n$ although the Hamiltonian diffeomorphism need not be non-degenerate. From this perspective, the conditions of Theorem B are less restrictive. This discrepancy is reflected by the difference in the definitions of pseudo-rotations of $\mathbb{CP}^n$ and Reeb pseudo-rotations of $S^{2n-1}$ which we adopt here. Of course, the higher-dimensional Franks theorem from [Sh22] allows us to just require $\varphi$ to have finitely many periodic orbits along with a minor non-degeneracy condition, but this is a substantial extra step using a machinery unavailable in the contact setting.

The paper is organized as follows. In Section 2 we set our conventions and notation. The relevant facts from Floer theory, mostly quite standard, are assembled in Section 3. In Section 4 we state and discuss in detail the three key results used in the proofs of the main theorems. We prove the main results of the paper in Section 5 and the Crossing Energy Theorem in Section 6. Finally, in the Appendix (Section 7) we recall an argument from [Fr99, FM] and prove the local version of the Le Calvez–Yoccoz theorem.

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## 2. Conventions and notation

In this section we set our conventions and notation, which are mainly similar to the ones used in [GG20].

### 2.1. Hamiltonians and the action functional.

**2.1. Hamiltonians and the action functional.** Even though Theorems A and B concern with Reeb flows on the sphere $S^{2n-1}$, $2n-1\geq 3$, it is convenient for the sake of future references to work with more general Liouville domains than star-shaped domains in $\mathbb{R}^{2n}$. Thus let $\alpha$ be the contact form on the boundary $M=\partial W$ of a Liouville domain $W^{2n\geq 4}$. For the sake of simplicity we will assume that $c_1(TW)\mid_{\pi_2}=0$. As usual denote by $\widehat{W}$ the symplectic completion of $W$, i.e.,

$$\widehat{W}=W\cup_M M\times[1,\infty)$$

with the symplectic form $\omega$ extended to $M\times[1,\infty)$ as $d(r\alpha)$, where $r$ is the coordinate on $[1,\infty)$. Sometimes it is convenient to extend the function $r$ to a collar of $M=\partial W$ in $W$. Thus we can think of $\widehat{W}$ as the union of $W$ and $M\times[1-\epsilon,\infty)$ for small $\epsilon>0$ with $M\times[1-\epsilon,1]$ lying in $W$ and the symplectic form given by the same formula.

*Example* 2.1. In this paper we are mainly interested in contact forms $\alpha$ on the standard contact sphere $M=S^{2n-1}$. In this case, we can take a star-shaped domain $W\subset\widehat{W}=\mathbb{R}^{2n}$ as the Liouville domain.

Unless specifically stated otherwise, most of Hamiltonians $H\colon\widehat{W}\to\mathbb{R}$ considered in this paper depend only on $r$ outside $W$, i.e., $H=h(r)$ on $M\times[1,\infty)$, where the function $h\colon[1,\infty)\to\mathbb{R}$ is required to meet the following conditions:

- $H$ is constant on $W$ and $h$ is monotone increasing;
- $h$ is convex, i.e., $h''\geq 0$, and $h''>0$ on $(1,r_{\max})$ for some $r_{\max}>1$ depending on $h$;
- $h(r)$ is linear, i.e., $h(r)=ar-c$, when $r\geq r_{\max}$.

In other words, the behavior of $h$ changes from a constant on $W$ to convex in $r$ on $M\times[1,r_{\max}]$, and strictly convex on the interior, to linear in $r$ on $M\times[r_{\max},\infty)$. When needed, we will denote $r_{\max}$ by $r_{\max}(h)$ to indicate its dependence on $h$.

In what follows, we will refer to $a$ as the *slope* of $H$ (or $h$) and write $a=\mathit{slope}(H)$. The slope is always assumed to be outside the action spectrum of $\alpha$, i.e., $a\notin\mathcal{S}(\alpha)$. We call $H$ *admissible* when $H|_W=\mathit{const}<0$ and *semi-admissible* when $H|_W\equiv 0$. (This terminology differs somewhat from the standard usage and we emphasize that *admissible Hamiltonians are not semi-admissible.*) When $H$ satisfies only the last of these conditions, we call it *linear at infinity.*

The difference between admissible and semi-admissible Hamiltonians is just an additive constant: $H-H|_W$ is semi-admissible when $H$ is admissible. Hence the two Hamiltonians have the same filtered Floer homology up to an action shift. Our reason for introducing semi-admissible Hamiltonians is that in the proofs of the main results we need to work with the Floer homology of a fixed Hamiltonian and its iterates rather than symplectic homology, and in this case semi-admissible Hamiltonians have a more convenient normalization.

The Hamiltonian vector field $X_H$ is determined by the condition

$$\omega(X_H, \cdot) = -dH$$

and, on $M \times [1, \infty)$,

$$X_H = h'(r)R_\alpha,$$

where $R_\alpha$ is the Reeb vector field. Hence every $T$-periodic orbit $x$ of the Reeb flow with $T < a$ gives rise to a 1-periodic orbit $\tilde{x} = (x, r)$ of $H$, where

$$h'(r) = T. \tag{2.1}$$

Clearly, $\tilde{x}$ lies in the shell $1 < r < r_{\max}$.

The action functional $\mathcal{A}_H$ is defined as

$$\mathcal{A}_H(z) = \int_z \hat{\alpha} - \int_{S^1} H(z(t))\, dt,$$

where $z\colon S^1 = \mathbb{R}/\mathbb{Z} \to \widehat{W}$ is a smooth loop in $\widehat{W}$ and $\hat{\alpha}$ is the Liouville primitive $\alpha_W$ of $\omega$ on $W$ and $\hat{\alpha} = r\alpha$ on $M \times [1, \infty)$. More explicitly, when $z\colon S^1 \to M \times [1, \infty)$, we have

$$\mathcal{A}_H(z) = \int_{S^1} r(z(t))\alpha\big(z'(t)\big)\, dt - \int_{S^1} h\big(r(z(t))\big)\, dt.$$

Thus when $z = \tilde{x} = (x, r)$ is a 1-periodic orbit of $H$, the action can be expressed as a function of $r$ only:

$$\mathcal{A}_H(\tilde{x}) = A_H(r),$$

where

$$A_H\colon [1, \infty) \to [0, \infty) \text{ is given by } A_H(r) = rh'(r) - h(r). \tag{2.2}$$

Sometimes we will also denote this *action function* by $A_h$. This is a monotone increasing function, for

$$A'_H(r) = h'(r) + rh''(r) - h'(r) = rh''(r) \geq 0.$$

It is easy to see that

$$\max A_H = A_H(r_{\max}) = c \geq a. \tag{2.3}$$

Here the first equality follows from the fact that $A_H$ is monotone increasing and the second one from that $h$ is linear, i.e., $h(r) = ar - c$, on $[r_{\max}, \infty)$. To prove the inequality, note first that

$$h(r_{\max}) \leq \int_1^{r_{\max}} h'(r)\, dr \leq a(r_{\max} - 1).$$

Hence, since $h'(r_{\max}) = a$, we have

$$\begin{aligned} c &= ar_{\max} - h(r_{\max}) \\ &\geq ar_{\max} - a(r_{\max} - 1) \\ &= a. \end{aligned}$$

Thus, when $r \geq r_{\max}$, the function $A_H$ is constant: $A_H(r) = A_H(r_{\max}) = c$. For this reason, in what follows we will usually limit the domain of this function to $[1, r_{\max}]$.

While the function $A_H$ expresses the Hamiltonian action as a function of $r$ we will also need another variant $\mathfrak{a}_H$ of an action function, expressing the Hamiltonian action as a function of the period $T$, i.e., the contact action. In other words, the function $\mathfrak{a}_H$ translates the contact action to the Hamiltonian action. Thus

$$\mathfrak{a}_H = A_H \circ (h')^{-1}\colon [0,\, a] \to [0,\, \max A_H = A_H(r_{\max})]$$

is more specifically defined by the condition

$$\mathfrak{a}_H(T) = A_H(r), \text{ where } h'(r) = T. \tag{2.4}$$

Since $H$ is (semi-)admissible, $h'$ is one-to-one on $[0,\, r_{\max}]$, and the inverse $(h')^{-1}$ is defined on $[0,\, a]$.

Then using the chain rule, we have

$$\mathfrak{a}'_H(T) = r := (h')^{-1}(T) \text{ and } 1 \leq \mathfrak{a}'_H \leq r_{\max}.$$

Thus $\mathfrak{a}_H$ a strictly monotone increasing convex $C^1$-function, which is $C^\infty$ on $(0,\, a)$, with $\mathfrak{a}''_H = \infty$ at $T = 0$ and $T = a$. Furthermore,

$$\mathfrak{a}_{H_1} \leq \mathfrak{a}_{H_0} \text{ on } [0,\, \mathit{slope}(H_0)] \text{ whenever } H_1 \geq H_0. \tag{2.5}$$

This inequality is not obvious. A simple way to prove (2.5) is as follows. First, note that that $-A_H(r)$ is the ordinate of the intersection with the vertical axis of the tangent line to the graph of $h$ at $(r, h(r))$. Next, we have $\mathfrak{a}_{H_1}(T) = A_{H_1}(r_1)$ and $\mathfrak{a}_{H_2}(T) = A_{H_2}(r_2)$, where $h'_1(r_1) = T = h'_2(r_2)$. As a consequence, the two tangent lines have the same slope $T$. The tangent line to the graph of $h_2$ lies above the tangent line to the graph of $h_1$; for it passes through the point $(r_2, h_2(r))$ which is above the graph of $h_1 \leq h_2$. Therefore, $A_{H_1}(r_1) \leq A_{H_2}(r_2)$.

Concluding this section we note that $\tilde{x}$ is well-defined only as a 1-periodic orbit of the Hamiltonian flow $\varphi^t_H$ of $H$. This orbit corresponds to the whole circle $\Gamma = \tilde{x}(S^1)$ of fixed points of the time-one map $\varphi_H$ (aka 1-periodic points of $\varphi_H$). These orbits however have the same action, mean index, etc. In what follows we will ignore this terminological ambiguity.

2.2. **Conley–Zehnder index and dynamical convexity.** We refer the reader to, e.g., [SZ] for the definition and a detailed discussion of the *Conley–Zehnder index* and to, e.g., [GG20, Sec. 4] or [GM, Sec. 2] or [Lo]. Here we normalize the Conley–Zehnder index, denoted throughout the paper by $\mu$, by requiring the flow for $t \in [0,\, 1]$ of a small positive definite quadratic Hamiltonian $Q$ on $\mathbb{R}^{2m}$ to have index $m$. More generally, when $Q$ is small and non-degenerate, the flow has index equal to $(\mathit{sgn} Q)/2$, where $\mathit{sgn} Q$ is the signature of $Q$. In other words, the Conley–Zehnder index of a non-degenerate critical point of a $C^2$-small autonomous Hamiltonian $H$ on $\mathbb{R}^{2m}$ is equal to $m - \mu_{\text{Morse}}$, where $\mu_{\text{Morse}} = \mu_{\text{Morse}}(H)$ is the Morse index of $H$.

We denote by $\mu_\pm\colon \widetilde{\mathrm{Sp}}(2m) \to \mathbb{Z}$ the upper and lower semi-continuous extensions of the Conley–Zehnder index. The mean index of $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ is defined as

$$\hat{\mu}(\Phi) = \lim_{k\to\infty} \frac{\mu_\pm\big(\Phi^k\big)}{k}.$$

This is the unique, up to normalization, homogeneous continuous quasi-morphism

$$\hat{\mu}\colon \widetilde{\mathrm{Sp}}(2m) \to \mathbb{R};$$

cf. [BG]. It is a standard fact (see, e.g., [SZ]) that

$$\hat{\mu}(\Phi) - m \leq \mu_-(\Phi) \leq \mu_+(\Phi) \leq \hat{\mu}(\Phi) + m \tag{2.6}$$

and that the first and the last inequalities are strict when $\Phi$ is non-degenerate.

The assumption that $c_1(TW)\mid_{\pi_2}=0$ guarantees that these invariants are also defined for a contractible periodic orbit $x$ of the Reeb flow on $M$ or a Hamiltonian flow on $\widehat{W}$, which we denote by $\mu(x)$, $\mu_\pm(x)$ and $\hat{\mu}(x)$ or $\mu(\tilde{x})$, etc. We note that since the Conley–Zehnder index of a closed Reeb orbit is defined via a trivialization of the contact structure, dealing with Reeb flows everywhere above we should set $m=n-1$ where $2n=\dim W$. On the other hand, for Hamiltonian flows, $m=n$.

A feature of Reeb flows central to our results is that of dynamical convexity.

**Definition 2.2.** The Reeb flow on a $(2n-1)$-dimensional contact manifold is said to be *dynamically convex* if for every closed contractible Reeb orbit $x$ (or equivalently when $\pi_1$ has no torsion, every simple closed contractible Reeb orbit) $\mu_-(x)\geq n+1$.

As is shown in [HWZ], the Reeb flow on a strictly convex hypersurface in $\mathbb{R}^{2n}$ is dynamically convex. The converse is not true; see [CE22a, CE22b]. The geodesic flow of a Finsler metric on $S^2$ with curvature meeting a certain pinching condition is dynamically convex, [HP]. While the notion of dynamical convexity understood literally as in Definition 2.2 encapsulates an important class of Reeb flows on the spheres, it is not entirely clear if it is of serious relevance for other contact manifolds, especially in higher dimensions. We are not aware of any non-trivial examples of dynamically convex geodesic flows on manifolds of dimension $n>2$. However, some geodesic flows are close to being dynamically convex. For instance, all closed geodesics on the standard round sphere $S^{n\geq 3}$ have index greater than on equal to $n-1$. (For geodesic flows, the Morse index is equal to the Conley–Zehnder index $\mu_-$.) It is not difficult to show that the same is true for the geodesic flow of a Finsler metric which is $C^2$-close to the round metric on $S^n$.

We refer the reader to, e.g., [GG20, Sec. 4.2] for a very detailed treatment of dynamical convexity and also to, e.g., [AM, ALM, DL$^2$W, GM, GGMa] for other relevant notions and results.

## 3. Preliminaries

In this section we recall basic definitions and results from Floer theory used in the proofs of Theorems A and B. None of the results stated here are really new and most of them are quite standard and can be traced in some form to the original works, [CFH, Vi], or found in, e.g., [BO09a, BO09b, CO]. When necessary, we give more specific references.

3.1. **Floer equation.** Fix an almost complex structure $J$ on $\widehat{W}$ satisfying the following conditions:

- $J$ is compatible with $\omega$, i.e., $\omega(\cdot, J\cdot)$ is a Riemannian metric, and on the cone $M\times[1,\infty)$ we have
- $Jr\partial/\partial r=R_\alpha$ and
- $J$ preserves $\ker(\alpha)$.

Note that the last two conditions are equivalent to

$$dr\circ J=-r\alpha. \tag{3.1}$$

We call such almost complex structures *admissible*. If the second and the third conditions are satisfied only outside a compact set and in addition $J$ can be time-dependent and 1-periodic in time within a compact set, we call $J$ *admissible at infinity*.

Let now $H$ be a Hamiltonian linear at infinity and let $J$ be an admissible at infinity almost complex structure. For our purposes it is convenient to have the $L^2$-anti-gradient of $\mathcal{A}_H$ adopted as the Floer equation:

$$\partial_s u = -\nabla_{L^2}\mathcal{A}_H(u), \tag{3.2}$$

where $u\colon \mathbb{R}\times S^1 \to \widehat{W}$ with coordinates $s$ on $\mathbb{R}$ and $t$ on $S^1$. Thus the function $s\mapsto \mathcal{A}_H\big(u(s,\cdot)\big)$ is decreasing. More explicitly this equation reads

$$\partial_s u - J\big(\partial_t u - X_H(u)\big) = 0. \tag{3.3}$$

Note that in contrast with the standard conventions the leading term of this equation is not the $\bar\partial$-operator but the $\partial$-operator. In other words, when $H\equiv 0$, solutions of (3.3) are anti-holomorphic curves rather than holomorphic curves. Nonetheless the standard properties of the solutions of the Floer equation readily translate to our setting, e.g., via the change of variables $s\mapsto -s$. We will often refer to solutions of the Floer equation as *Floer cylinders*.

Recall also that the energy of $u$ is by definition

$$E(u) = \int_{S^1\times\mathbb{R}} \|\partial_s u\|^2\, dt\, ds.$$

When $u$ is asymptotic to $\tilde{x} = (r^+, x)$ at $-\infty$ and $\tilde{y} = (r^-, y)$ at $+\infty$, we have

$$E(u) = \mathcal{A}_H(\tilde{x}) - \mathcal{A}_H(\tilde{y}) = A_H(r^+) - A_H(r^-), \tag{3.4}$$

where $r^+ \geq r^-$ since (3.2) is an anti-gradient Floer equation; see Section 4.1.

We will make extensive use of two properties of Floer cylinders $u$ for admissible or semi-admissible Hamiltonians $H$ and admissible almost complex structures $J$.

The first one is the standard maximum principle asserting that the function $r\circ u$ cannot attain a local maximum in the domain mapped by $u$ into the cone $M\times[1,\infty)$ where $r$ is defined. (We refer the reader to, e.g., [Vi] and also [FS, Sec. 2] for a direct proof of this fact.) Of course, the same is true for Hamiltonians linear at infinity and $J$ admissible at infinity in the domain where the Hamiltonian is a linear function of $r$ and $J$ is admissible. Moreover, the maximum principle also holds for continuation Floer trajectories when $h(r) = a(s)r - c(s)$ as long as the slope $a$ is a non-decreasing function of $s$; there are however no constraints on the function $c(s)$. The maximum principle is crucial to having Floer cylinders and continuation solutions of the Floer equation contained in a compact region of $\widehat{W}$, and thus the Floer homology and continuation maps for homotopies with non-decreasing slope defined.

In particular, let $u$ be a solution of the Floer equation asymptotic to 1-periodic orbits $(x^+, r^+)$ at $-\infty$ and $(x^-, r^-)$ at $+\infty$. Note that $r^+ > r^-$ – hence the notation – since with our conventions $A_H$ is an increasing function and the action is decreasing along $u$. Then the maximum principle implies that

$$\sup_{\mathbb{R}\times S^1} r\big(u(s,t)\big) \leq r^+ = r\big(u(-\infty,t)\big). \tag{3.5}$$

The second fact we need, which we will refer to as *Bourgeois–Oancea monotonicity*, is much less standard and goes back to [BO09b, p. 654]; see also [CO, Lemma 2.3]. It asserts that

$$\max_{t\in S^1} r\big(u(s,t)\big) \geq r^- = r\big(u(\infty,t)\big) \tag{3.6}$$

for all $s \in \mathbb{R}$. For the sake of completeness, we prove (3.6) in Section 6.2.5, closely following the argument in [CO]. Note that as a consequence of the maximum principle and (3.6), the left-hand side of (3.6) is a monotone decreasing function of $s$ ranging from $r^+$ at $-\infty$ to $r^-$ at $\infty$.

## 3.2. **Filtered Floer and symplectic homology.**

3.2.1. *Floer homology and continuation maps.* Let $H$ be a Hamiltonian $H$ linear at infinity such that, as usual, $\mathit{slope}(H) \notin \mathcal{S}(\alpha)$ and let $I \subset \mathbb{R}$ be an action interval. Then, regardless of whether $H$ is non-degenerate or not, the filtered (contractible) Floer homology $\mathrm{HF}^I(H)$ over a fixed ground field $\mathbb{F}$ is defined as long as the end-points of $I$ are outside the action spectrum $\mathcal{S}(H)$ of $H$. Throughout the paper we will always assume the latter condition to be met by $I$. For the sake of brevity, we will write

$$\mathrm{HF}^\tau(H) := \mathrm{HF}^{(-\infty,\,\tau]}(H).$$

To define $\mathrm{HF}^I(H)$ it suffices to replace $H$ by a small compactly supported non-degenerate perturbation. With our conventions the Floer homology is graded by the Conley–Zehnder index. Thus a non-degenerate minimum of $H$ with small Hessian gives rise to a generator of degree $n$ and a non-degenerate closed Reeb orbit $x$ with Conley–Zehnder index $m$ gives rise to two generators $\check{x}$ and $\hat{x}$ with indices $m$ and $m+1$, respectively. (The Floer complex is discussed in more detail in Section 3.3.) We will usually suppress the grading in the notation.

Let $H_s$, $s \in \mathbb{R}$, be a homotopy between two linear at infinity Hamiltonians $H_0$ and $H_1$, i.e., $H_s$ is a family of linear at infinity Hamiltonians such that $H_s = H_0$ when $s$ is close to $-\infty$ and $H_s = H_1$ when $s$ is close to $+\infty$. (In what follows we will take the liberty to have homotopies parametrized by $[0,\,1]$ rather than $\mathbb{R}$.) There are two situations where a homotopy gives rise to a map in Floer homology.

The first one is when all Hamiltonians $H_s$ have the same slope. Then the homotopy induces a continuation map

$$\mathrm{HF}^I(H_0) \to \mathrm{HF}^{I+C}(H_1)$$

shifting the action filtration by

$$C = \int_{-\infty}^{\infty} \max_{z \in \widehat{W}} \max\{0, -\partial_s H_s(z)\}\, ds.$$

As a consequence, $\mathrm{HF}(H) = \mathrm{HF}^\tau(H)$ for $\tau > \operatorname{supp} \mathcal{S}(H)$; see Lemma 3.2. Moreover, it is well-known and not hard to show that $\mathrm{HF}(H)$ does not change as long as $\mathit{slope}(H)$ stays outside of $\mathcal{S}(\alpha)$.

The second one is when $H_s$ is monotone increasing, i.e., the function $s \mapsto H_s(z)$ is monotone increasing for all $z \in \widehat{W}$. In this case, the function $s \mapsto \mathit{slope}(H_s)$ is also monotone increasing. Furthermore, while $\mathit{slope}(H_0)$ and $\mathit{slope}(H_1)$ are still required to be outside $\mathcal{S}(\alpha)$, the intermediate slopes $\mathit{slope}(H_s)$ can pass through the points of $\mathcal{S}(\alpha)$. A monotone increasing homotopy induces a map

$$\mathrm{HF}^I(H_0) \to \mathrm{HF}^I(H_1)$$

preserving the action filtration.

In both cases the fact that the continuation Floer trajectories are confined to a compact set is a consequence of the maximum principle; see Section 3.1.

The Floer homology is insensitive to small perturbations of the Hamiltonian and the action interval. To be more precise, fix a linear at infinity Hamiltonian $H$ and an interval $I$. (Here as usual we require that $\mathit{slope}(H) \notin \mathcal{S}(\alpha)$ and the end-points of $I$ are not in $\mathcal{S}(H)$.) Assume that the slope of $H'$ is sufficiently close to the slope of $H$ and $H'$ is $C^0$-close to $H$ on the complement of the domain where they both are linear functions of $r$, and that the end-points of $I'$ are close to $I$. Then there is a natural isomorphism of the Floer homology groups

$$\mathrm{HF}^I(H) \cong \mathrm{HF}^{I'}(H').$$

Floer homology carries the pair-of-pants product

$$\mathrm{HF}^{\tau_0}_{m_0}(H_0) \otimes \mathrm{HF}^{\tau_1}_{m_1}(H_1) \to \mathrm{HF}^{\tau_0+\tau_1}_{m_0+m_1-n}(H_0 + H_1),$$

where as always we have assumed that the slopes of $H_0$ and $H_1$ and their sum are not in $\mathcal{S}(\alpha)$; see, e.g., [AS].

With our conventions, when $F$ is a semi-admissible Hamiltonian and $\delta > 0$ is small, there is a natural isomorphism

$$\mathrm{HF}^{\delta}_{*}(F) \cong \mathrm{H}_{n+*}(W; \partial W). \tag{3.7}$$

Furthermore, under this isomorphism, the fundamental class

$$[W, \partial W] \in \mathrm{H}_{2n}(W, \partial W) \cong \mathbb{F} \cong \mathrm{HF}^{\delta}_{n}(F) \tag{3.8}$$

corresponds to a unit for the pair-of-pants product. To be more precise, fix a linear at infinity Hamiltonian $H$ and $\tau \notin \mathcal{S}(H)$. For the sake of simplicity, assume in addition that for a semi-admissible Hamiltonian $F$ as above, $\mathit{slope}(F)$ and also $\delta > 0$ are sufficiently small. Then the composition of maps

$$\mathrm{HF}^{\tau}(H) = \mathrm{HF}^{\tau}(H) \otimes \mathrm{HF}^{\delta}_{n}(F) \xrightarrow{\cong} \mathrm{HF}^{\tau+\delta}(H + F) \cong \mathrm{HF}^{\tau}(H) \tag{3.9}$$

with the middle arrow given by the pair-of-pants product is the identity map.

3.2.2. *Invariance.* The results of this section are somewhat less standard although the methods are. So far, throughout the above discussion, it was sufficient to have Hamiltonians to only be linear at infinity. (The Hamiltonian $F$ in (3.7) and (3.9) is an exception.) In what follows, however, it becomes essential to require Hamiltonians to be semi-admissible.

Namely, let $H_0 \leq H_1$ be two such Hamiltonians. For the sake of simplicity, we will assume furthermore that $r_{\max}(h_1) = r_{\max}(h_0)$ and use $r_{\max}$ to denote both of these parameters. (This condition can be replaced by that $r_{\max}(h_1) \leq r_{\max}(h_0)$ with suitable wording modifications.)

Let

$$f = \mathfrak{a}_{H_1} \circ \mathfrak{a}_{H_0}^{-1} \colon [0,\, A_{H_0}(r_{\max})] \to [0,\, A_{H_1}(r_{\max})].$$

The function $f$ is monotone as a composition of two monotone increasing functions. Furthermore, as is easy to see,

$$f\big(\mathcal{S}(H_0)\big) = f\big([0,\, A_{H_0}(r_{\max})]\big) \cap \mathcal{S}(H_1), \tag{3.10}$$

i.e., $f$ gives rise to a one-to-one correspondence between the action spectra as long as the target is in the range of $f$.

**Proposition 3.1.** *We have $f(\tau) \leq \tau$ for all $\tau$. Furthermore, for all $\tau < A_{H_0}(r_{\max})$ and not in $\mathcal{S}(H_0)$, there are isomorphisms of Floer homology groups*

$$\mathrm{HF}^{\tau}(H_0) \xrightarrow{\cong} \mathrm{HF}^{f(\tau)}(H_1). \tag{3.11}$$

*These isomorphisms are natural in the sense that they commute with maps induced by inclusion of action intervals, monotone homotopies, etc. For instance, whenever* $\beta$ *is in the range of* $f$*, the composition*

$$\mathrm{HF}^{\beta}(H_0) \to \mathrm{HF}^{\beta}(H_1) \to \mathrm{HF}^{f^{-1}(\beta)}(H_0),$$

*where the first arrow comes from a monotone increasing homotopy and the second is the inverse of* (3.11)*, is induced by the inclusion of action intervals:* $\beta \leq f^{-1}(\beta)$*.*

Of course, a similar result holds for admissible Hamiltonians, but then we also have to take into account the shift of actions. For the sake of completeness we include a proof of the proposition.

*Proof.* The fact that $f(\tau) \leq \tau$ follows from that $\mathfrak{a}_{H_1} \leq \mathfrak{a}_{H_0}$ when $H_1 \geq H_0$; see (2.5). We construct the isomorphism (3.11) in two steps.

Let $r_0 \in [1, r_{\max})$ be uniquely determined by the condition that $A_{H_0}(r_0) = \tau$. Pick an intermediate semi-admissible Hamiltonian $H_{01} = h_{01}(r)$ with $r_{\max}(h_{01}) = r_{\max}(h_0) = r_{\max}(h_1)$ and the following properties

- $h_0 \leq h_{01} \leq h_1$;
- $h_{01} = h_0$ on $[1, r_0]$;
- $\mathit{slope}(h_{01}) = \mathit{slope}(h_1)$.

We note that $\tau \notin \mathcal{S}(H_{01})$.

Then the monotone increasing linear homotopy from $H_0$ to $H_{01}$ induces an isomorphism

$$\mathrm{HF}^{\tau}(H_0) \xrightarrow{\cong} \mathrm{HF}^{\tau}(H_{01}). \tag{3.12}$$

This is a consequence of the fact that by the maximum principle (see Section 3.1) all Floer cylinders for $H_0$ and $H_{01}$ and Floer continuation trajectories starting at 1-periodic orbits with action less than or equal $\tau$ lie on the region $r \leq r_0$ or, to be more precise, $W \cup \big(M \times [1, r_0]\big)$; cf. Section 3.3.

The second step is based on the following standard lemma which essentially goes back to [CFH, Vi].

**Lemma 3.2.** *Let* $F_s$ *be a homotopy of Hamiltonians linear at infinity such that all Hamiltonians* $F_s$ *have the same slope, and let* $I_s$ *be a family of intervals continuously depending on* $s$ *such that for all* $s$ *the end-points of* $I_s$ *are outside* $\mathcal{S}(F_s)$*. Then there is a natural isomorphism*

$$\mathrm{HF}^{I_0}(F_0) \xrightarrow{\cong} \mathrm{HF}^{I_1}(F_1). \tag{3.13}$$

The lemma is proved by breaking down the homotopy into a concatenation of homotopies such that for each of them either the interval or the Hamiltonian is independent of $s$. For a fixed Hamiltonian and varying interval, the assertion follows directly from the definition. For a fixed interval and varying Hamiltonian, it is a consequence of the fact that the "inverse" homotopy induces the inverse map. This is the point where the condition that $\mathit{slope}(F_s) = \mathit{const}$, and hence a homotopy need not necessarily be monotone increasing, enters the picture.

In the setting of the proposition, let $F_s$ be a linear monotone increasing homotopy from $F_0 = H_{01}$ to $F_1 = H_1$. Denote by $f_s = \mathfrak{a}_{F_s} \circ \mathfrak{a}_{F_0}^{-1}$ the resulting family of maps and note that $f_s(\tau) \notin \mathcal{S}(F_s)$ by (3.10) since $\tau \notin \mathcal{S}(F_0)$. Applying the lemma to the family of intervals $I_s = (-\infty, f_s(\tau)]$, we obtain an isomorphism

$$\mathrm{HF}^{\tau}(H_{01}) \xrightarrow{\cong} \mathrm{HF}^{f_1(\tau)}(H_1) = \mathrm{HF}^{f(\tau)}(H_1). \tag{3.14}$$

In the last identity we have used the fact that $f = f_1$ on $[0, \tau]$, i.e., in self-explanatory notation $f_{H_0 H_1} = f_{H_{01} H_1}$ on this interval, since $h_{01} = h_0$ on $[1, r_0]$.

The desired isomorphism (3.11) is now defined as the composition of the isomorphisms (3.12) and (3.14). The last assertion of the proposition follows immediately from this construction. □

*Remark* 3.3. The condition that $\mathit{slope}(F_s) = \mathit{const}$ in Lemma 3.2 is essential. For instance, let $F_s$ be a monotone increasing homotopy from $F_0$ to $F_1$. Let $I_s = (-\infty, \tau]$ where $\tau > \operatorname{supp}_{s\in\mathbb{R}} \max \mathcal{S}(F_s)$. Then $\mathrm{HF}^I(F_0) = \mathrm{HF}(F_0)$ and $\mathrm{HF}^I(F_1) = \mathrm{HF}(F_1)$ are not in general isomorphic when $\mathit{slope}(F_s)$ crosses $\mathcal{S}(\alpha)$ as one can see already from the example of the round sphere. However, one can somewhat relax this condition by requiring that $\mathit{slope}(F_s)$ stays outside $\mathcal{S}(\alpha)$.

3.2.3. *Local Floer homology.* When $x$ is an isolated $T$-periodic orbit of $\alpha$ with $T < \mathit{slope}(H)$ the 1-periodic orbit $\tilde{x}$ of $H$ is also isolated as a 1-periodic orbit of the flow of $H$. As a consequence, we obtain an isolated circle $\Gamma = \tilde{x}(S^1)$ of fixed points of $\varphi_H$. We denote by $\mathrm{HF}(\tilde{x})$ the *local Floer homology* of $\Gamma$; see [Fl89a, Fl89b] and also [Fe, GG10, McL]. By definition, the support $\operatorname{supp} \mathrm{HF}(\tilde{x})$ is the range of degrees for which this homology is non-trivial. Clearly, by (2.6) with $m = n - 1$,

$$\operatorname{supp} \mathrm{HF}(\tilde{x}) \subset [\mu_-(x),\, \mu_+(x) + 1] \subset [\hat{\mu}(x) - n + 1,\, \hat{\mu}(x) + n]. \tag{3.15}$$

For instance, when $x$ is non-degenerate the support of $\mathrm{HF}(\tilde{x})$ comprises exactly two points: $\mu(x)$ and $\mu(x) + 1$. More generally, $\mathrm{HF}_*(\tilde{x}) = \mathrm{HF}_*(\psi) \oplus \mathrm{HF}_{*-1}(\psi)$, where $\psi$ is the germ of the Poincaré return map of $x$; see [Fe]. However, we will not use this fact.

3.2.4. *Symplectic homology.* Let $I$ be an interval, and as always assume that the end-points of $I$ are not in $\mathcal{S}(\alpha)$. The *symplectic homology* $\mathrm{SH}^I(\alpha)$ is defined as

$$\mathrm{SH}^I(\alpha) := \varinjlim_{H} \mathrm{HF}^I(H), \tag{3.16}$$

where the limit is over all Hamiltonians linear at infinity and such that $H|_W < 0$. Since admissible (but not semi-admissible) Hamiltonians form a co-final family, we can limit $H$ to this class. When working with this definition, it is useful to keep in mind that

$$\mathcal{S}(H) \to \{0\} \cup \mathcal{S}(\alpha)$$

uniformly on compactly intervals. Clearly, $\mathrm{SH}^I(\alpha)$ is a $\mathbb{Z}$-graded vector space over $\mathbb{F}$ with the grading coming from the Conley–Zehnder index. For the sake of brevity we will write $\mathrm{SH}^\tau(\alpha)$ when $I = (-\infty, \tau]$.

*Remark* 3.4. In (3.16), we could have required that $H|_W \leq 0$ rather than that $H|_W < 0$, or equivalently required $H$ to be semi-admissible or admissible. This would result in the same groups $\mathrm{SH}^I(\alpha)$, but also keep us from having convenient choices of co-final sequences. For instance, let $H$ be a semi-admissible Hamiltonian. Pick two sequences of positive numbers: $\lambda_i \to \infty$ and $\epsilon_i \to 0$. Then the sequence $H_i = \lambda_i H - \epsilon_i$ is co-final in the class of admissible Hamiltonians. However, neither this sequence nor the sequence $\lambda_i H$ is co-final when semi-admissible Hamiltonians is added to the class.

From a somewhat different perspective, $\mathrm{SH}^\tau(\alpha)$ can also be defined as follows; cf. [Vi]. For $\tau \notin \mathcal{S}(\alpha)$, consider the function $F_\tau\colon \widehat{W} \to [0, \infty)$ given by

$$F_a(z) = \begin{cases} 0 & z \in W, \\ \tau\big(r(z) - 1\big) & z \in M \times [1, \infty). \end{cases}$$

While this function is only continuous, its Floer homology is obviously defined (by continuity). For instance, we can set

$$\mathrm{HF}^I(F_\tau) := \varinjlim_{H \leq F_\tau} \mathrm{HF}^I(H)$$

where the limit is taken over admissible or semi-admissible Hamiltonians bounded from above by $F_\tau$. It is not hard to see that

$$\mathcal{S}(H) \to \{0\} \cup \big(\mathcal{S}(\alpha) \cap [0, \tau]\big),$$

and hence it suffices to require that the end-points of $I$ are not in $\{0\}\cup\big(\mathcal{S}(\alpha)\cap[0, \tau]\big)$.

**Corollary 3.5.** *For $I \subset \mathbb{R}$, we have*

$$\mathrm{HF}^I(F_\tau) = \mathrm{SH}^{(-\infty,\, \tau] \cap I}(\alpha).$$

*In particular,* $\mathrm{HF}(F_\tau) = \mathrm{SH}^\tau(\alpha)$.

This corollary is essentially a consequence of Proposition 3.1 with some wording modifications; its proof is routine and for the sake of brevity we omit it.

A well-known, but crucial for our purposes, fact is that $\mathrm{SH}(\alpha)$ vanishes whenever $W$ is displaceable in $\widehat{W}$. For instance, $\mathrm{SH}(\alpha) = 0$ when $W$ is a ball in $\widehat{W} = \mathbb{R}^{2n}$. In the latter case, vanishing of $\mathrm{SH}(\alpha)$ is established in [Vi]. The general case is proved in [CFO] via Rabinowitz–Floer homology and direct proofs are given in [Su16] and [GS].

On the level of specific Hamiltonians this fact is reflected by the following lemma which follows from the definition of symplectic homology as a direct limit and Remark 3.4.

**Lemma 3.6.** *Assume that $\mathrm{SH}(\alpha) = 0$. Then for any semi-admissible Hamiltonian $H$ and any $I \subset \mathbb{R}$ there exist constants $\lambda \geq 1$ and $C \geq 0$ such that the natural inclusion/homotopy map*

$$\mathrm{HF}^I(H) \to \mathrm{HF}^{I+C}(\lambda H)$$

*is zero.*

3.3. **Floer complexes and graphs.** The proofs of the main theorems require working on the level of Floer complexes rather than Floer homology. The construction of the Floer complex is quite standard and in this section we just briefly spell out the necessary definitions.

3.3.1. *Non-degenerate case.* Assume first that the Reeb flow of $\alpha$ is non-degenerate. Then a (semi-)admissible Hamiltonian $H$ is Morse–Bott non-degenerate except for the critical set $W$. To deal with this minor issue, we fix an autonomous $C^2$-small perturbation $\tilde{H}$ of $H$ such that $\tilde{H} - H$ is supported in a small neighborhood of $W$ and $\tilde{H}$ is Morse. Now $\tilde{H}$ is Morse–Bott non-degenerate and all constant 1-periodic orbits of $\tilde{H}$ are non-degenerate.

We will define the *Floer complex* $\mathrm{CF}(H)$ by applying the standard Morse–Bott complex construction using "cascades" to the non-constant 1-periodic orbits of $H$;

see, e.g., [BH10, BH13, Bo, BO09a, Fu, HN] and references therein. Let $x$ be a closed Reeb orbit of $\alpha$ with period $T < a$ and let $\tilde{x}(x,r)$ be the corresponding orbit of $H$; see Section 2.1. All non-constant 1-periodic orbits of $H$ have this form. Here we think of $\tilde{x}$ as a, not necessarily simple, 1-periodic orbit of the flow of $H$. It gives rise to a family of fixed points of $\varphi_H$ parametrized by the circle $\Gamma_x = \tilde{x}(S^1) \subset \widehat{W}$ which we identify with $x(S^1) \subset M$. Fix a Morse function $f_x$ on $\Gamma$ with exactly one maximum and one minumum and a Riemannian metric which we suppress in the notation. (Note that different orbits $x$ can give rise to the same set $\Gamma$ and in this case we are allowed to take different functions $f_x$, although this is not necessary for our purposes.)

Each orbit $\tilde{x}$ gives rise to two generators $\hat{x}$ and $\check{x}$ of $\mathrm{CF}(H)$ with grading $|\hat{x}| = \mu(x) + 1$ and $|\check{x}| = \mu(x)$, corresponding to the maximum and the minimum of $f_x$. Each constant orbit of $\tilde{H}$ gives rise to one generator. (In the proofs we are only interested in the part of the complex where the actions and indices are large. Clearly, the generators coming from $W$ do not contribute to this part.)

The Floer differential $\partial_{Fl}$ counts cascades: concatenations of integral curves of $-\nabla f_x$ and Floer trajectories. For instance, when $|\hat{x}| = |\check{y}| + 1$, a cascade from $\hat{x}$ to $\check{y}$ comprises an integral curve on $-\nabla f_x$ from $\hat{x}$ to some point $z \in \Gamma_x$, a solution of the Floer equation asymptotic at $-\infty$ to a parametrization of $\tilde{x}$ with $\tilde{x}(0) = z$ and to some parametrization of $\tilde{y}$ at $+\infty$ and finally an integral curve of $-\nabla f_y$ from $\tilde{y}(0)$ to $\check{y}$. The coefficient with which $\check{y}$ enters $\partial_{Fl}\hat{x}$ is an algebraic number of such (unparametrized) cascades. Cascades from $\hat{x}$ to $\hat{y}$ or from $\check{x}$ to $\check{y}$ or $\hat{y}$ are defined in a similar fashion, as long as the degree difference is 1, but now at least one of the integral curves is constant. Note also that $\check{x}$ enters $\partial_{Fl}\hat{x}$ with zero coefficients since the Morse function $f_x$ is perfect.

While the generators of the complex are essentially determined by $H$ (and its perturbation $\tilde{H}$), the differential depends in addition on some auxiliary data: the functions $f_x$ and metrics on $\Gamma_x$ and an almost complex structure $J$ admissible at infinity. To ensure regularity, it suffices to take $J$ generic in this class, [FHS]. In particular, we can take $J$ arbitrarily $C^\infty$-close to a fixed admissible almost complex structure. The Floer complex $\mathrm{CF}(H)$ is still filtered by the action functional $\mathcal{A}_H$. Hence, while the generators $\hat{x}$ and $\check{x}$ lie on the same action level, the differential $\partial_{Fl}$ is strictly action decreasing.

We find it convenient to think in terms of the *Floer graph* of $H$; see [ÇGG22a]. The vertices of the graph are the generators of the complex and two generators are connected by an arrow whenever one of them enters the Floer differential of the other with non-zero coefficient. The *length of the arrow* is by definition the action difference. Keeping in mind that the generators outside $W$ come in pairs, we will also say that $\tilde{x}$ (or $x$) is connected by an arrow to $\tilde{y}$ (or to $y$) when $\hat{x}$ or $\check{x}$ is connected to $\hat{y}$ or $\check{y}$. The Floer graph carries the same information as the Morse–Bott Floer complex of $H$. In particular, it also depends on all the auxiliary structures and choices used in the construction of the Floer Morse–Bott differential.

3.3.2. *Degenerate case.* Assume now that the Reeb flow of $\alpha$ is degenerate and let $H$ be a (semi-)admissible Hamiltonian for $\alpha$. As always we require that $\mathit{slope}(H) \notin \mathcal{S}(\alpha)$. Then the Floer complex of $H$ is defined by replacing it with a $C^\infty$-small perturbation $\tilde{H}$. Here we specify two ways to do this.

In the first one, we simply take as $\tilde{H}$ a $C^\infty$-small autonomous maximally non-degenerate perturbation of $H$ which has the same or nearly the same slope as $H$. Then $\tilde{H}$ is Morse–Bott and we can apply the construction from the previous section.

The second approach is based on perturbing the contact form rather than directly $H$. To this end, fix a $C^\infty$-small non-degenerate perturbation $\tilde{\alpha} = g\alpha$ of $\alpha$, where $g\colon M \to \mathbb{R}$ is $C^\infty$-close to 1. Thus $\tilde{\alpha}$ is the contact form on the hypersurface $\{r = g\}$ in $\widehat{W}$. (We may assume that $g \geq 1$ or extend $r$ to a collar of $M$ in $W$.) As a result, while the Liouville domain $W$ is slightly effected, we can keep $\widehat{W}$ unchanged. The $r$-coordinate is however also effected and $H$ is no longer admissible for $\tilde{\alpha}$. We replace it with the new Hamiltonian $\tilde{H}$ obtained by the change of variables $r \mapsto r/g$ (and also perturb it on a neighborhood of $W$ as in the non-degenerate case). Now the resulting Hamiltonian $\tilde{H}$ is Morse–Bott. We fix the necessary auxiliary structures and apply the construction from Section 3.3.1 to $\tilde{H}$. By definition, the Floer complex of $H$ is the Floer complex of $\tilde{H}$.

In both cases, the Hamiltonian $\tilde{H}$ is $C^\infty$-close to $H$ on compact sets and in the first construction we can even have $\tilde{H} = H$ at infinity.

Assume next that the closed Reeb orbits of $\alpha$ with period smaller than *slope*$(H)$ are isolated. Then so are the non-constant 1-periodic orbits of $H$. As a consequence, all non-constant 1-periodic orbits of $\tilde{H}$ arise from non-constant 1-periodic orbits $\tilde{x}$ of $H$ splitting into Morse–Bott non-degenerate orbits. In other words, non-constant 1-periodic orbits of $\tilde{H}$ and the corresponding generators of $\mathrm{CF}(\tilde{H})$ come in clusters labeled by the orbits $H$. Limiting the Floer differential to a cluster we obtain a complex and the homology of this complex is the local Floer homology $\mathrm{HF}(\tilde{x})$; see Section 3.2.3.

The *reduced Floer graph* of $H$ is then defined as follows; cf. [ÇGG22a, Sec. 5.2]. The vertices are the 1-periodic orbits of $H$ or equivalently the underlying closed Reeb orbits of $\alpha$. Each vertex is also labeled by $\mathrm{HF}(\tilde{x})$. In addition, the graph has an extra vertex corresponding to $W$, which is labeled by

$$\mathrm{HF}^\delta(H) = \mathrm{HF}^\delta(\tilde{H}) = \mathrm{H}(W; \partial W)[-n]$$

for a sufficiently small $\delta > 0$; see (3.7) and (3.8).

The arrows of the reduced Floer graph are defined by using the action filtration spectral sequence and then collapsing it into one complex as in [GG20, Sec. 2.1.3 and 2.5]. Namely, the $E_2$-term of this spectral sequence is

$$\mathcal{C} = \bigoplus_x \mathrm{HF}(\tilde{x}) \oplus \mathrm{HF}^\delta(H) \tag{3.17}$$

Then [GG20, Lemma 2.8] gives a way to organize the higher level differentials in the spectral sequence into one differential $\partial\colon E_2 \to E_2$ so that the resulting homology is isomorphic to $\mathrm{HF}(H)$ or, to be more precise, to the graded space associated with the filtration of $\mathrm{HF}(H)$ by the images of the maps $\mathrm{HF}^\tau(H) \to \mathrm{HF}(H)$. Denote by

$$\partial_{\tilde{x}\tilde{y}}\colon\ \mathrm{HF}(\tilde{x}) \to \mathrm{HF}(\tilde{y})$$

the $\mathrm{HF}(\tilde{y})$-component of the restriction of $\partial$ to $\mathrm{HF}(\tilde{x})$. The vertices $x$ and $y$ or equivalently $\tilde{x}$ and $\tilde{y}$ are connected by an arrow if $\partial_{\tilde{x}\tilde{y}} \neq 0$ and then the arrow is labeled by this map. Thus the graph carries exactly the same information as the complex $(\mathcal{C}, \partial)$ together with the decomposition (3.17). The length of an arrow is again the action difference. The arrows to or from the vertex $W$ are defined in a similar fashion.

*Example* 3.7. Clearly, the range of degrees of the generators in the cluster corresponding to $x$ is contained in

$$[\mu_-(x),\ \mu_+(x)+1] \subset [\hat{\mu}(x)-n+1,\ \hat{\mu}(x)+n].$$

Hence, $x$ and $y$ are never connected by an arrow when $\hat{\mu}(x)-\hat{\mu}(y)>2n$.

Likewise, assume that every Floer cylinder $u$ asymptotic to $\tilde{x}$ has energy $E(u)>\sigma$; see Section 4.1 for the precise definition. Then all arrows to/from $x$ have length greater than $\sigma$. We will repeatedly use both of these facts in the proofs of the main theorems.

*Remark* 3.8. Even when $\alpha$ is non-degenerate, the reduced Floer graph differs from the Floer graph in two ways. First, in the reduced Floer graph the vertices $\check{x}$ and $\hat{x}$ are lumped together. (The same is true for the generators coming from the critical points of $\tilde{H}$ in $W$.) But equally importantly the reduced Floer graph may have fewer arrows. This can be the case already when the construction is applied to a Morse function, and hence the two graphs have exactly the same vertex set; cf. [GG20, Rmk. 2.10]. Furthermore, it is not clear to us to what extent the reduced Floer graph depends on the auxiliary data: the perturbation $\tilde{H}$, the Morse–Bott data, the almost complex structure, etc.

## 4. Background results

In this section we discuss three results central to the proofs of the main theorems. These are the Crossing Energy Theorem for admissible Hamiltonians (Theorem 4.1); Theorem 4.3 stating, generally speaking, that the barcode of a (semi-)admissible Hamiltonian is a priori bounded; and finally the Index Recurrence Theorem (Theorem 4.7).

### 4.1. Crossing energy theorem.

The first key ingredient of the proofs is the Crossing Energy Theorem (Theorem 4.1). To state this result, we start by recalling some terminology.

Let $z$ be a closed Reeb orbit of $\alpha$ with period $T$. We say that $z$ is *isolated* (as a periodic orbit) if for every $T'>T$ it is isolated among periodic orbits with period less than $T'$. Clearly, all periodic orbits of $\alpha$ are isolated if and only if for every $T'$ the number of periodic orbits with period less than $T'$ is finite. A stronger requirement is that $z$ is *isolated as an invariant set* or *locally maximal*, i.e., $z$ has a neighborhood $V$ which contains no invariant sets other than the image $z(\mathbb{R}/T\mathbb{Z})$. We call $V$ an *isolating neighborhood*. These definitions extend verbatim to any flow.

For instance, a non-degenerate periodic orbit is isolated as a periodic orbit but not necessarily as an invariant set. A hyperbolic periodic orbit is isolated as an invariant set.

Let $H(x,r)=h(r)$ be a (semi-)admissible Hamiltonian with $a=\mathit{slope}(H)>T$ and let $\tilde{z}=(z,r_*)$, where

$$h'(r_*)=T$$

by (2.1), be the corresponding 1-periodic orbit of the Hamiltonian flow of $H$. In particular, $r_*<r_{\max}$; see Section 2.1. (Note that $\tilde{z}$ is isolated as a 1-periodic orbit of $H$ if $z$ is isolated; $\tilde{z}$ is Morse–Bott non-degenerate when $z$ is non-degenerate; and $\tilde{z}$ is never isolated as an invariant set.) Then

$$\tilde{z}^k=(z^k,r_*)$$

is also a 1-periodic orbit of $kH$ or equivalently a $k$-periodic orbit of $H$.

Let $u\colon \mathbb{R}\times S^1 \to \widehat{W}$ be a Floer cylinder of $H$. We say that $u$ is *asymptotic* to $\tilde{z}$ at $-\infty$ if there exists a sequence $s_i \to -\infty$ such that $u(s_i,\cdot) \to \tilde{z}$ in the $C^1$-sense, up to the choice of the initial condition on $\tilde{z}$ which might depend on $s_i$. This is equivalent to $u(s,\cdot)\to\tilde{z}$ in the $C^\infty$-sense as $s\to-\infty$ when $z$ is non-degenerate, [Bo]. (Likewise, $u$ is asymptotic to $\tilde{z}$ at $+\infty$ when $s_i\to+\infty$, etc.)

In general, $u$ can be asymptotic to more than one orbit $\tilde{z}$ at the same end. However, $\mathcal{A}_H(\tilde{z}) = \lim \mathcal{A}_H\big(u(s_i,\cdot)\big)$, and hence $\mathcal{A}_H(\tilde{z})$ is independent of the choice of $\tilde{z}$. Furthermore, (3.4) holds: $E(u)$ is the difference of actions of the orbits which $u$ is asymptotic to at $\pm\infty$. It is a standard fact that $u$ is asymptotic to some 1-periodic orbits of $H$ at $\pm\infty$ if and only if $E(u)<\infty$; see [Sa99, Sec. 1.5].

Next, assume that $z$ is isolated as a closed Reeb orbit or, equivalently, $\tilde{z}$ is isolated as a 1-periodic orbit of the flow of $H$. Then, as is easy to see, $\tilde{z}$ is unique and $u(s,\cdot)\to\tilde{z}$ as $s\to-\infty$ in the $C^1$-sense, up to the choice of an initial condition on $\tilde{z}$ which might depend on $s$. This is a consequence of the fact that

$$E\big(u|_{(-\infty,s_i]\times S^1}\big) \to 0 \text{ as } s_i\to-\infty$$

since $\mathcal{A}_H\big(u(s,\cdot)\big)$ is a monotone function of $s$ and of the argument in [Sa99, Sec. 1.5]; see also Remark 6.4. Therefore, for every tubular neighborhood $U\subset\widehat{W}$ of $\tilde{z}(S^1)$, there exists $s_0\in\mathbb{R}$ such that $u\big((-\infty,\, s_0]\times S^1\big)\subset U$, and for $s\le s_0$ the loop $u(s,\cdot)$ is homotopic to $\tilde{z}$ in $U$. For instance, when $\tilde{z}$ is $k$-iterated, the free homotopy class of $u(s,\cdot)$ is the $k$th multiple of a generator of $\pi_1(U)\cong\mathbb{Z}$.

**Theorem 4.1** (Crossing Energy Theorem)**.** *Assume that $z$ is a $T$-periodic orbit of $\alpha$ which is locally maximal as an invariant set. Let $H(r,x)=h(r)$ be a semi-admissible or admissible Hamiltonian with slope$(H)>T$, meeting the additional requirement that*

$$h'''\ge 0 \text{ on } [1,\, r_*+\delta] \tag{4.1}$$

*for some $\delta>0$. Fix an admissible almost complex structure $J$. Then there exists $\sigma>0$ such that $E(u)\ge\sigma$ for any $k\in\mathbb{N}$ and any Floer cylinder $u\colon\mathbb{R}\times S^1\to\widehat{W}$ of $kH$ asymptotic, at either end, to $\tilde{z}^k$.*

A few remarks are due regarding the statement of this theorem, which we will prove in Section 6. First of all, the point that $\sigma$ is independent of $k$ is crucial for our purposes. Furthermore, without it, for a fixed $k$ the theorem would follow immediately from a suitable variant of Gromov compactness and it would be enough to assume that $z^k$ is isolated as a periodic orbit.

Secondly, fix $k$ and $0<\sigma'<\sigma$, and let $V$ be a small tubular neighborhood of $\tilde{z}^k(S^1)$. Consider a $C^\infty$-small $k$-periodic in time, non-degenerate perturbation $\tilde{H}$ of $kH$ in the class specified in Section 3.3 and a $C^\infty$-small generic $k$-periodic perturbation of $J$. The 1-periodic orbit $\tilde{z}^k$ of $kH$ splits into several non-degenerate periodic orbits of $\tilde{H}$ contained in $V$. It follows again from a suitable version of the Gromov compactness theorem (see, e.g., [Fi]) that every Floer cylinder of $\tilde{H}$ asymptotic to any of these orbits at either end has energy greater than $\sigma'$. Thus, replacing $\sigma$ by, say, $\sigma'=\sigma/2$, we obtain the following result.

**Corollary 4.2.** *Let $z$, $H$ and $J$ be as in Theorem 4.1. Assume that all closed Reeb orbits of $\alpha$ are isolated. Then there exists $\sigma>0$ such that for all $k\in\mathbb{N}$ and for any choice of auxiliary data every arrow to/from $\tilde{z}^k$ in the Floer graph of $kH$ has length greater than $\sigma$.*

4.2. **Vanishing of symplectic homology.** The result we need here translates vanishing of symplectic homology to a certain quantitative property of the filtered Floer homology of the sequence $kH$. More specifically, it asserts that the homology $\mathrm{HF}^I(kH)$ is uniformly unstable, i.e., there is a uniform upper bound on the longest bar, over larger and larger range of action as $k \to \infty$, whenever $\mathrm{SH}(\alpha) = 0$.

**Theorem 4.3.** *Assume that* $\mathrm{SH}(\alpha) = 0$*. Fix* $b > 0$ *and let* $H$ *be a semi-admissible Hamiltonian with* $a := \mathit{slope}(H) > b$*. Then there exists a constant* $C_{bar} > 0$ *depending only on* $H$ *such that for any interval* $I \subset (-\infty, kb]$ *and every sufficiently large* $k \in [1, \infty)$ *the inclusion/quotient map*

$$\mathrm{HF}^I(kH) \to \mathrm{HF}^{I+C_{bar}}(kH)$$

*is zero. In particular, every bar ending below* $kb$ *has length less than* $C_{bar}$*.*

In particular, this theorem applies when $\alpha$ is a contact form on $M \cong S^{2n-1}$ supporting the standard contact structure. Indeed, in this case, we can think of $M$ as the boundary of a star-shaped domain $W \subset \widehat{W} = \mathbb{R}^{2n}$; and $\mathrm{SH}(\alpha) = 0$ since $W$ is displaceable in $\widehat{W}$; see Section 3.2.4.

*Remark* 4.4. This theorem is closely related to [GS, Prop. 3.5], which is essentially due to Kei Irie, asserting that for some constant $C > 0$ depending only on $\alpha$ the map $\mathrm{SH}^I(\alpha) \to \mathrm{SH}^{I+C}(\alpha)$ is zero for any interval $I$ if and only if $\mathrm{SH}(\alpha) = 0$. The proof of Theorem 4.3 follows roughly the same path as the proof of that proposition with natural complications arising from the argument having to deal with a specific Hamiltonian rather than a direct limit. We also note that Theorem 4.3 is considerably stronger than Lemma 3.6. The key difference is that the constant $C_{bar}$ here can be taken independent of $I \subset (-\infty, kb]$ and $k$, as long as $k$ is sufficiently large. Moreover, $k$ need not be an integer here.

*Remark* 4.5. As stated, the theorem does not hold for admissible Hamiltonians. For the sequence $kH$ is then decreasing on $W$ and one has to account for this fact by adding a constant, linearly increasing with $k$, to $kH$ or to $C_{bar}$. Furthermore, the requirement that $I \subset (-\infty, kb]$ is essential; for in general $\mathrm{HF}(kH) \neq 0$ even though $\mathrm{SH}(\alpha) = 0$.

*Proof.* Let $a = \mathit{slope}(H)$. By the long exact sequence it suffices to prove the theorem for a semi-infinite interval, which we denote by

$$(-\infty, \beta] := I \subset (-\infty, kb].$$

Hence our goal is to show that the map

$$\mathrm{HF}^\beta(kH) \to \mathrm{HF}^{\beta+C_{bar}}(kH)$$

is identically zero for all large $k$ and $\beta \leq kb$ and some constant $C_{bar} > 0$. We carry out the argument in three steps.

*Step 1.* Let $\epsilon > 0$ and $\delta > 0$ be sufficiently small. Then, by (3.7) and (3.8) $\mathrm{HF}^\delta_*(\epsilon H) \cong \mathrm{H}_{*+n}(W, \partial W)$ and, in particular, $\mathrm{HF}^\delta_n(\epsilon H) \cong \mathbb{F}$. On the other hand, by Lemma 3.6, the natural inclusion/homotopy map $\iota\colon \mathrm{HF}^\delta_n(\epsilon H) \to \mathrm{HF}^C_n(\lambda H)$ is identically zero for some $C > 0$ and a sufficiently large $\lambda$ since $\mathrm{SH}(\alpha) = 0$.

Next, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{HF}^{\beta}(kH)\otimes \mathrm{HF}^{\delta}_{n}(\epsilon H) & \longrightarrow & \mathrm{HF}^{\beta+\delta}\big((k+\epsilon)H\big) \\ \downarrow{\scriptstyle id\otimes\iota} & & \downarrow{\scriptstyle \Psi} \\ \mathrm{HF}^{\beta}(kH)\otimes \mathrm{HF}^{C}_{n}(\lambda H) & \longrightarrow & \mathrm{HF}^{\beta+C}\big((k+\lambda)H\big) \end{array}$$

Here $\Psi$ is induced by a monotone increasing homotopy of the Hamiltonians and the inclusion of action intervals, and the horizontal arrows are given by the pair-of-pants product. The top horizontal arrow in this diagram is an isomorphism since the generator of $\mathrm{HF}^{\delta}_{n}(\epsilon H)\cong \mathbb{F}$ is a "unit" with respect to the pair-of-pants product; see Section 3.2.1. Therefore, $\Psi = 0$ since $\iota = 0$. Furthermore,

$$\mathrm{HF}^{\beta+\delta}\big((k+\epsilon)H\big)\cong \mathrm{HF}^{\beta}(kH)$$

when $\delta > 0$ and $\epsilon > 0$ are sufficiently small. Composing this isomorphism with $\Psi$, we conclude that the map

$$\Phi\colon\ \mathrm{HF}^{\beta}(kH)\to \mathrm{HF}^{\beta+C}\big((k+\lambda)H\big),$$

again induced by a monotone increasing homotopy of the Hamiltonians and the inclusion of action intervals, is also zero. This completes the first step of the proof.

*Remark* 4.6. Continuing Remark 4.4, note that to prove [GS, Prop. 3.5] we could simply apply this argument to an admissible Hamiltonian $H_k$ in place of $kH$ from a cofinal sequence and then pass to the limit. However, for the iterates of a specific Hamiltonian and an unbounded action range an additional reasoning is needed. This is in essence the question of uniform convergence and interchanging the limits; cf. Remark 4.4.

*Step 2.* By Proposition 3.1, a monotone increasing homotopy from $kH$ to $(k+\lambda)H$ induces an isomorphism in the filtered homology

$$\mathrm{HF}^{\tau}(kH)\to \mathrm{HF}^{f(\tau)}\big((k+\lambda)H\big).$$

Here, as in Section 3.2.2,

$$f\colon [0,\ A_{kH}(r_{\max})] = [0,\ kA_H(r_{\max})] \to [0,\ A_{(k+\lambda)H}(r_{\max})] = [0,\ (k+\lambda)A_H(r_{\max})]$$

is the function $\mathfrak{a}_{(k+\lambda)H}\circ \mathfrak{a}_{kH}^{-1}$. We claim that

$$\tau \geq f(\tau) \geq \tau - \lambda h(r_{\max}). \tag{4.2}$$

The exact value of the right-hand side in (4.2) is inessential for our purposes. However, it is important that the right-hand side, in contrast with the function $f$, is independent of $k$.

Postponing the proof of the claim, let us finish the proof of the theorem. First observe that $\beta + C$ is in the range of $f$, i.e.,

$$\beta + C \leq f\big(kA_H(r_{\max})\big)$$

when $k$ is large enough. Indeed, by (2.3) with $H$ replaced by $kH$ and $a$ replaced by $ka$, we have

$$kA_H(r_{\max}) \geq ka.$$

Also recall that $\beta \leq kb$. Thus, by (4.2), $\beta + C$ is in the range of $f$ whenever

$$kb + C \leq ka - \lambda h(r_{\max}).$$

This condition is automatically satisfied when $k$ is sufficiently large since $b < a$.

Furthermore, since $f$ is monotone increasing, applying (4.2) to $\tau = \beta + C + \lambda h(r_{\max})$, we see that

$$\beta + C + \lambda h(r_{\max}) \geq f^{-1}(\beta + C). \tag{4.3}$$

Consider now the sequence of maps

$$\mathrm{HF}^{\beta}(kH) \xrightarrow{\Phi=0} \mathrm{HF}^{\beta+C}\big((k+\lambda)H\big) \to \mathrm{HF}^{f^{-1}(\beta+C)}(kH) \to \mathrm{HF}^{\beta+C+\lambda h(r_{\max})}(kH).$$

Here $f^{-1}(\beta+C) \geq \beta + C > \beta$ and, by Proposition 3.1, the composition of the first two maps is the inclusion-induced map

$$\mathrm{HF}^{\beta}(kH) \to \mathrm{HF}^{f^{-1}(\beta+C)}(kH),$$

which is then identically zero, for $\Phi = 0$. The action level $f^{-1}(\beta + C)$ depends on $k$, but (4.3) provides an upper bound which does not. Hence, setting

$$C_{bar} = C + \lambda h(r_{\max}),$$

we conclude that the map

$$\mathrm{HF}^{\beta}(kH) \to \mathrm{HF}^{\beta+C_{bar}}(kH)$$

is also zero.

*Step 3.* To complete the argument, it remains to prove (4.2). Consider the linear monotone increasing homotopy $F_s := (k + s\lambda)H$, $s \in [0, 1]$, from $F_0 = kH$ to $F_1 = (k + \lambda)H$. (Strictly speaking, we should replace here $s$ by a monotone increasing function on $\mathbb{R}$ equal to 0 near $-\infty$ and 1 near $+\infty$. However, this technicality does not affect the result of the calculation.) Setting

$$f_s := \mathfrak{a}_{F_s} \circ \mathfrak{a}_{F_0}^{-1}, \tag{4.4}$$

we need to bound the change of $f_s(\tau)$ as $s$ ranges from 0 to 1.

Recall from (2.4) that $\mathfrak{a}_{F_s}(T) = A_{F_s}\big(r(s)\big)$, where

$$(k + s\lambda)h'(r) = T. \tag{4.5}$$

(In other words, for a closed Reeb orbit with period $T$, the corresponding 1-periodic orbit of $F_s$ lies on the level $r = r(s)$.) We also note that the domain of $\mathfrak{a}_{F_s}$ changes with $s$ increasing from $[0,\, ka]$ for $s = 0$ to $[0,\, (k + \lambda)a]$ for $s = 1$. Below we always assume that $T$ is in the range $[0,\, ka]$ of $\mathfrak{a}_{F_0}^{-1}$.

Differentiating (4.5) with respect to $s$, we have

$$\lambda h'(r) + (k + s\lambda)h''(r)r' = 0. \tag{4.6}$$

Furthermore, recall that by (2.2)

$$\mathfrak{a}_{F_s}(T) = A_{F_s}(r) = (k + s\lambda)\big(rh'(r) - h(r)\big).$$

Differentiating again, we see that

$$\begin{aligned}\frac{d}{ds}\mathfrak{a}_{F_s}(T) &= \lambda\big(rh'(r) - h(r)\big)\\ &\quad + (k + s\lambda)\big(r'h'(r) + rh''(r)r' - h'(r)r'\big)\\ &= \lambda\big(rh'(r) - h(r)\big) + (k + s\lambda)rh''(r)r'.\end{aligned}$$

By (4.6), the last term in this expression is $-\lambda rh'(r)$, and hence

$$\begin{aligned}\frac{d}{ds}\mathfrak{a}_{F_s}(T) &= \lambda\big(rh'(r) - h(r)\big) + (k + s\lambda)rh''(r)r' \\ &= \lambda\big(rh'(r) - h(r)\big) - \lambda rh'(r) \\ &= -\lambda h(r).\end{aligned}$$

Thus this derivative is non-positive and

$$\left|\frac{d}{ds}\mathfrak{a}_{F_s}(T)\right| \leq \lambda h(r_{\max})$$

for all $T \in [0,\, a]$. Therefore, by (4.4),

$$0 \geq \frac{df_s}{ds} = \frac{d\mathfrak{a}_{F_s}}{ds} \circ \mathfrak{a}_{F_0}^{-1} \geq -\lambda h(r_{\max})$$

on $[0,\, kA_H(r_{\max})]$. Integrating with respect to $s$ and using the fact that $f_0(\tau) = \tau$ and $f_1 = f$, we obtain (4.2). □

4.3. **Index recurrence.** Index recurrence has a very different flavor from the first two results of this section and is essentially a symplectic linear algebra phenomenon. Let, as in Section 2.2, $\mu_\pm$ be the upper and lower semicontinuous extensions of the Conley–Zehnder index to the universal covering $\widetilde{\mathrm{Sp}}(2m)$. We need to introduce some further invariants of $\Phi$, playing a central role in the index recurrence theorem. Abusing notation, we will automatically extend all invariants from $\mathrm{Sp}(2m)$ to $\widetilde{\mathrm{Sp}}(2m)$ by applying them to the end-point.

Consider first a totally degenerate symplectic linear map $A \in \mathrm{Sp}(2m)$. (In other words, $A$ is unipotent: all eigenvalues of $A$ are equal to 1.) Then we can write $A$ as $A = \exp(JQ)$, where $Q$ is symmetric and all eigenvalues of $JQ$ are equal to 0; see, e.g., the proof of [GG20, Lemma 4.2]. We will view $Q$ as a quadratic form. It can be symplectically decomposed into a sum of terms of four types:

- the identically zero quadratic form on $\mathbb{R}^{2\nu_0}$,
- the quadratic form $Q_0 = p_1q_2 + p_2q_3 + \cdots + p_{d-1}q_d$ in Darboux coordinates on $\mathbb{R}^{2d}$, where $d \geq 1$ is odd,
- the quadratic forms $Q_\pm = \pm(Q_0 + p_d^2/2)$ on $\mathbb{R}^{2d}$ for any $d$.

(This variant of the Williamson normal forms is taken from [AG, Sect. 2.4].) Clearly, $\dim\ker Q_0 = 2$ and $\dim\ker Q_\pm = 1$. Let $b_*(Q)$, where $* = 0, \pm$, be the number of the $Q_0$ and $Q_\pm$ terms in the decomposition. Let us also set $b_*(A) := b_*(Q)$ and $\nu_0(A) := \nu_0(Q)$. These are symplectic invariants of $Q$ and $A$, and the geometric multiplicity $\nu_{geom}$ of the eigenvalue 1 is

$$\nu_{geom} = 2(b_0 + \nu_0) + b_+ + b_-.$$

It is not hard to show (cf. [GG20, Lemma 4.2]) that for a path $\Phi$ with $\Phi(1) = A$, we have

$$\mu_+(\Phi) = \hat{\mu}(\Phi) + b_0 + b_+ + \nu_0 \quad \text{and} \quad \mu_-(\Phi) = \hat{\mu}(\Phi) - b_0 - b_- - \nu_0. \tag{4.7}$$

These formulas readily extend to all paths. Namely, every $\Phi \in \widetilde{\mathrm{Sp}}(2m)$ can be written (non-uniquely) as a product of a loop $\varphi$ and the direct sum $\Psi_0 \oplus \Psi_1$ where $\Psi_0 \in \widetilde{\mathrm{Sp}}(2m_0)$ is a totally degenerate (for all $t$) path $\Psi_0(t) = \exp(JQt)$ and $\Psi_1 \in \mathrm{Sp}(2m_1)$ is non-degenerate. In particular, $m_0 = \nu(\Phi)$ and $m_0 + m_1 = m$. (Note that $\varphi$ can be absorbed into $\Psi_1$ unless $m_1 = 0$.)

Then we set

$$b_*(\Phi) := b_*(\Psi_0) \text{ for } * = 0, \pm \text{ and } \nu_0(\Phi) := \nu_0(\Psi_0).$$

These are symplectic invariants of $\Phi$, and

$$\mu_+(\Phi) = \hat{\mu}(\varphi) + \mu(\Psi_1) + b_0 + b_+ + \nu_0$$

and

$$\mu_-(\Phi) = \hat{\mu}(\varphi) + \mu(\Psi_1) - b_0 - b_- - \nu_0.$$

Let also $2\nu_{alg}(\Phi)$ be the algebraic multiplicity of the eigenvalue 1 of $\Phi(1)$. Clearly, $\nu_{geom}/2 \leq \nu_{alg} \leq m$ and

$$\nu_{alg} \geq \nu_0 + b_0 + b_+ + b_- \geq b_+ - b_-.$$

**Theorem 4.7** (Index Recurrence, [GG20]). *Let $\Phi_0, \ldots, \Phi_q$ be a finite collection of elements in $\widetilde{\mathrm{Sp}}(2m)$ with $\hat{\mu}(\Phi_i) > 0$ for all $i$. Then for any $\eta > 0$ and any $\ell_0 \in \mathbb{N}$, there exists an integer sequence $d_s \to \infty$ and $q$ integer sequences $k_{is}$, $i = 0, \ldots, q$, going to infinity as $s \to \infty$ and such that for all $i$ and $s$, and all $\ell \in \mathbb{N}$ in the range $1 \leq \ell \leq \ell_0$, we have*

(i) $\big|\hat{\mu}(\Phi_i^{k_{is}}) - d_s\big| < \eta$,
(ii) $\mu_\pm(\Phi_i^{k_{is}+\ell}) = d_s + \mu_\pm(\Phi_i^\ell)$,
(iii) $\mu_+(\Phi_i^{k_{is}-\ell}) = d_s - \mu_-(\Phi_i^\ell) + \big(b_+(\Phi_i^\ell) - b_-(\Phi_i^\ell)\big)$.

*In particular, $\mu_+(\Phi_i^{k_{is}-\ell}) \leq d_s - \mu_-(\Phi_i^\ell) + \nu_{alg}(\Phi_i^\ell)$, and $\mu(\Phi_i^{k_{is}-\ell}) = d_s - \mu(\Phi_i^\ell)$ when $\Phi_i$ is non-degenerate. Furthermore, for any $N \in \mathbb{N}$ we can make all $d_s$ and $k_{is}$ divisible by $N$.*

This theorem asserts, in particular, that every arbitrarily long segment starting at $\mu_\pm(\Phi_i)$ and ending at $\mu_\pm(\Phi_i^{\ell_0})$ will reoccur infinitely many times in the sequence $\mu_\pm(\Phi_i^k)$, $k \in \mathbb{N}$, up to a common shift independent of $i$. (Hence the name of the name of the theorem.) When the paths are non-degenerate we can take the symmetric segment $\mu(\Phi_i^{-\ell_0}), \ldots, \mu(\Phi_i^{\ell_0})$ with $\mu(\Phi^0)$ omitted. We also note that when $\Phi$ is dynamically convex, i.e., $\mu_-(\Phi_i) \geq m + 2$, we have $\mu_+(\Phi_i^{k_{is}-\ell}) \leq d_s - 2$ in (iii).

Theorem 4.7 can be easily derived from the Common Jump Theorem from [Lo, LZ] or proved independently. We refer the reader to [GG20, Sec. 5] for a direct proof of a more precise result than stated here. Note that for one non-degenerate path $\Phi := \Phi_0$ the assertion readily follows from Kronecker's theorem – it is enough to pick the iterations $k_s := k_{0s}$ so that all elliptic eigenvalues of $\Phi^{k_s}$ are sufficiently close to 1. In a similar vein, the more general case of several non-degenerate paths ultimately relies on Minkowski's theorem on simultaneous homogeneous approximations.

## 5. Proofs of the main theorems

In this section we prove Theorems A and B. When $2n - 1 = 1$, Theorem A holds trivially. Hence, we can assume in both of the proofs that $2n - 1 \geq 3$.

### 5.1. Proof of Theorem A: Hyperbolic periodic orbits and multiplicity.

We argue by contradiction. Assume that the Reeb flow has only finitely many simple periodic orbits with positive mean index. We denote these orbits by $x_0, x_1, \ldots, x_q$. Let $T_i$ be the period of $x_i$. Without loss of generality we can assume that $z = x_0$ is a hyperbolic orbit with $\hat{\mu}(z) > 0$ and $\mu(z) \geq 3$.

In the notation from Section 2, let $H$ be a semi-admissible Hamiltonian such that the slope $a$ of $H$ is greater than $\max T_i$. In particular, $T_0 < a$. The 1-periodic orbits of $kH$ have the form

$$\tilde{x}_i^j(k) := (x_i^j, r_{ij}(k)),$$

where $x_i^j$ is the $j$-th iterate of $x_i$ and $r_{ij}(k) \in (0, \infty)$ is determined by the condition

$$kh'\big(r_{ij}(k)\big) = jT_i; \tag{5.1}$$

see (2.1). Note that for every fixed $i$ the range of $j$ is $\big[1, \lfloor ka/T_i \rfloor\big]$. In particular, it is always non-empty since $T_i < a$; the range grows linearly with $k$ and can be larger than $k$.

For technical reasons it will also be useful to require that

$$\max_i \hat{\mu}(z)T_i / \hat{\mu}(x_i) < a. \tag{5.2}$$

Finally, we also need to take $a$ sufficiently large so that Theorem 4.3 applies with $b$ to be specified later; see (5.4).

When the iteration order $k$ of the Hamiltonian is clear from the context, we will suppress it in the notation and simply write $\tilde{x}_i^j$ for $\tilde{x}_i^j(k)$. For $i = 0$ we will denote the corresponding orbit by $\tilde{z}^j$. Note that, in spite of the notation, in general $\tilde{x}_i^j$ is not the $j$-th iterate of $\tilde{x}_i^1$. However, an exception is the case of $j = k$. Namely, by (5.1), $r_{ik}(k) = r_{i1}(1)$, and hence

$$\tilde{x}_i^k(k) = (x_i^k, r_{i1}(1)).$$

Denoting $r_* = r_{01}(1)$, we have, in particular,

$$\tilde{z}^k(k) = (z^k, r_*), \text{ where } h'(r_*) = T_0. \tag{5.3}$$

The orbit $\tilde{z}^k(k)$ gives rise to two generators $\check{z}^k$ and $\hat{z}^k$ of degrees $|\check{z}^k| = \mu(z^k)$ and $|\hat{z}^k| = \mu(z^k) + 1$ respectively in the Floer complex of $kH$.

Throughout the proof we will need to make sure that the pair $(H, r_*)$ meets the conditions of Theorem 4.1. It is easy to show that such Hamiltonians exist. For instance, let us start with $H$ such that, in addition,

$$h''' \geq 0 \text{ on } [1,\, \rho]$$

for some $\rho > 1$. At this point we do not necessary have $r_* \leq \rho$. Consider, however, the family of Hamiltonians $\lambda H$ with $\lambda \geq 1$. For each of these Hamiltonians $r_* = r_*(\lambda)$ is determined by

$$h'(r_*) = T_0/\lambda,$$

and hence $r_* \to 1$ as $\lambda \to \infty$. For $\lambda$ large enough, we have $r_* < \rho$. Replacing $H$ by $\lambda H$ and keeping the notation, we have a semi-admissible Hamiltonian $H$ meeting the requirements of Theorem 4.1. Thus $E(u) > \sigma$ for any non-constant Floer cylinder for $kH$ asymptotic to $(z^k, r^*)$ at either end, where the constant $\sigma > 0$ is independent of $k$ and $u$.

Fix

$$b > \mathcal{A}_H(\tilde{z}) = r_*T_0 - h(r_*). \tag{5.4}$$

We will further require that $a = \mathit{slope}(H) > b$, and hence Theorem 4.3 applies. Again, it is easy to see that $H$ meeting all of the above requirements exist. Let $C_{bar} > 0$ be as in that theorem. In other words, the map

$$\mathrm{HF}^I(kH) \to \mathrm{HF}^{I+C_{bar}}(kH)$$

vanishes for large $k$ and any $I \subset (-\infty, kb]$.

In the proof, we will apply the Index Recurrence Theorem (Theorem 4.7) with $m = n-1$ to the linearized Reeb flows $\Phi_i$ along $x_i$, including the flow along $z = x_0$. In the theorem, we can take any $\eta < \min\{\sigma/C, 1/2\}$, where the constant $C$ is to be specified later (see (5.11)), and any $\ell_0 \in \mathbb{N}$ with

$$\ell_0 > \frac{n+3}{\min_{i\geq 1} \hat{\mu}(x_i)}. \tag{5.5}$$

We will need to have $s$ sufficiently large so that $d_s$ and $k_{is}$ are also large. For the sake of brevity, suppressing $s$ in the notation, we set $d = d_s$, $k = k_{0s}$ and $k_i = k_{is}$ for $i = 1, \dots, q$.

The key to the proof is the following result.

**Lemma 5.1.** *The length of the shortest arrow to/from $\hat{z}^k$ in the Floer complex of $kH$ goes to infinity as $s \to \infty$, and hence $k \to \infty$.*

The theorem readily follows from the lemma. Indeed, let $A = \mathcal{A}_{kH}(\tilde{z}^k)$ and $I = [A - 2C_{bar},\, A + 2C_{bar}]$. Then $\hat{z}^k$ is closed in $\mathrm{CF}^I(kH)$ when $k$ is large enough. Furthermore, $[\hat{z}^k] \neq 0$ in $\mathrm{HF}^I(kH)$ and its image in $\mathrm{HF}^{I+C_{bar}}(kH)$ is also non-zero. This contradicts Theorem 4.3. (Alternatively, one can use here a variant of [ÇGG21, Prop. 3.8].)

*Proof of Lemma 5.1.* Clearly, $\hat{z}^k$ is not connected by an arrow to any orbit $y$ with $\hat{\mu}(y) \leq 0$ or any generator arising from $W$ when $k$ is large; see Section 3.3. Hence, to prove the lemma, it suffices to show that the "index distance" from $\hat{z}^k(k)$ to $\operatorname{supp} \mathrm{HF}\left(\tilde{x}_i^j(k)\right)$ is greater than or equal to 2 or the action difference between the action of $\tilde{z}^k(k)$ and $\tilde{x}_i^j(k)$ becomes arbitrarily large as $k \to \infty$ or is smaller than $\sigma$. (We emphasize that all 1-periodic orbits here are taken for the same Hamiltonian $kH$ where $k = k_{0s}$. However, we usually suppress this role of $k$ in the notation. Thus $\check{z}^k := \check{z}^k(k)$, $\hat{z}^j := \hat{z}^j(k)$, $\tilde{x}_i^j := \tilde{x}_i^j(k)$, etc.)

First, note that since $z$ is hyperbolic, the mean index of $z^j$ is equal to its Conley–Zehnder index:

$$\hat{\mu}(z^j) = j\,\hat{\mu}(z) = \mu(z^j) = j\mu(z) \in \mathbb{Z}. \tag{5.6}$$

In particular, by Condition (i) of Theorem 4.7,

$$d = \hat{\mu}(z^k) = k\,\hat{\mu}(z) = \mu(z^k)$$

since $0 < \eta < 1/2$, and hence

$$|\check{z}^k| = d \text{ and } |\hat{z}^k| = d + 1.$$

For the sake of brevity, we will write $y := x_i^j$ and $x = x_i$. Setting

$$l = j - k_i,$$

we will treat separately several cases determined by the value of $l$ and whether or not $z = x$ (i.e., $i = 0$) or $z \neq x$ (i.e., $i \geq 1$).

*Case 0: $z = x$, i.e., $i = 0$.* Clearly, the orbits $\check{z}^k$ and $\hat{z}^k$ are not connected by a Floer arrow even though the index difference is one. Next, due to the requirement that $\mu(z) \geq 3$ and (5.6), for $j \neq k$ the degree difference between $\check{z}^k$ or $\hat{z}^k$ and $\check{z}^j$ or $\hat{z}^j$ is at least two. Hence, neither of the orbits $\check{z}^k$ or $\hat{z}^k$ is connected by a Floer arrow to either $\check{z}^j$ or $\hat{z}^j$. From now on we will assume that $i \geq 1$.

*Case 1: $l = 0$, i.e., $j = k_i$, and $i \geq 1$.* This case is central to the proof of the theorem and the argument amounts to controlling the action difference.

Recall that $k = k_{0s}$ and $j = k_{is}$ depend on the parameter $s$, and $k \to \infty$ and $j \to \infty$ as $s \to \infty$. By Condition (i) of Theorem 4.7, we have

$$\frac{j}{k} \to \frac{\hat{\mu}(z)}{\hat{\mu}(x)} \quad \text{as} \quad s \to \infty.$$

Consider now the periodic orbits $\tilde{y} = \big(x^j, r^* := r_{ij}(k)\big)$ and $\tilde{z}^k = (z^k, r_*)$ of $H^{\#k}$. Here $r_*$ is determined by (5.3) and

$$kh'(r^*) = jT$$

by (5.1). In other words,

$$h'(r^*) = \frac{j}{k}T \to \frac{\hat{\mu}(z)}{\hat{\mu}(x)}T \quad \text{as} \quad s \to \infty. \tag{5.7}$$

We will break down the argument into two subcases: either

$$\frac{T}{\hat{\mu}(x)} = \frac{T_0}{\hat{\mu}(z)}, \tag{5.8}$$

or these two ratios are distinct. Here $T = T_i$ is the Reeb action of $x$ and $T_0$ is the Reeb action of $z$.

We will show that for a large $s$, and hence $k$ and $j$, (5.8) results in a short action gap (smaller than $\sigma$) between 1-periodic orbits $\tilde{y}$ and $\tilde{z}^k$ of $kH$ and the inequality leads to a large action gap going to infinity as $k \to \infty$. By rescaling $\alpha$, we can assume without loss of generality that

$$T_0 = \hat{\mu}(z) = d. \tag{5.9}$$

Let us first focus on the former subcase – the equality, (5.8). Then we also have

$$T = \hat{\mu}(x). \tag{5.10}$$

Also note that the limit in (5.7) is then equal to $T_0$.

Let

$$(h')^{\mathrm{inv}} \colon [0,\, a] \to [1,\, r_{\max}]$$

be the inverse function of $h'$, where $a$ is the slope of $H$. This is a continuous function, which is $C^\infty$-smooth on $(0,\, a)$. (At the end-points 0 and $a$ the derivative of this function is $+\infty$.) Set

$$C_1 := 2\,\frac{d(h')^{\mathrm{inv}}}{d\tau}(T_0) < \infty.$$

Then, since $jT/k \to T_0$, when $k$ is large enough we have

$$|r_* - r^*| \leq C_1 |T_0 - jT/k|.$$

Set

$$C_2 = \max_{[1,\, r_{\max}]} |A_h'| = \max_{[1,\, r_{\max}]} r|h''(r)| < \infty.$$

Then

$$\begin{aligned}\left|\mathcal{A}_{kH}(\tilde{y})-\mathcal{A}_{kH}(\tilde{z}^k)\right| &= \left|kA_h(r^*)-kA_h(r_*)\right|\\ &\leq C_2k|r^*-r_*|\\ &\leq C_1C_2k|T_0-jT/k|\\ &= C_1C_2|kT_0-jT|\\ &= C_1C_2|k\,\hat{\mu}(z)-j\,\hat{\mu}(x)|\\ &\leq C_1C_2\eta.\end{aligned}$$

Here we used Condition (i) of Theorem 4.7 and the convention (5.9). Without this convention, the upper bound would be $\big(C_1C_2T_0/\,\hat{\mu}(z)\big)\eta$.

Therefore,

$$\left|\mathcal{A}_{kH}(\tilde{y})-\mathcal{A}_{kH}(\tilde{z}^k)\right|<\sigma,$$

when

$$C\eta<\sigma,\ \text{ where } C:=C_1C_2T_0/\,\hat{\mu}(z), \tag{5.11}$$

and $\tilde{y}$ and $\tilde{z}^k$ are not connected by a Floer arrow by the Crossing Energy Theorem (Theorem 4.1).

Next, let us focus on the subcase where (5.8) fails. Again, without loss of generality we can require (5.9). Now, in place of (5.10), we have

$$|T-\hat{\mu}(x)|>\delta$$

for some $\delta>0$.

Set

$$c_1:=\min_{[0,\,a]}\left|\frac{d(h')^{\mathrm{inv}}(\tau)}{d\tau}\right|>0.$$

Let $\xi>0$ be so small that

$$1+\xi\leq(h')^{\mathrm{inv}}\big(\,\hat{\mu}(z)T/\,\hat{\mu}(x)\big)\leq r_{\max}-\xi$$

and

$$1+\xi\leq r_*=(h')^{\mathrm{inv}}(T_0)\leq r_{\max}-\xi.$$

Such $\xi$ exists due to (5.2). Then, by (5.7), $r^*\in\Gamma$ and, by construction, $r_*\in\Gamma$, where $\Gamma:=[1+\xi,\,r_{\max}-\xi]$. Let

$$c_2:=\min_{\Gamma}|A_h'|\geq(1+\xi)\min_{\Gamma}|h''|>0.$$

We emphasize that both of the constants $c_1$ and $c_2$ are strictly positive and the interval with end-points $r_*$ and $r^*$ is contained in $\Gamma$ when $k$ is large. Therefore,

$$\begin{aligned}\left|\mathcal{A}_{kH}(\tilde{y})-\mathcal{A}_{kH}(\tilde{z}^k)\right| &= \left|kA_h(r^*)-kA_h(r_*)\right|\\ &\geq c_2k|r^*-r_*|\\ &\geq c_1c_2k|T_0-jT/k|\\ &= c_1c_2|kT_0-jT|\\ &\geq c_1c_2\delta j-c_1c_2|k\,\hat{\mu}(z)-j\,\hat{\mu}(x)|\\ &\geq c_1c_2\delta j-c_1c_2\eta.\end{aligned}$$

As a consequence, the length of an arrow between $\tilde{z}^k$ and $\tilde{y}$, if it exists, is bounded from below by

$$c_1c_2\delta j-c_1c_2\eta\to\infty.$$

This completes the proof of the lemma in Case 1.

*Remark* 5.2. Clearly condition (5.8) is extremely non-generic. However, we cannot use a small perturbation to eliminate it, for the contradiction assumption that the Reeb flow of $\alpha$ has finitely many simple periodic orbits is already non-generic. In fact, under this assumption, one would expect resonance relations of this type to hold; see, e.g., [GG20, Sec. 6] and references therein.

It remains to deal with the situation where

$$l = j - k_i \neq 0$$

and $i \geq 1$, which is handled via the index gap by Theorem 4.7. Note that in the setting of that theorem

$$\ell = |l|.$$

Recall also that, by (3.15),

$$\operatorname{supp} \mathrm{HF}(\tilde{y}) \subset \mathcal{I} := [\mu_-(y),\, \mu_+(y) + 1] \subset [j\, \hat{\mu}(x) - n + 1,\, j\, \hat{\mu}(x) + n].$$

*Case 2: $\ell := |l| > \ell_0$ and $i \geq 1$.* Let us show that neither of the orbits $\check{z}^k$ or $\hat{z}^k$ is connected by a Floer arrow to $\tilde{y}$. Assume first that $l$ is positive, i.e., $l > \ell_0$. Then

$$\mathcal{I} \subset k_i\, \hat{\mu}(x) + [l\, \hat{\mu}(x) - n + 1,\, l\, \hat{\mu}(x) + n] \subset d + [l\, \hat{\mu}(x) - n,\, l\, \hat{\mu}(x) + n + 1],$$

where the second inclusion relies on the requirement that $0 < \eta < 1$ together with the fact that $|k_i\, \hat{\mu}(x) - d| < \eta$ by Condition (i) from Theorem 4.7. Thus the distance from $|\check{z}^k| = d$ or $|\hat{z}^k| = d + 1$ to $\mathcal{I}$, and hence to $\operatorname{supp} \mathrm{HF}(\tilde{y})$, is at least 2; for $l\, \hat{\mu}(x) - n \geq \ell_0\, \hat{\mu}(x) - n \geq 2$ by (5.5). When $l < -\ell_0$, the argument is similar.

*Case 3: $\ell := |l| \leq \ell_0$ and $i \geq 1$.* Arguing as in Example 3.7, we claim that under the assumptions of the theorem, the index difference between $|\hat{z}^k| = d + 1$ and $\operatorname{supp} \mathrm{HF}(\tilde{y})$ is again at least 2, and hence $\hat{z}^k$ is not connected to $\tilde{y}$ by a Floer arrow. To see this, assume first that $l > 0$. Then, by Condition (ii) of Theorem 4.7 with $l = \ell$,

$$\mathcal{I} \subset [d + \mu_-(x^\ell),\, \infty) \subset [d + 3,\, \infty),$$

and the distance from $|\hat{z}^k| = d + 1$ to $\mathcal{I}$, and hence to $\operatorname{supp} \mathrm{HF}(\tilde{y})$, is at least 2. When $l < 0$, setting $l = -\ell$, we have

$$\mathcal{I} \subset (-\infty,\, d - \mu_-(x^\ell) + \nu_{\scriptscriptstyle alg}(x^\ell) + 1] \subset (-\infty,\, d - 1]$$

by Condition (iii). Therefore, the distance from $|\hat{z}^k| = d + 1$ to $\operatorname{supp} \mathrm{HF}(\tilde{y}) \subset \mathcal{I}$ is also at least 2. This completes the proof of the lemma and the proof of Theorem A. ☐

*Remark* 5.3. Note that in Cases 0–2 of the proof, the orbits $\check{z}^k$ and $\hat{z}^k$ play similar roles and neither one of them is connected to $\tilde{y}$ by a (short) arrow. Moreover, no dynamical convexity type condition other than that $\mu(z) \geq 3$ was used in these steps. However, in Case 3 the index conditions enter the picture and the symmetry between the orbits $\check{z}^k$ and $\hat{z}^k$ breaks down. We have chosen to work with $\hat{z}^k$ and impose the assumption that $\mu_-(x) \geq \max\big\{3,\, 2 + \nu_{\scriptscriptstyle alg}(x)\big\}$. Alternatively, one could have used here $\check{z}^k$, requiring instead that $\mu_-(x) \geq 3 + \nu_{\scriptscriptstyle alg}(x)$ for all closed orbits including $x = z$. However, when $\alpha$ is degenerate this condition is more restrictive and does not appear to automatically follow from dynamical convexity. Furthermore, note that in the statement of Theorem A and also in this modification of the theorem, we could have replaced $\nu_{\scriptscriptstyle alg}$ by $b_+ - b_-$.

5.2. **Proof of Theorem B: Non-existence of locally maximal orbits.** The proof follows the same line of reasoning as the proof of Theorem A and we will only outline the argument, emphasizing the necessary modifications and skipping some details. We will also keep the notation from that proof.

Thus let $z = x_0$ be a non-degenerate, locally maximal orbit of the Reeb flow and let $x_1, \ldots, x_q$ be the remaining simple closed orbits. By the dynamical convexity condition, $\hat{\mu}(x_i) \geq 2$ for all $i = 0, \ldots, q$. As in the proof of Theorem A, it is easy to find an admissible Hamiltonian $H$ such that the Crossing Energy Theorem (Theorem 4.1) holds for $\tilde{z}$. Furthermore, the Index Recurrence Theorem (Theorem 4.7) applies to the orbits $z = x_0, \ldots, x_q$. As before, $k := k_{0s}$. Our goal is to prove an analogue of Lemma 5.1. Namely, we will show that at least one of the orbits $\check{z}^k$ or $\hat{z}^k$ cannot be connected by a short arrow to any other closed orbit.

Due to the non-degeneracy requirement, every Reeb orbit $x_i^j$ gives rise to two generators $\check{x}_i^j$ and $\hat{x}_i^j$ in $\mathrm{CF}(kH)$ with $|\check{x}_i^j| = \mu(x_i^j)$ and $|\hat{x}_i^j| = \mu(x_i^j) + 1$.

Recall that, by Condition (i) of Theorem 4.7,

$$|d - \hat{\mu}(x_i^{k_{is}})| < \eta$$

and that, by non-degeneracy,

$$|\,\hat{\mu}(x_i^j) - \mu(x_i^j)| < n - 1.$$

for any $j \neq 0$. In particular,

$$|d - \hat{\mu}(z^k)| < \eta$$

and

$$|\mu(z^k) - \hat{\mu}(z^k)| < n - 1.$$

Set

$$\ell_0 := n + 1.$$

Then, similarly to Case 2 of Lemma 5.1, when $\ell := |j - k_{is}| > \ell_0$, the degree difference between $\check{z}^k$ or $\hat{z}^k$ and $\check{x}_i^j$ or $\hat{x}_i^j$ is greater than or equal to 2.

Next, as in Case 3, assume that $0 < \ell = |j - k_{is}| \leq \ell_0$. We claim that then $\check{z}^k$ is not connected to $\check{x}_i^j$ and $\hat{x}_i^j$ when $d < \hat{\mu}(z^k)$ and that $\hat{z}^k$ is not connected to either of these orbits when $d > \hat{\mu}(z^k)$.

To see this, note first that when $j > k_{is}$ the range of $|\check{x}_i^j|$ is $[d + n + 1, \infty)$ and the range of $|\hat{x}_i^j|$ is $[d + n + 2, \infty)$ by Condition (ii) of Theorem 4.7. Likewise, when $j < k_{is}$, the range of $|\check{x}_i^j|$ is $(-\infty, d - n - 1]$ and the range of $|\hat{x}_i^j|$ is $(-\infty, d - n]$ by Condition (iii).

When $d < \hat{\mu}(z^k)$, the range of $|\check{z}^k|$ is $[d - n + 2, d + n - 1]$ and the range of $|\hat{z}^k|$ is $[d - n + 3, d + n]$. Thus the degree difference between $\check{z}^k$ and $\check{x}_i^j$ or $\hat{x}_i^j$ is at least 2.

When $d > \hat{\mu}(z^k)$, the range of $|\check{z}^k|$ is $[d - n + 1, d + n - 2]$ and the range of $|\hat{z}^k|$ is $[d - n + 2, d + n - 1]$. Therefore, in this case, the degree difference between $\hat{z}^k$ and $\check{x}_i^j$ or $\hat{x}_i^j$ is at least 2.

Note that here we allow $i$ to be 0, and hence we have in particular shown that $\check{z}^k$ or $\hat{z}^k$ is not connected by an arrow with $\check{z}^j$ and $\hat{z}^j$ for any $j$. (This is an analogue of Case 0 of Lemma 5.1, which is now incorporated into Cases 2 and 3.)

The remaining case is where $j = k_{is}$ and $i \geq 1$. This case is handled exactly as Case 1 in the proof of Lemma 5.1: the action difference is either less than $\sigma$, which is impossible by Theorem 4.1 applied to $z$, or goes to infinity as $k \to \infty$, which is also impossible by Theorem 4.3. This contradiction completes the proof of Theorem B. □

## 6. Proof of the crossing energy theorem

In this section we prove the Crossing Energy Theorem – Theorem 4.1. The proof comprises two major components. The first one, Theorem 6.1, deals with the location of Floer trajectories. Roughly speaking, the goal of this part is to show that a small energy Floer trajectory cannot enter the region where $r$ is close to 1 with a lower bound on $r$ independent of the order of iteration. This part is completely new. The second component gives then an energy lower bound, also independent of the order of iteration. The proof of the second part is quite similar to the argument in [ÇGG21, GG14, GG18].

### 6.1. Location constraints.

The key new ingredient in the proof of the Crossing Energy Theorem is the following result which, under a minor additional condition on $H$, limits the range of $r \circ u$.

Before stating the result we need to set some new conventions. Throughout the proof, it is convenient to adopt a different Hamiltonian iteration procedure. Namely, passing to the $k$th iterate of $H$ (or to be more precise of $\varphi_H$), rather than looking at 1-periodic orbits of the Hamiltonian $kH$, we will look at the $k$-periodic orbits of $H$, changing the time range from $S^1 = \mathbb{R}/\mathbb{Z}$ to $S^1_k = \mathbb{R}/k\mathbb{Z}$. We will refer to the resulting Hamiltonian as $H^{\sharp k}$; cf. [GG14, GG18]. A time-dependent almost complex structure is then also assumed to be $k$-periodic in time rather than 1-periodic. This modification does not effect the Floer complexes, the action and the action filtration, the energy of a Floer trajectory, etc., with an isomorphism given by an $(s,t)$-reparametrization.

Furthermore, whenever $H|_W = const$ a part of a solution $u$ of the Floer equation is a (anti-)holomorphic curve. Hence, the standard monotonicity (see, e.g., [Si]) implies that solutions with small energy can only enter a narrow collar of $M = \partial W$ in $W$. Thus extending $r$ to the collar as a function to $(1-\eta, 1] \subset (0, \infty)$, for some small $\eta > 0$, as in Section 2.1 we can assume without loss of generality that the composition $r \circ u \colon \mathbb{R} \times S^1_k \to (0, \infty)$ is always defined. By a suitable variant of the Gromov compactness theorem (see, e.g., [Fi]) the same is true for a $C^\infty$-small perturbation of $H^{\sharp k}$.

It is convenient throughout the proof to think of $r$ as taking values in $(0, \infty)$ rather than in $[1, \infty)$ or $[1-\eta, \infty)$. Likewise, we extend $h$ to a smooth function on $(0, \infty)$ by setting $h|_{(0,1]} \equiv 0$.

In what follows, $H$ is admissible or semi-admissible and $J$ is admissible. In particular, $H$ and $J$ are independent of time. As above, we will assume that all Floer cylinders $u$ we consider have sufficiently small energy and hence are contained $M \times (1-\eta, \infty)$ for some small $\eta > 0$.

**Theorem 6.1.** *Let $H(r,x) = h(r)$ be a semi-admissible Hamiltonian. Assume furthermore that $1 < r_*^- \leq r_*^+$ and $\delta > 0$ are such that $1 < r_*^- - \delta$ and $r_*^+ + \delta \leq r_{\max}$, and*

$$h''' \geq 0 \text{ on } [1, r_*^+ + \delta] \subset [1, r_{\max}). \tag{6.1}$$

*Fix an admissible almost complex structure $J$. Then there exists $\sigma > 0$ such that for any $k \in \mathbb{N}$ and any Floer cylinder $u \colon \mathbb{R} \times S^1_k \to \widehat{W}$ of $H^{\sharp k}$ with energy $E(u) \leq \sigma$*

*and asymptotic, at either end, to a periodic orbit in* $[r_*^-, r_*^+] \times M$*, the image of* $u$ *is contained in* $(r_*^- - \delta, r_*^+ + \delta) \times M$*.*

*Remark* 6.2. By the target compactness theorem from [Fi], the assertion of the theorem still holds with perhaps a smaller value of $\sigma$ when $H^{\sharp k}$ and $J$ are replaced by their compactly supported, $k$-periodic in time $C^\infty$-small perturbations. (The size of the perturbation may depend on $k$.) Likewise, one can use perturbations of the type specified in Section 3.3.2.

The proof of Theorem 6.1 is an easy application of the following technical result. Recall from Section 2.1 that the action function $A_h$ is defined as $A_h(r) = rh'(r) - h(r)$, where having extended $h$ to $(0, \infty)$ as 0 on $(0, 1]$ we also have $A_h \equiv 0$ on $(0, 1]$.

**Theorem 6.3.** *Let* $H(r, x) = h(r)$ *be a semi-admissible Hamiltonian and* $J$ *be an admissible almost complex structure. There exist* $\epsilon > 0$ *and* $C > 0$ *such that for any* $k \in \mathbb{N}$ *and any Floer cylinder* $u\colon \mathbb{R} \times S^1_k \to \widehat{W}$ *of* $H^{\sharp k}$ *with* $E(u) \leq \epsilon$ *and contained in the region where* $h'''(r) \geq 0$*, we have*

$$\inf_{\mathbb{R}\times S^1_k} r\big(u(s,t)\big) \geq r^- - \frac{C \cdot (r^+)^{3/4}}{\sqrt{A_h(r^+)}} E(u)^{5/8}, \tag{6.2}$$

*where* $r^\pm = r\big(u(\mp\infty, t)\big)$ *and* $A_h(r^+) = r^+h'(r^+) - h(r^+)$*.*

*Remark* 6.4. Let us specify how $\epsilon > 0$ and $C > 0$ in Theorem 6.3 are chosen. By [Sa99, Sec. 1.5], there exist $\epsilon' > 0$ and $C' > 0$, depending only on $H$ and $J$, such that for any Floer cylinder $u\colon \mathbb{R} \times S^1_k \to \widehat{W}$, the point-wise upper bound

$$\big\|\partial_s u(s,t)\big\| < C' E(u)^{1/4} \tag{6.3}$$

holds whenever $E(u) < \epsilon'$. (This is an instance when it is more convenient to work with $H^{\sharp k}$ than $kH$; for the proof of (6.3) is local in the domain of $u$, but the constants $\epsilon'$ and $C'$ depend on $H$ and its first and second derivatives.) In the proof of the Theorem 6.3 we take $\epsilon = \epsilon'$ and $C = \sqrt{4C'}$.

Let us now prove Theorem 6.1 assuming Theorem 6.3.

*Proof of Theorem 6.1.* Let $\epsilon > 0$ be as in Theorem 6.3. Choose $0 < \sigma' < \epsilon$ such that $r^+ - r^- < \delta/2$ for any $k \in \mathbb{N}$ and any Floer cylinder $u$ of $H^{\sharp k}$ with energy $E(u) < \sigma'$, which is asymptotic at at least one of the ends to a $k$-periodic orbit in $[r_*^-, r_*^+] \times M$. Such a constant $\sigma'$ exists since $A_h' = rh'' > 0$ on $(r_*^- - \delta, r_*^+ + \delta)$. Note that we can apply Theorem 6.3 to any $u$ as above with $E(u) < \sigma'$. Next we choose $0 < \sigma < \sigma'$ such that

$$\sigma^{5/8} < \frac{\delta\sqrt{A_h(r_*^-)}}{2C \cdot (r_*^+)^{3/4}}.$$

It follows from (6.2) and the maximum principle, (3.5), that the image of $u$ is contained in $(r_*^- - \delta, r_*^+ + \delta) \times M$ whenever $E(u) < \sigma$. This completes the proof Theorem 6.1 with $\sigma$ chosen as above. ☐

*Remark* 6.5. Note that in this argument at least one of the points $r^+$ and $r^-$ is in the interval $[r_*^-, r_*^+]$ but *a priori* we do not know which one. When this is $r^+$, a small energy trajectory $u$ is contained in $(r_*^- - \delta, r_*^+] \times M$. However, this is not necessarily the case when $r^+ \notin [r_*^-, r_*^+]$, e.g., if $r^- = r_*^- = r_*^+ < r^+$.

6.2. **Proof of Theorem 6.3: The $r$-range.** We carry out the proof in four steps. First, we show that the average in $t \in S^1_k$ of $r\big(u(s,t)\big)$ satisfies the differential inequality (6.4). In the second step, we establish lower and upper bounds on this average. This is where the third derivative condition on $h$ becomes essential. The goal of the third step is to prove inequality (6.9), which is central to the proof and gives an upper bound on the amount of time $u(s,t)$ spends below a certain fixed $r$-level. Finally, in the last step, we finish the proof of the theorem by combining the key inequality, (6.9), with the pointwise upper bound (6.3) on $\|\partial_s u\|$.

6.2.1. *Step 1.* Fix a semi-admissible Hamiltonian $H(x,r) = h(r)$, an iteration $k \in \mathbb{N}$ and a Floer cylinder $u\colon \mathbb{R} \times S^1_k \to \widehat{W}$ of $H^{\sharp k}$ as in Theorem 6.3. Let us denote by $u_s := u(s,\cdot)\colon S^1_k \to \widehat{W}$ the $s$-slice of the cylinder $u$. In this step we show that

$$\frac{d}{ds}\int_0^k r(u_s)\,dt \leq -kA_h(r^-) + \int_0^k A_h\big(r(u_s)\big)\,dt. \tag{6.4}$$

Using (3.1) and the Floer equation, (3.3), we have

$$\begin{aligned}
\partial_s(r(u)) &= dr(\partial_s u)\\
&= dr\big(J(\partial_t u - X_H(u))\big)\\
&= -r(u)\alpha(\partial_t u) + r(u)\alpha(X_H(u))\\
&= -r(u)\alpha(\partial_t u) + r(u)h'(r(u))\\
&= -\big[r(u)\alpha(\partial_t u) - h(r(u))\big] + \big[r(u)h'(r(u)) - h(r(u))\big].
\end{aligned}$$

Integrating over $S^1_k$ gives

$$\frac{d}{ds}\int_0^k r(u_s)dt = -\mathcal{A}_{H^{\sharp k}}(u_s) + \int_0^k A_h\big(r(u_s)\big)\,dt. \tag{6.5}$$

Recall that by (3.2) the action is decreasing along Floer cylinders. Hence we have

$$\mathcal{A}_{H^{\sharp k}}(u_s) \geq \mathcal{A}_{H^{\sharp k}}(u_\infty) = kA_h(r^-). \tag{6.6}$$

Then the desired inequality, (6.4), follows from (6.5) and (6.6).

6.2.2. *Step 2.* From now on we assume that the image of $u$ is contained in a region where $h'''(r) \geq 0$. In this step we show that

$$\frac{A_h(r^-)}{A_h(r^+)}kr^+ \leq \inf_{s\in\mathbb{R}}\int_0^k r(u_s)\,dt \leq kr^+. \tag{6.7}$$

The upper bound for the middle term follows from the maximum principle, (3.5). Let us prove the lower bound.

*Remark* 6.6. Note that since $E(u) = k(A_h(r^+) - A_h(r^-))$, the lower bound in (6.7) can also be written as $kr^+ - E(u)r^+/A_h(r^+)$.

We will need the following claim, where in the setting of the theorem we can take $r_0 = r^+_* + \delta$.

*Claim 1:* The function $A_h(r)/r$ is non-decreasing on $[1, r_0]$, provided that $h''' \geq 0$ on this interval.

Assuming the claim, let us prove the first inequality in (6.7), i.e.,

$$\frac{A_h(r^-)}{A_h(r^+)}kr^+ \leq \int_0^k r(u_s)\,dt \tag{6.8}$$

for all $s \in (-\infty, \infty]$. First, note that by the claim

$$\frac{A_h(r^-)}{A_h(r^+)} kr^+ \leq kr^-.$$

Hence (6.8) holds at $s = \infty$. It remains to verify (6.8) at the critical points of the function

$$s \mapsto \int_0^k r(u_s)\, dt.$$

Thus let $s_0$ be a critical point. Using (6.4), the maximum principle (3.5) and the claim above, we have

$$kA_h(r^-) \leq \int_0^k A_h\big(r(u_{s_0})\big)\, dt \leq \frac{A_h(r^+)}{r^+} \int_0^k r(u_{s_0})\, dt.$$

It follows that (6.8) holds at the critical points as well. This completes the proof of (6.7), assuming the claim.

*Proof of Claim 1.* Recall that $A_h(r) = rh' - h$. Dividing by $r$ and differentiating, we have

$$\frac{d}{dr}\big(A_h(r)/r\big) = \frac{rA_h' - A_h}{r^2} = \frac{r^2h'' - rh' + h}{r^2}.$$

Thus $d\big(A_h(r)/r\big)/dr \geq 0$ whenever $f = r^2h'' - rh' + h \geq 0$. Since $h''' \geq 0$, the function $h''$ is monotone increasing. This together with the fact that $h'(1) = 0$ implies that

$$h'(r) \leq (r-1)h''(r).$$

Indeed,

$$h'(r) = \int_1^r h''(\xi)\, d\xi \leq \int_1^r h''(r)\, d\xi = (r-1)h''(r).$$

Hence,

$$f = r(rh'' - h') + h \geq r\big(rh'' - (r-1)h''\big) + h = rh'' + h \geq 0.$$

This argument shows that in fact $A_h(r)/r$ is strictly increasing on $[1, r_0]$ unless $h \equiv 0$, which in the setting of the theorem is ruled out by that $h'' > 0$ on $(1, r_{\max})$. □

6.2.3. *Step 3.* Let $\rho > 0$ and $\mu(s, \rho) \in [0, k]$ be the total time that the slice $u_s$ spends under the level $r = r^+ - \rho$. In other words,

$$\mu(s, \rho) := \operatorname{Leb}\big\{t \mid r(u(s,t)) \leq r^+ - \rho\big\}.$$

The purpose of this step is to prove the inequality

$$\mu(s, \rho)\rho \leq \frac{r^+}{A_h(r^+)} E(u). \tag{6.9}$$

By the maximum principle, (3.5), we have

$$\int_0^k r(u_s)\, dt \leq \big(k - \mu(s,\rho)\big)r^+ + \mu(s,\rho)(r^+ - \rho) = kr^+ - \mu(s,\rho)\rho. \tag{6.10}$$

Next, combining (6.7), Remark 6.6 and (6.10), we obtain

$$kr^+ - \frac{r^+}{A_h(r^+)} E(u) \leq kr^+ - \mu(s,\rho)\rho,$$

from which (6.9) immediately follows.

6.2.4. *Step 4.* In this step we finish the proof of Theorem 6.3. Let $\epsilon > 0$ and $C' > 0$ be as in Remark 6.4, and assume that $E(u) < \epsilon$. If $r^- \leq \inf_{\mathbb{R}\times S^1_k} r(u)$, there is nothing to show. Hence we can assume without loss of generality that $r^- > \inf r(u)$. Fix $\eta \in \big(0,\, r^- - \inf r(u)\big)$.

Next, we choose $(s_0, t_0) \in \mathbb{R} \times S^1_k$ with $r\big(u(s_0,t_0)\big) < r^- - \eta$. Notice that by the Bourgeois–Oancea monotonicity property, (3.6), the slice $u_{s_0}$ has to rise at least to the level $r = r^-$. We will need the following claim.

*Claim 2:* For every $t_1 \in S^1_k$ with $r\big(u(s_0,t_1)\big) \geq r^- - \eta/2$, we have

$$\eta/2 < C'\sqrt{r^+}E(u)^{1/4}|t_1 - t_0|.$$

First, let us prove Theorem 6.3 assuming the claim. Set $\rho = (r^+ - r^-) + \eta/2$. From the claim and Bourgeois–Oancea monotonicity, (3.6), we obtain

$$\eta/2 \leq C'\sqrt{r^+}E(u)^{1/4}\mu(s_0,\rho). \tag{6.11}$$

Combining (6.11) with (6.9), we see that

$$\eta^2/4 \leq (r^+ - r^-)\eta/2 + \eta^2/4 \leq \frac{C' \cdot (r^+)^{3/2}}{A_h(r^+)}E(u)^{5/4}.$$

Next, we take the square root and get

$$\eta \leq \frac{\sqrt{4C'} \cdot (r^+)^{3/4}}{\sqrt{A_h(r^+)}}E(u)^{5/8}. \tag{6.12}$$

Since (6.12) holds for all $\eta \in (0,\, r^- - \inf r(u))$, it holds in particular for $\eta = r^- - \inf r(u)$. Setting $C = \sqrt{4C'}$ completes the proof of the theorem.

*Proof of Claim 2.* We have

$$\eta/2 < r\big(u(s_0,t_1)\big) - r\big(u(s_0,t_0)\big) = \int_{t_0}^{t_1} dr\big(\partial_t u(s_0,t)\big)\,dt. \tag{6.13}$$

Let us rewrite the integral using the identity

$$dr(\partial_t u) = r(u)\alpha(\partial_s u) + r(u)\alpha\big(JX_H(u)\big) = r(u)\alpha(\partial_s u),$$

which follows from (3.1), the Floer equation (3.3) and the condition that $\alpha(JX_H) = 0$. Then (6.13) turns into

$$\begin{aligned}
\eta/2 &< \int_{t_0}^{t_1} r\big(u(s_0,t)\big)\alpha\big(\partial_s u(s_0,t)\big)\,dt \\
&\leq \int_{t_0}^{t_1} r\big(u(s_0,t)\big)\frac{\|\partial_s u(s_0,t)\|}{\big\|R_\alpha\big(u(s_0,t)\big)\big\|}\,dt \\
&= \int_{t_0}^{t_1} \sqrt{r\big(u(s_0,t)\big)}\,\|\partial_s u(s_0,t)\|\,dt \\
&\leq \sqrt{r^+}\max_{t_0\leq t\leq t_1}\|\partial_s u(s_0,t)\| \cdot |t_1 - t_0| \\
&\leq C\sqrt{r^+}E(u)^{1/4}|t_1 - t_0|,
\end{aligned}$$

where $R_\alpha$ is the Reeb vector field of $\alpha$. In the third line we have used the identity

$$\|R_\alpha\|^2 = \omega(R_\alpha, JR_\alpha) = r.$$

The fourth line follows from the maximum principle, (3.5), and the fifth line is a consequence of Remark 6.4. This concludes the proof of the claim and the theorem. □

6.2.5. *Proof of the Bourgeois–Oancea monotonicity,* (3.6). Here, for the sake of completeness, we include the proof of (3.6) closely following the argument from [CO, Lemma 2.3]. Arguing by contradiction, assume that there exists $s_0 \in \mathbb{R}$ such that

$$\max_{t \in S^1_k} r\big(u(s_0, t)\big) < r^-.$$

It follows from the maximum principle, (3.5), that

$$\max_{t \in S^1_k} r\big(u(s, t)\big) \leq r^-$$

for all $s \in [s_0, \infty)$. Using (6.4) and the condition that $A'_h(r) \geq 0$ we see that

$$\frac{d}{ds} \int_0^k r(u_s)\, dt \leq -kA_h(r^-) + \int_0^k A_h\big(r(u_s)\big)\, dt \leq 0$$

for all $s \in [s_0,\, \infty)$. On the other hand, by the assumption, we have

$$\int_0^k r\big(u(s, t)\big)\, dt \leq kr^- - \epsilon.$$

for $s = s_0$ and some $\epsilon > 0$. Then the same is true for all $s \in [s_0,\, \infty)$. This is a contradiction with the fact that the left-hand side converges to $kr^-$, completing the proof of (3.6). □

6.3. **Energy bounds.** There are three sufficiently different approaches to the proof of crossing energy type results. The first one is based on target Gromov compactness; see [Fi]. This approach is used in the original proof in [GG14] and then in [GG18, ÇGG21] and more recently and in a more sophisticated form in [Pr]. The second approach uses the upper bound (6.3) from, e.g., [Sa90, Sa99], quoted in Remark 6.4 and is also closely related on the conceptual level to (the proof of) Gromov compactness. This method is pointed out in [ÇGG21, Rmk. 6.4]. The third approach utilized in [GGMz] is based on finite-dimensional approximations in Morse theory. This approach does not fit well with general Floer theory techniques, but either of the first two methods can be used to prove Theorem 4.1. Here we have chosen the second one, which is more hands-on and direct, albeit somewhat less general.

*Proof of Theorem 4.1.* Let $\tilde{z} = (z, r_*)$ and $H$ be as in the statement of the theorem. Let also $u$ be a Floer cylinder of $H^{\natural k}$ asymptotic to $\tilde{z}^k$ at either end. Our goal is to show that the energy $E(u)$ cannot be arbitrarily small. Clearly, to this end, we can require $E(u)$ to be sufficiently small which we will do several times throughout the process. Furthermore, we can assume that $k$ is large enough, since for a fixed $k \in \mathbb{N}$ the desired lower bound directly follows from Gromov compactness; see Section 4.1.

Setting $r_*^- = r_* = r^+$ and taking a sufficiently small $\delta > 0$, we see that by (4.1) the condition (6.1) of Theorem 6.1 is satisfied. As a consequence, the image of $u$ is contained in $(r_* - \delta,\, r_* + \delta) \times M$ when $E(u)$ is sufficiently small. From now on we will assume that this is the case.

Next, (6.3) from Remark 6.4 holds since $E(u)$ is assumed to be small. Therefore,

$$\big\|\partial_s u(s, t)\big\| < C' E(u)^{1/4} =: \epsilon,$$

for some constant $C'$. (This $\epsilon$ is unrelated to $\epsilon$ and $\epsilon'$ in Remark 6.4.) By the Floer equation, (3.3),

$$\|\partial_t u - X_H\| < \epsilon,$$

for all $s \in \mathbb{R}$, where $X_H = h' R_\alpha$. (This is another point where it is more convenient to work with $H^{\sharp k}$ than with $kH$; for $X_{H^{\sharp k}} = X_H$ is independent of $k$.) Denote by $v$ the projection of $u$ to $M$. Then we also have

$$\big\|\partial_t v - h'(r(u))R_\alpha\big\| < \epsilon. \tag{6.14}$$

Fix an isolating neighborhood $V$ of $z$. Without loss of generality, we can assume that $V$ is a closed tubular neighborhood of $z(\mathbb{R})$ in $M$. We note that $v$ cannot be entirely contained in $V$ or, equivalently, $u$ cannot be contained in $U := (r_* - \delta,\, r_* + \delta) \times V$. Indeed, if this were the case, $u$ would be asymptotic to some orbit $\tilde{z}^m$ of $H^{\sharp k}$ with $m \neq k$ at the other end. This however is impossible; for the free homotopy classes of these two orbits are different in $U$ since the free homotopy classes of $z^k$ and $z^m$ are different in $V$. Hence the image of $v$ intersects $\partial V$.

Furthermore, it is not hard to see that, since $V$ is isolating, there exists $\tau_0 > 0$ and $\eta > 0$ such that

$$\max_{\tau \in [-\tau_0,\, \tau_0]} d\big(V, \varphi_\alpha^\tau(x)\big) > \eta \tag{6.15}$$

for all $x \in \partial V$. Here $d$ is some fixed distance on $M$ and $\varphi_\alpha$ is the Reeb flow of $\alpha$.

Since $u$ is asymptotic to $\tilde{z}^k$, for some $s \in \mathbb{R}$ the curve $v(s, S^1_k)$ is contained in $V$ while $v(s,t) \in \partial V$. Without loss of generality, we may assume that $t = 0$. Thus, $v(s,t) \in V$ for all $t \in S^1_k$ and $v(s,0) \in \partial V$. Let us reparametrize the map $t \mapsto v(s,t)$ by using the change of variables $t = t(\tau)$ so that

$$t'(\tau) = h'\big(r(u(s,t))\big),$$

and set

$$\gamma(\tau) = v(s, t(\tau)).$$

The curve $\gamma$ is parametrized by the circle $S^1_{\tau_k}$ with

$$\frac{k}{h'(r_* + \delta)} \leq \frac{k}{\max_{t \in S^1_k} h'\big(r(u(s,t))\big)} \leq \tau_k \leq \frac{k}{\min_{t \in S^1_k} h'\big(r(u(s,t))\big)} \leq \frac{k}{h'(r_* - \delta)}.$$

Here the lower bound follows from the maximum principle and the upper bound, which we do not need, from Theorem 6.1. We will take $k$ large enough so that $\tau_k \geq 2\tau_0$. The image of $\gamma$ is contained in $V$ and $x := \gamma(0) \in \partial V$.

We infer from (6.14) and Theorem 6.1 that

$$\big\|\dot{\gamma}(\tau) - R_\alpha\big(\gamma(\tau)\big)\big\| < \frac{\epsilon}{\min_{t \in S^1_k} h'\big(r(u(s,t))\big)} \leq \frac{\epsilon}{h'(r_* - \delta)},$$

where the dot stands for the derivative with respect to $\tau$. This is the point where Theorem 6.1 becomes crucial. By the standard Gronwall inequality type argument (see, e.g., [Br15b, GG18]) and the above upper bound,

$$d\big(\gamma(t), \varphi^\tau(x)\big) \leq \frac{e^{C|\tau|}\epsilon}{h'(r_* - \delta)}, \tag{6.16}$$

for some constant $C$. (In particular, $\gamma$ is a pseudo-orbit of the Reeb flow, but we will not directly use this fact.) Combining (6.15) and (6.16), we obtain

$$\eta \leq \frac{e^{C|\tau|}\epsilon}{h'(r_* - \delta)}$$

for some $\tau \in [-\tau_0, \tau_0]$. As a consequence,

$$\eta \leq \frac{e^{C\tau_0}\epsilon}{h'(r_* - \delta)},$$

which gives a lower bound on $\epsilon$ and hence on $E(u)$, independent of $k$. More explicitly, we have

$$E(u) \geq \left[\eta h'(r_* - \delta)e^{-C\tau_0}/C'\right]^4 > 0.$$

This concludes the proof of Theorem 4.1. □

## 7. Appendix: The local Le Calvez–Yoccoz theorem in dimension two

For the sake of completeness, here we include a topological proof of the local version of the Le Calvez–Yoccoz theorem, [LCY], closely following [Fr99, Prop. 3.1] and [FM].

To state the result, let us recall a few classical notions from dynamics. The index $i(\varphi, z)$ of an isolated fixed point $z$ of a smooth map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined as the Brower degree of the associated map

$$\partial B^n(z, \epsilon) \to \partial B^n(0, 1), \qquad y \mapsto \frac{\varphi(y) - y}{\|\varphi(y) - y\|},$$

for any $\epsilon > 0$ small enough. Here $B^n(z, \epsilon)$ is the ball of radius $\epsilon$ centered at $z$. The definition extends to isolated fixed points of smooth maps of manifolds by employing arbitrary local coordinates. If $\varphi\colon M \to M$ is a diffeomorphism, we denote by $\mathrm{inv}(\varphi, U)$ the largest $\varphi$-invariant subset contained in $U \subset M$, i.e.

$$\mathrm{inv}(\varphi, U) := \bigcap_{k\in\mathbb{Z}} \varphi^k(U).$$

A compact $\varphi$-invariant subset $\Lambda$ is called *locally maximal* when $\mathrm{inv}(\varphi, U) = \Lambda$ for some neighborhood $U \subset M$ of $\Lambda$; cf. Section 4.1. A fixed point $z$ of $\varphi$ is called *irrationally elliptic* when the eigenvalues of the linear map $d\varphi(z)$ lie on the unit circle of the complex plane and are not roots of 1. This is equivalent to that $\varphi$ is elliptic and all iterates $\varphi^k$ or $\varphi$ are non-degenerate at $z$.

**Theorem 7.1** ([Fr99, FM])**.** *An irrationally elliptic fixed point of a surface symplectomorphism is never a locally maximal invariant set.*

Applying this theorem to the Poincaré return map of a closed irrationally elliptic orbit in dimension three, we obtain the following result.

**Corollary 7.2.** *A closed irrationally elliptic orbit of a volume-preserving flow on a 3-manifold is never a locally maximal set.*

*Proof of Theorem 7.1.* Let $z$ be an irrationally elliptic fixed point of a surface symplectomorphism $\varphi$. In particular, $d\varphi^m(z)$ has eigenvalues in $S^1 \setminus \{1\}$ for all $m \geq 1$, and $z$ is an isolated fixed point of $\varphi^m$. Its indices are given by

$$i(\varphi^m, z) = 1, \qquad \forall m \geq 1, \tag{7.1}$$

see, e.g., [KH, p. 320]. Let us assume by contradiction that $z$ is a locally maximal $\varphi$-invariant subset. Under this assumption, a theorem due to Easton [Ea] implies that $z$ possesses so-called isolating blocks in the sense of Conley theory. More precisely, following Franks and Misiurewicz [Fr99, Sec. 3], there exist an arbitrarily small connected compact neighborhood $V \subset M$ of $z$ and an arbitrarily small compact neighborhood $W \subset V \setminus \{z\}$ of the exit set

$$V^- := \{y \in V \mid \varphi(y) \notin \textit{interior}(V)\}$$

such that $\mathrm{inv}(\varphi, \overline{V \setminus W}) = \{z\}$, $\varphi(W) \cap \overline{V \setminus W} = \varnothing$, and $(V, W)$ is homeomorphic to a finite simplicial pair. Since $\varphi$ is area-preserving, we must have $V^- \neq \varnothing$, and thus

$$W \neq \varnothing.$$

If $\partial V \subset V^-$, since $\phi$ is area-preserving, we must have $\varphi(V) = V$, but this would contradict the local maximality of $z$. Therefore, $\partial V \not\subset V^-$ and we can choose $W$ small enough so that

$$\partial V \not\subset W. \tag{7.2}$$

We consider the induced quotient map

$$\widetilde{\varphi} : V/W \to V/W.$$

Notice that this map has precisely two fixed points: $z$ and $[W]$. Since $(V, W)$ is a finite simplicial pair, we can apply Lefschetz fixed-point theorem to the iterates of $\widetilde{\varphi}$ and obtain

$$\mathrm{tr}(\widetilde{\varphi}_0^m) - \mathrm{tr}(\widetilde{\varphi}_1^m) + \mathrm{tr}(\widetilde{\varphi}_2^m) = i(\varphi^m, z) + i(\widetilde{\varphi}^m, [W]), \tag{7.3}$$

where $\widetilde{\varphi}_k : \mathrm{H}_k(V/W; \mathbb{Q}) \to \mathrm{H}_k(V/W; \mathbb{Q})$ is the homomorphism induced by $\widetilde{\varphi}$. Since $V/W$ is connected, we have

$$\mathrm{tr}(\widetilde{\varphi}_0) = 1.$$

By (7.2), $\mathrm{H}_2(V/W; \mathbb{Q}) = 0$. Therefore,

$$\mathrm{tr}(\widetilde{\varphi}_2) = 0.$$

For any sufficiently small neighborhood $U \subset V/W$ of $[W]$, we have $\widetilde{\varphi}(U) = [W]$, and hence the index of the fixed point $[W]$ is given by

$$i(\widetilde{\varphi}^m, [W]) = 1.$$

Putting these equations together and using (7.1), we see that equation (7.3) becomes

$$\mathrm{tr}(\widetilde{\varphi}_1^m) = -1.$$

This is impossible. Indeed, as is not hard to show, for every linear endomorphism $L$ of a finite-dimensional vector space, $\mathrm{tr}(L^m) \geq 0$ for infinitely many integers $m \geq 1$. □

*Remark* 7.3. The requirement that $z$ is irrationally elliptic in Theorem 7.1 is essential. It is easy to construct an area-preserving diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with a non-degenerate rationally elliptic fixed point $z$ such that $\varphi^k$ has a monkey saddle at $z^k$, for some $k \in \mathbb{N}$, and hence $z$ is locally maximal.

*Remark* 7.4. It is worth pointing out that the proof of Theorem 7.1 is purely two-dimensional. The dimension assumption is crucially used in the fact that the homology $\mathrm{H}_*(V/W; \mathbb{Q})$ is concentrated in only two degrees. As we have noted in the introduction it is not known if Theorem 7.1 holds in higher dimensions. In other words, there may exist a Hamiltonian diffeomorphism $\varphi$ with an elliptic fixed point $z$ which is non-degenerate for all iterates $\varphi^m$ and locally maximal. This seems particularly likely when the linearization $D\varphi$ at $z$ is not required to be semi-simple.

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