On marked boundary rigidity for surfaces

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(joint work with Colin Guillarmou)

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  $\ell_g : \partial_{\mathrm{in}} SM \to [0, \infty]$
  $\ell_g(x_0, v_0) =$ length of the geodesic γ_{v_0}

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  (i.e. we identify $g \sim \phi^* g$, for all diffeomorphisms $\phi : M \to M$, $\phi|_{\partial M} = \mathrm{id}$)

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Boundary rigidity

In general, there is no rigidity

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Simple Riemannian manifolds are balls (B^n, g) such that, at each point $x \in B^n$, the Riemannian exponential map $\exp_x : K_x \to B^n$ is a diffeomorphism.

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- Pestov-Uhlmann, 2004: True if $\dim(M) = 2$
- Open if $\dim(M) > 2$, but several partial results (Stefanov-Uhlmann, Burago-Ivanov, Pestov-Sharafutdinov, etc.)

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  S_g determines M and the conformal class $\left\{e^{\omega} g \;\middle|\; \omega|_{\partial M} \equiv 0\right\}$

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- Several recent results in higher dimension (Pestov-Sharafutdinov, Stefanov-Uhlmann, Stefanov-Uhlmann-Vasy, etc.)

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α curve in M joining x and y

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$\pi : \widetilde{M} \to M$ universal cover of M
$\widetilde{g} = \pi^* g$
md_g is equivalent to $d_{\widetilde{g}} : \partial\widetilde{M} \times \partial\widetilde{M} \to [0, \infty)$

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Assume one of the following:

(i) g_1 has $K_{g_1} < 0$ and makes ∂M convex,
g_2 has no conjugate points, trapped set of zero Liouville measure, and makes ∂M convex;

(ii) g_1 makes ∂M convex, has no conjugate points, and hyperbolic trapped set;
$\|g_2 - g_1\|_{C^2}$ is small enough.

Then $g_2 = \phi^* g_1$ for some diffeomorphism $\phi : M \to M$, $\phi|_{\partial M} = \mathrm{id}$

Thank you for your attention!