On the boundary rigidity problem for surfaces

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(joint work with Colin Guillarmou and Leo Tzou)

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  $\sigma_g : U \subseteq \partial_{\mathrm{in}} SM \to \partial_{\mathrm{out}} SM$
  $\sigma_g(x, v) = \phi_{\tau_g(x,v)}(x, v)$

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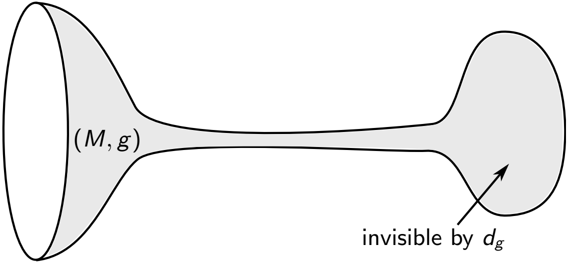

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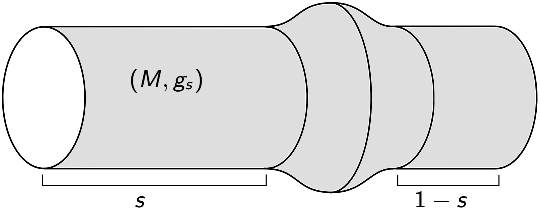

Lens data of (M, g_s) independent of $s \in [0, 1]$

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- Croke-Otal, 1990: True if $\dim(M) = 2$ and g has negative curvature.
- Pestov-Uhlmann, 2004: True if $\dim(M) = 2$.
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Remark. On simple manifolds (B^n, g), the scattering map σ_g and the boundary distance d_g are equivalent.

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- Burago-Ivanov, 2010: Boundary rigidity holds for nearly flat subdomains of $\mathbb{R}^n$.

Rigidity on non-simple manifolds

Theorem (Guillarmou, M., Tzou, 2017)

Let (M_i, g_i), $i = 1, 2$, *compact oriented surfaces with no conjugate points, no trapped set, isometric boundaries, and same scattering map* $\sigma_{g_1} = \sigma_{g_2}$. *Then* $\exists\ \phi : M_1 \to M_2$ *and* $\rho \in C^\infty(M_1)$ *such that* $\rho|_{\partial M_1} \equiv 0$ *and* $\phi^* g_2 = e^\rho g_1$.

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- Calderon's problem (Lassas-Uhlmann, Belishev 2003): $\mathcal{H}_g$ determines M and the conformal class of g □

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Corollary

Let g_1, g_2 be Riemannian metrics on B^2 with no conjugate points and $d_{g_1} = d_{g_2}$. Then $\exists\ \phi \in \mathrm{Diff}(M)$ such that $\phi|_{\partial M} = \mathrm{id}$ and $\phi^* g_2 = g_1$.

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- Croke's trick: (σ_g, τ_g) determine g within its conformal class

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- (M, g) as in the statement. Our theorem implies that σ_g determines M and the conformal class of g.

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- This can be used to show that $(\sigma_{g'}, \tau_{g'})$ determine g', and thus g.

Thank you for your attention!