Spectral characterizations of Besse and Zoll Reeb flows

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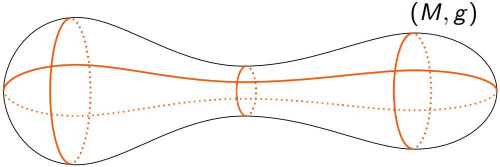
Joint work with:

- Stefan Suhr
- Daniel Cristofaro-Gardiner
- Viktor Ginzburg, Basak Gurel

The closed geodesics conjectures



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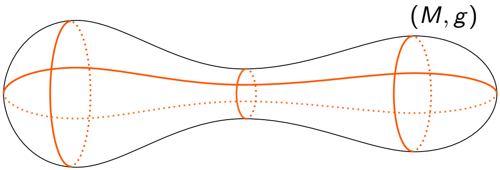

- *Every closed Riemannian manifold* (M, g) *of* $\dim(M) \geq 2$ *has infinitely many closed geodesics.*
- *Every closed Finsler manifold* (M, F) *has at least* $\dim(M)$ *many closed geodesics.*

Widely open for $M = S^n$ (except S^2)

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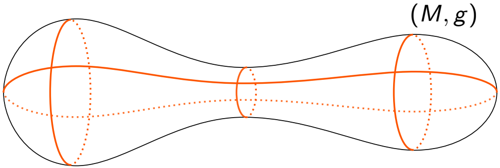

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Subconjecture: *Every closed* (M, g) *or* (M, F) *with* $\dim(M) > 2$ *has at least two closed geodesics.*

Open for $M = S^n$ (except $1 \leq n \leq 4$).

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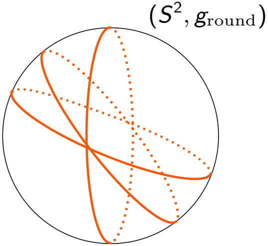
Example:



$$\sigma_{\mathrm{p}}(S^2, g_{\mathrm{round}}) = \{2\pi\}.$$

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Theorem (Mazzucchelli, Suhr, 2017; claimed by Lusternik, 1960s)
The conjecture is true for (S^2, g).

Indeed, slightly more is true: if every simply closed geodesic of (S^2, g) has length ℓ, then every geodesic of (S^2, g) is simply closed and has length ℓ.

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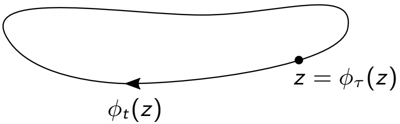

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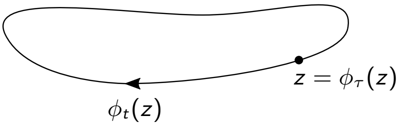

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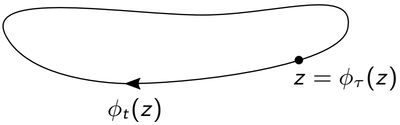

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Example: $Y = S^*M$ unit cotangent bundle of (M, F) or (M, g),
λ Liouville form,
ϕ_t geodesic flow

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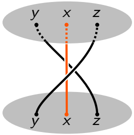

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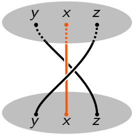
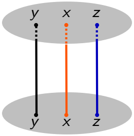


- (Y,λ) is Zoll when every Reeb orbit is periodic with the same minimal period τ,

  i.e. $\quad \phi_\tau = \mathrm{id}, \quad \mathrm{fix}(\phi_t) = \varnothing \;\; \forall t \in (0,\tau).$



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Example: ellipsoid

$$Y = E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \tfrac{|z_1|^2}{a} + \tfrac{|z_1|^2}{b} = \tfrac{1}{\pi} \right\} \quad a, b > 0$$

$$\lambda = \frac{i}{4} \sum_{j=1,2} \left(z_j \, d\overline{z}_j - \overline{z}_j \, dz_j \right)$$

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- If $b/a \in \mathbb{Q}$ then (Y, λ) is Besse
- If $a = b$ then (Y, λ) is Zoll

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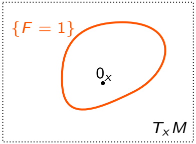
Riemannian and Finsler surfaces

(M^2, F) closed Finsler surface



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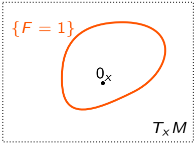

Corollary. $\sigma(M^2, F) \subset r\mathbb{Z}$ for some $r > 0$ if and only if F is Besse and $M = S^2$ or $\mathbb{R}P^2$.

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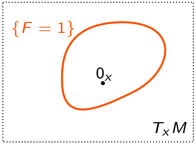

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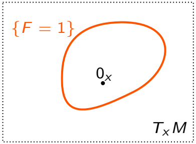

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Corollary.

- *If M is orientable, then $\sigma(M, g) \subset r\mathbb{Z}$ for some $r > 0$ if and only if $M = S^2$ and g Zoll.*
- *If M is non-orientable, then $\sigma(M, g) \subset r\mathbb{Z}$ for some $r > 0$ if and only if $M = \mathbb{R}P^2$ and g has constant curvature.*

(Hard) open questions

(Y^{2n+1}, λ) closed contact manifold of dimension $2n + 1 > 3$
$\sigma_{\mathrm{p}}(Y, \lambda)$ = prime action spectrum
$\sigma(Y, \lambda)$ = action spectrum

- (Weinstein's conjecture) Does (Y, λ) have closed Reeb orbits?
- If yes, does it have more than one?

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- If yes, does it have more than one?
- If yes, does $\sigma_{\mathrm{p}}(Y, \lambda) = \{\tau\}$ implies that (Y, λ) is Zoll?
- If yes, does $\sigma(Y, \lambda) \subset r\mathbb{N}$ for some $r > 0$ implies that (Y, λ) is Besse?

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$Y \subset \mathbb{C}^{n+1}$ convex hypersurface enclosing 0

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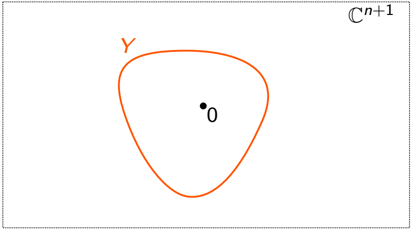

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Theorem (Ginzburg, Gürel, Mazzucchelli, 2019)

- $c_k = c_{k+n}$ *for some* k *if and only if* (Y, λ) *is Besse.*

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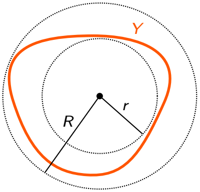
- *$c_k = c_{k+n}$ for some k if and only if (Y, λ) is Besse.*
- *$c_1 = c_{n+1}$ if and only if (Y, λ) is Zoll.*
- *Assume Y is δ-pinched for some $\delta \in (1, \sqrt{2}]$.*
  Then $\sigma(Y, \lambda) \cap (c_1, \delta^2 c_1) = \varnothing$ if and only if (Y, λ) is Zoll.



$$\frac{R}{r} < \delta$$

Proof that $c_k = c_{k+n}$ implies Besse

- $a \in (1,2)$
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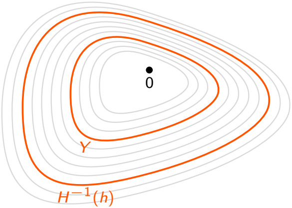

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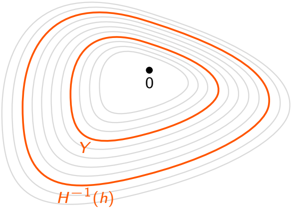

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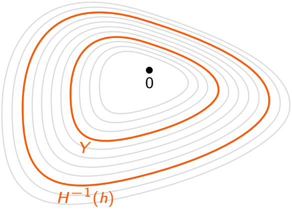

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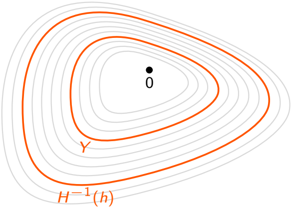

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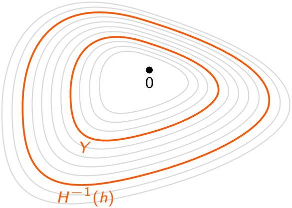

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  $$\Psi : L_0^b(S^1, \mathbb{C}^{n+1}) \to \mathbb{R}, \quad \Psi(\dot\Gamma) = \int_{S^1} \big(\langle i\dot\Gamma, \Gamma\rangle - H^*(-i\dot\Gamma)\big)\, dt, \quad b = \tfrac{a}{a-1}$$

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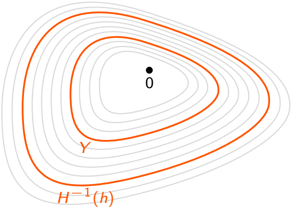
  $H : \mathbb{C}^{n+1} \to \mathbb{R}$ such that $H|_Y \equiv 1$ and $H(\lambda\,\cdot\,) = \lambda^a H$.



γ τ-periodic Reeb orbit on Y

$\Gamma(t) = h^{1/a}\gamma(\tau t)$ Hamiltonian 1-periodic orbit on $H^{-1}(h)$, for some unique $h = h(\tau)$

- $H^* : \mathbb{C}^{n+1} \to \mathbb{R}$ dual function to H
  $$H(w) = \max_z \big(\langle w, z\rangle - H(z)\big)$$
- Clarke action functional
  $$\Psi : L^b_0(S^1, \mathbb{C}^{n+1}) \to \mathbb{R}, \quad \Psi(\dot\Gamma) = \int_{S^1} \big(\langle i\dot\Gamma, \Gamma\rangle - H^*(-i\dot\Gamma)\big)\,dt, \quad b = \tfrac{a}{a-1}$$
- $\mathrm{Crit}(\Psi) \setminus \{0\} = \big\{\dot\Gamma \;\big|\; \Gamma \text{ 1-periodic Hamiltonian orbits}\big\}$
  $$\Psi(\dot\Gamma) = f(\tau) := \tfrac{a}{2}\big(\tfrac{a-2}{2}\tau\big)^{(2-a)/a}$$

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- Apply Lusternik-Schnirelmann theory:

  If $c_k = c_{k+n} = c$ then $e^n|_U \neq 0$ for all $U \subset W^{1,b}(\mathbb{R}/c\mathbb{Z}, Y)$ S^1-invariant neighborhood of the space of c-periodic Reeb orbits

Proof that $c_k = c_{k+n}$ implies Besse

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  Every sufficiently small neighborhood $W \subset W^{1,b}(\mathbb{R}/c\mathbb{Z}, Y)$ of the space of c-periodic Reeb orbits has non-zero cohomology $H^{2n+1}(W)$.

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$$\begin{tikzcd}
H^{2n+1}(U) \arrow[r] \arrow[d, "\pi_*"'] & H^{2n+1}(W) \arrow[r] & H^{2n+1}(U') \arrow[d, "\pi_*"] \\
H^{2n}_{S^1}(U) \arrow[rr] & & H^{2n}_{S^1}(U')
\end{tikzcd}$$

$U \supseteq W \supseteq U'$ neighborhoods of the space of c-periodic Reeb orbits;

U, U' are S^1-invariant

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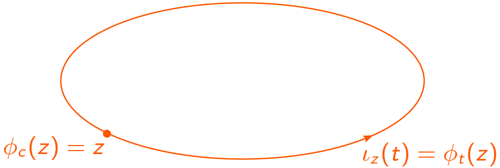

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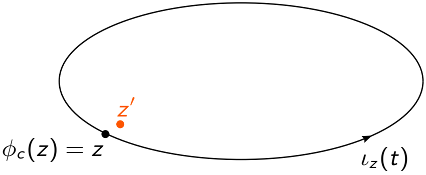

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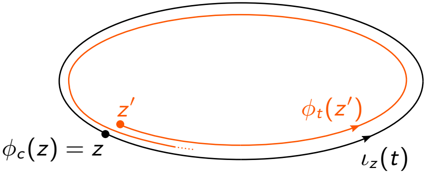

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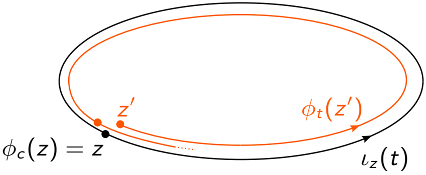

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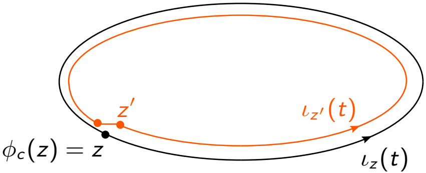

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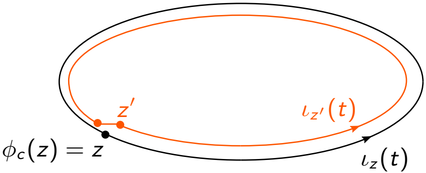

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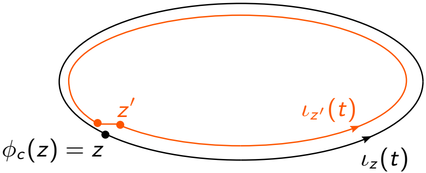

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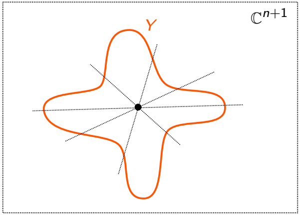
- $H^{2n+1}(W) \cong H^{2n+1}(Z) = 0$. $\square$

Besse and Zoll Reeb flows in higher dimension

(Y^{2n+1}, λ) restricted contact type hypersurface of $\mathbb{C}^{n+1}$



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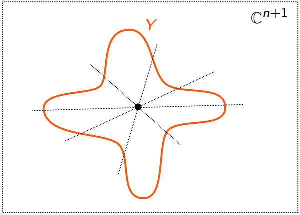

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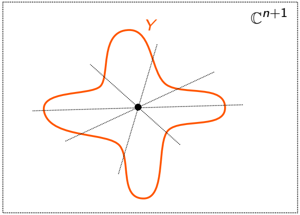

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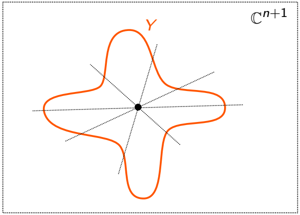

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Theorem (Ginzburg, Gürel, Mazzucchelli, 2019) *If $\sigma(Y, \lambda)$ is discrete and $c_k(Y) = c_{k+n}(Y) =: c$ for some $k \geq 1$, then (Y, λ) is Besse and c is a common period for its closed Reeb orbits.*

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- α_m, β_m generators of $H^{i_m}_{S^1}(\Lambda M, M)$ and $H^{i_m+2n-2}_{S^1}(\Lambda M, M)$
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Theorem (Ginzburg, Gürel, Mazzucchelli, 2019)

The following conditions are equivalent:

(i) $c(\alpha_1) = c(\beta_1)$

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If $M = S^n$ with $n \neq 3$, then (i) can be replaced by:

(i') $c(\alpha_m) = c(\beta_m)$ *for some* $m \geq 1$

Thank you for your attention!