Closed geodesics on reversible Finsler 2-spheres

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Joint work with:

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- Michele Marini (SISSA)
- Stefan Suhr (Ruhr Universität Bochum)

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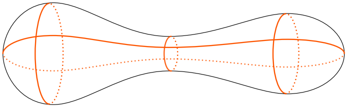
- Closed geodesics $=$ projections of periodic orbits of ϕ_t



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Every closed Riemannian manifold (M, g) *of* $\dim(M) \geq 2$ *has* *infinitely many* *closed geodesics.*

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Open for manifolds with the rational cohomology of a CROSS:

$\mathbb{R}P^n$, S^n for $n \geq 3$
$\mathbb{C}P^n$ for $n \geq 2$
$\mathbb{H}P^n$ for any n
$\mathrm{Ca}P^2$

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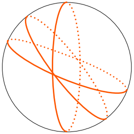
the closed geodesics conjecture holds even with a Finsler metric [Lu 2015]

However, there exists a Finsler 2-sphere (the Katok sphere) with only two closed geodesics!

Katok counterexample for Finsler spheres

- Unit 2-sphere with round metric (S^2, g)

  Every geodesic is closed with length 2π



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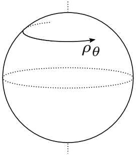

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$$\zeta(t) := \rho_{\alpha t} \circ \gamma(t)$$
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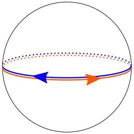
- The only closed geodesics of (S^2, F) are the two the two oriented equators



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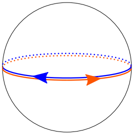

- F is non reversible: $F(x, v) \neq F(x, -v)$ for some $(x, v) \in TM$

Closed geodesics on reversible Finsler manifolds

Conjecture. *Every closed, reversible, Finsler, rational cohomology CROSS* (M, F) *of dimension at least* 2 *has infinitely many closed geodesics.*

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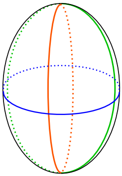
The main difficulty was to extend the theorem of the three simple closed geodesics of Lusternik-Schnirelmann for Riemannian S^2.



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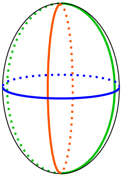
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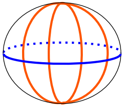
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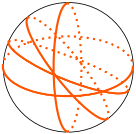
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(iii) for all $W \subset \Omega$ C^1-small neighborhood of the simple closed geodesics of length $\ell > 0$, there exists $\epsilon > 0$ and $t > 0$ such that

$$\phi_t(\Omega^{<\ell+\epsilon}) \subset \Omega^{<\ell} \cup W$$

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[Grayson 1989] In the Riemannian case (i.e. $F(x, v) = \sqrt{g_x(v, v)}$)
$\gamma_t := \phi_t(\gamma_0) : S^1 \hookrightarrow S^2$ solution of the PDE

$$\partial_t \gamma_t(s) = \kappa_t(s) n_t(s)$$

κ_t = signed curvature of γ_t
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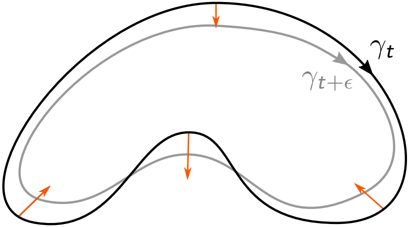

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$$dL(\gamma)\eta = \langle \nabla L(\gamma), \eta \rangle$$
$$\langle \eta, \xi \rangle = \int_{S^1} g(\eta(t), \xi(t))\, \|\dot\gamma(t)\|_g\, dt$$

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Lusternik-Schnirelmann theory

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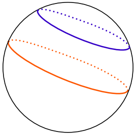
- If $\tau_1 \smile \tau_2 \neq 0$ in $H^*(\Omega, \Omega^{<\epsilon})$ and $c(\tau_1) = c(\tau_1 \smile \tau_2)$, then $\tau_2|_U \neq 0$ in $H^*(U)$ for all neighborhoods $U \subset \Omega$ of the simple closed geodesics of length ℓ

Topology of $\Omega = \frac{\mathrm{Emb}(S^1,S^2)}{\mathrm{Diff}(S^1)}$

$G :=$ {great circles} $\subset C :=$ {circles}



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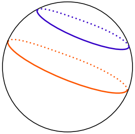
$G :=$ {great circles} $\subset C :=$ {circles}



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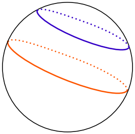

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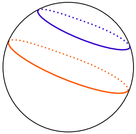

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- U contains all simple closed geodesics of length ℓ
- Since $\ell := c(\tau_i) = c(\tau_{i+1})$, Lusternik-Schnirelmann theorem implies that $\kappa|_U \neq 0$. Impossible since U is contractible! $\square$

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- If $\Sigma \subset T^*S^2$ is a fiberwise starshaped hypersurface equipped with the Liouville contact form $\lambda = p\,dq$ and invariant with respect to $(q, p) \mapsto (q, -p)$, does its Reeb flow have three closed orbits?

Thank you for your attention!