Analytic double fibration transforms

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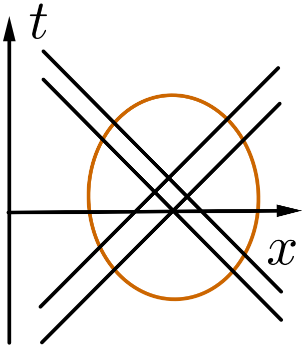
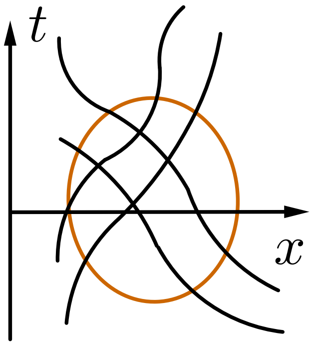
Light ray transform



Minkowski light rays



Lorentzian light rays

Light ray transform

IP for waves/relativity lead to Lorentzian light ray transform

$$Rf(\gamma) = \int f(\gamma(t))\, dt$$

where γ runs over the null geodesics for a Lorentzian metric g (light rays). It is invertible

- in Minkowski space $\mathbb{R}^{n+1}$ [Stefanov 1989]
- for analytic g with a foliation condition [Stefanov 2017]
- for certain time-independent (stationary/static) g [Feizmohammadi-Ilmavirta-Oksanen 2021]

Invertibility is open for small perturbations of Minkowski space.
Microlocal aspects: [Lassas-Oksanen-Stefanov-Uhlmann, Vasy, Wang, ...]

Bicharacteristic ray transform

[Oksanen-S-Stefanov-Uhlmann 2023]: for a real principal type[1] PDE, boundary measurements $\rightsquigarrow$ null bicharacteristic ray transform of unknown coefficients. This transform is given by

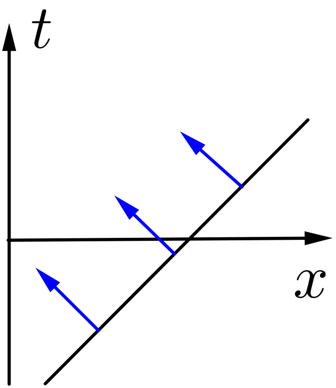
$$Rf(\gamma) = \int f(x(t))\,dt$$

where $\gamma(t) = (x(t), \xi(t))$ runs over all null bicharacteristics.

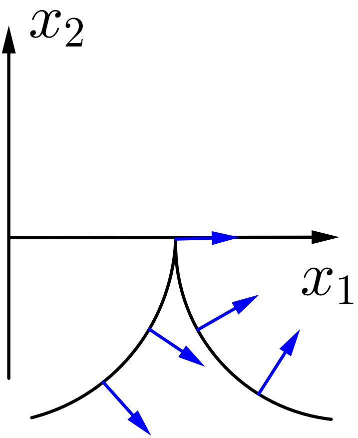
Examples. Geodesic/pseudo-Riemannian X-ray transform, light ray transform.

[1] real principal symbol + no trapped null bicharacteristics

Null bicharacteristics



Wave operator $\partial_t^2 - \Delta$



Tricomi operator $x_2\partial_{x_1}^2 + \partial_{x_2}^2$

Overview

Null bicharacteristic ray transforms fall under the microlocal double fibration approach to integral geometry [Guillemin 1975].

Under the Bolker condition ($\leftrightsquigarrow$ no conjugate points), one can recover C^∞ singularities (wave front set) of f from Rf. If coefficients are real-analytic, analytic microlocal analysis may yield local/global invertibility of R.

We make a systematic study of double fibration transforms (also for k-dim. submanifolds) and of the Bolker condition. We prove a theorem on the recovery of analytic singularities for Fourier integral operators (FIOs), and use this to invert R.

Overview

As consequences, we obtain inversion results for

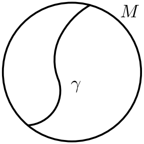
- geodesic X-ray transform
- null bicharacteristic ray transform
- generalized Radon transform over codim. k submanifolds

A nonvanishing analytic weight can be included in all results. Our method uses analytic continuation and is not stable (cf. [Koch-Rüland-S 2021]). Stable inversion might still be possible.

Geodesic X-ray transform

Let (M, g) be a compact Riemannian manifold with strictly convex boundary. M is nontrapping if any geodesic reaches ∂M in finite time. The geodesic X-ray transform of f is

$$If(\gamma) = \int_\gamma f(\gamma(t))\, dt$$



for geodesics γ. I is invertible if

- M has no conjugate points [Mukhometov 1977]
- $\dim(M) \geq 3$ and M is foliated by strictly convex manifolds [Uhlmann-Vasy 2016]

Conjecture

I is invertible on compact, strictly convex, nontrapping manifolds.

Geodesic X-ray transform

Theorem

Conjecture is true if $\dim(M) = 2$ and (M, g) is real-analytic.

Uses a local uniqueness result for analytic double fibration transforms [Stefanov-Uhlmann 2008, Mazzucchelli-S-Tzou 2023] combined with a strictly convex foliation when $\dim(M) = 2$ [Betelu-Gulliver-Littman 2002].

Null bicharacteristic ray transform

Let $p(x, \xi) \in C^\infty(T^*M)$ be real and homogeneous in ξ. Let

$$Rf(\gamma) = \int_\gamma f(x(t))\, dt$$

where $\gamma(t) = (x(t), \xi(t))$ runs over all null bicharacteristics.

Theorem

Let p be analytic, and suppose that

$$\nabla_\xi p \neq 0, \quad \det(\nabla_\xi^2 p) \neq 0 \qquad \text{on } p^{-1}(0).$$

Suppose that $\mathrm{supp}(f)$ is foliated by strictly pseudoconvex "timelike" hypersurfaces Γ_t. If $Rf = 0$, then $f = 0$.

Null bicharacteristic ray transform

Proved for light ray transforms in [Stefanov 2017].
Conditions for the family of null bicharacteristic curves:

- $\nabla_\xi p \neq 0 \iff$ curves have no cusps
- "timelike" means that Γ_t has many tangential curves
- strictly pseudoconvex means that Γ_t has many short almost tangential curves
- $\det(\nabla_\xi^2 p) \neq 0 \implies$ short curves have no conjugate points

Corollary

If P is analytic real principal type and above conditions hold, then the Cauchy data set of $P + V$ determines $V \in C^\infty(M)$.

Double fibrations: motivation

Let $M = \mathbb{R}^2$ and $\mathcal{G} = \{\text{lines in } \mathbb{R}^2\}$. The Radon transform R and its adjoint R^* satisfy

$$Rf(L) = \int_{x \in L} f(x), \qquad R^*F(x) = \int_{L \ni x} F(L).$$

Let $Z = \{(L, x) \,:\, x \in L\} = \{(L, x) \,:\, L \ni x\} \subset \mathcal{G} \times M$ be the incidence relation (or line-point relation). Then

$$\begin{array}{ccc} & Z & \\ {\scriptstyle \pi_{\mathcal{G}}} \swarrow & & \searrow {\scriptstyle \pi_M} \\ \mathcal{G} & & M \end{array}$$

with $\pi_{\mathcal{G}}, \pi_M$ submersions (=derivative surjective).

Double fibrations

Definition (Gel'fand-Graev-Shapiro 1969)

Let $\mathcal{G}$ and M be manifolds. A submanifold Z of $\mathcal{G} \times M$ is a double fibration if $\pi_{\mathcal{G}}$ and π_M are submersions.

For any $z \in \mathcal{G}$, one has a smooth manifold

$$G_z = \pi_M(\pi_{\mathcal{G}}^{-1}(z)).$$

Double fibration transform encodes integrals over the G_z:

$$R : C_c^\infty(M^{\text{int}}) \to C^\infty(\mathcal{G}), \quad Rf(z) = \int_{G_z} f \, d\omega_{G_z}$$

General curve families [Mazzucchelli-S-Tzou 2023]

For any double fibration ray transform R, there is

- a fiber bundle N over M, and
- a vector field Y on N

so that R integrates over projections of integral curves of Y.
The converse also holds under suitable conditions.

Examples:

- $N = SM$, $Y = X_g$ (geodesic X-ray transform)
- $N = p^{-1}(0)$, $Y = H_p$ (null bicharacteristic ray transform)

Other works for general curve families:
[Mukhometov 1977, Frigyik-Stefanov-Uhlmann 2008, Zhou 2016, ...]

Microlocal approach

[Guillemin '75]: R is an FIO. If Bolker condition holds (π_L is an injective immersion), R^*R is an elliptic ΨDO $\implies$ recovery of C^∞ singularities.

$$\begin{array}{ccccc} & & N^*Z \setminus 0 & & \\ & \swarrow^{\pi_L} & & \searrow^{\pi_R} & \\ T^*\mathcal{G} \setminus 0 & & & & T^*M \setminus 0 \end{array}$$

[Boman-Quinto '87, '93]: suggested to consider analytic case $\implies$ recovery of analytic singularities $\implies$ inversion of R.

Issue: clean intersection FIO calculus missing in analytic case. Alternative FBI transform method for specific transforms: [Stefanov-Uhlmann '08, Frigyik-Stefanov-Uhlmann '08, Krishnan-Stefanov '09, Stefanov '17, Homan-Zhou '17].

We prove recovery of analytic singularities for general double fibration transforms directly, via FBI transforms.

Main FIO result

Theorem

If R is an analytic double fibration transform and Bolker condition holds at $(z,\zeta,x,\eta) \in C := (N^*Z \setminus 0)'$, i.e.

- (global part) $\pi_L^{-1}(z,\zeta) = \{(z,\zeta,x,\eta)\}$
- (local part) $d\pi_L|_{(z,\zeta,x,\eta)}$ is injective

then

$$(z,\zeta) \notin WF_a(Rf) \implies (x,\eta) \notin WF_a(f).$$

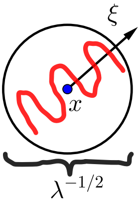
We give geometric characterizations for the Bolker condition:

- global part $\leftrightsquigarrow$ no conjugate points
- local part $\leftrightsquigarrow$ enough variations of tangents/normals of G_z

Analytic wave front set

Let $\lambda > 0$, and consider Gaussian wave packet

$$\psi_\lambda(y; x, \xi) = e^{i\lambda(y-x)\cdot\xi} e^{-\lambda\frac{|y-x|^2}{2}}$$



The (Gaussian) FBI transform of $f \in L^2(\mathbb{R}^n)$ is

$$T_\lambda f(x, \xi) = (f, \psi_\lambda(\,\cdot\,; x, \xi))_{L^2(\mathbb{R}^n)}$$

FBI transform gives a "phase space portait" of f.

Definition. f is analytic at (x_0, ξ_0), or $(x_0, \xi_0) \notin WF_a(f)$,[1] if

$$T_\lambda f(x, \xi) = O(e^{-c\lambda}) \text{ near } (x_0, \xi_0) \text{ as } \lambda \to \infty\ .$$

[1] Respectively, $(x_0, \xi_0) \notin WF(f)$ if $T_\lambda f = O(\lambda^{-\infty})$ near (x_0, ξ_0).

Ideas of proof

1. Localize in space using global part of Bolker condition.
2. Locally $Z = \{x'' = \phi(z, x')\}$. Consider model FIO

$$Tf(z) = \iint e^{i(\phi(z,x')-x'')\cdot\eta} a(z,x) f(x)\, d\eta\, dx$$

   Microlocalize in $\mathcal{G}$ via FBI transform L_λ, i.e.

$$L_\lambda Tf(z,\zeta) = \int K_\lambda(z,\zeta,x) f(x)\, dx = O(e^{-c\lambda})$$

   near $(\hat{z}, \hat{\zeta})$ since $(\hat{z}, \hat{\zeta}) \notin WF_a(Tf)$.
3. Simplify kernel K_λ by stationary phase, get

$$\int e^{i\lambda\psi(z,\zeta,x)} \hat{a}(z,\zeta,x;\lambda) f(x)\, dx = O(e^{-c\lambda}).$$

4. Show that LHS is a generalized FBI transform that detects $WF_a(f)$. Uses local Bolker condition.

Perspectives

1. Inversion without analyticity?
   (Energy estimates/Uhlmann-Vasy approach)
2. Violation of Bolker condition?
   [Monard-Stefanov-Uhlmann '15, Holman-Uhlmann '18, Felea-Gaburro-Greenleaf-Nolan '22, . . .]
3. Systems, tensors, . . . [Paternain-S-Uhlmann book '23]
4. Analytic FIO composition calculus?