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On topological and dynamical conditions imposing infinitely many periodic orbits in Hamiltonian dynamics

De certaines conditions topologiques et dynamiques imposant une infinité d'orbites périodiques en dynamique hamiltonienne

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Introduction

Introduction française

L'objet de cette thèse est l'étude des orbites périodiques de la dynamique Hamiltonienne. Prenons un instant pour définir ces termes. La dynamique Hamiltonienne désigne le domaine de la physique mathématique étudiant certains types de trajectoires que suivraient en particulier des systèmes physiques conservatifs selon les lois de la physique newtonienne classique. Supposons que nous étudions un système physique à n degrés de liberté $q_1, \dots, q_n$ que nous pouvons supposer réels (on pensera par exemple au cas de $n/3$ particules massives ayant pour degrés de liberté dans l'espace tridimensionnel leurs coordonnées spatiales). Selon le formalisme de la dynamique hamiltonienne, ses interactions physiques peuvent être décrites au moyen d'une fonction scalaire : le hamiltonien du système $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ prenant pour variables le temps $t \in \mathbb{R}$, la position $q := (q_1, \dots, q_n) \in \mathbb{R}^n$ et le moment $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ (dont l'expression dépend du système). En effet, l'évolution de la position q du système physique au cours du temps est décrite par le système d'équations différentielles non linéaire :

$$\dot{q} = \frac{\partial H_t}{\partial p}(q, p) \quad \text{et} \quad \dot{p} = -\frac{\partial H_t}{\partial q}(q, p),$$

où $\dot{q}$ et $\dot{p}$ désignent les dérivées temporelles respectives de q et p au temps t. Cette équation est la célèbre équation de Hamilton. Étant intéressés aux potentielles solutions périodiques d'une telle équation, nous ferons l'hypothèse que le hamiltonien est temporellement 1-périodique : $H_{t+1} = H_t$ pour tout $t \in \mathbb{R}$ et nous verrons plutôt t comme un élément de $S^1 := \mathbb{R}/\mathbb{Z}$.

L'équation de Hamilton que nous avons défini sur $\mathbb{R}^{2n}$ se généralise aux variétés symplectiques (M, ω), lesquelles sont des variétés M munies d'une 2-forme fermée non dégénérée ω appelée forme symplectique. En effet, l'équation précédente se réécrit

$$X_t \lrcorner \Omega = \mathrm{d}H_t \quad \text{avec} \quad \Omega := \sum_{j=1}^{n} \mathrm{d}q_j \wedge \mathrm{d}p_j \quad \text{et} \quad X_t := (\dot{q}, \dot{p}).$$

On étend ainsi la définition de l'équation de Hamilton aux hamiltoniens $H : S^1 \times M \to \mathbb{R}$ par $X_t \lrcorner \omega = \mathrm{d}H_t$. Le flot au temps 1 d'une telle équation différentielle sera l'un de nos principaux objets d'intérêt : il s'agit d'un difféomorphisme $\varphi : M \to M$. On désigne par difféomorphisme hamiltonien un tel difféomorphisme et on note $\mathrm{Ham}(M, \omega)$ l'ensemble des difféomorphismes hamiltoniens de la variété (M, ω). En effet, dans le cas où H_t dépend effectivement de t, l'étude générale des orbites périodiques se ramène à l'étude des points périodiques du difféomorphisme hamiltonien associé.

Dans cette thèse, nous nous intéressons aux conditions dynamiques ou topologiques imposant l'existence d'un nombre infini de trajectoires périodiques pour certains types de systèmes hamiltoniens. Dans une première partie, nous prolongeons la théorie de Givental et Théret basée sur les fonctions génératrices permettant d'étudier le cas des espaces projectifs complexes $M = \mathbb{C}\mathrm{P}^d$; nous retrouvons ainsi des résultats très récents au moyen d'outils bien plus élémentaires. Dans une seconde partie, nous nous intéressons aux flots géodésiques et démontrons de nouveaux résultats apportant des exemples de telles conditions dynamiques ou topologiques. Les résultats de cette seconde partie sont partiellement issus d'une collaboration avec Tobias Soethe. Les résultats de cette thèse ont fait l'objet d'articles : [**1**] et [**3**] pour ce qui concerne la première partie, [**2**] et [**4**] pour la seconde.

Partie 1 : étude des points périodiques des difféomorphismes hamiltoniens de $\mathbb{C}\mathrm{P}^d$ via les fonctions génératrices

On considère aussi l'espace projectif complexe $\mathbb{C}\mathrm{P}^d$ de dimension complexe d, variété symplectique dont la forme symplectique est la forme de Fubini-Study ω caractérisée par l'identité

$\pi^*\omega = i^*\Omega$ où $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ est l'application quotient, $i : \mathbb{S}^{2d+1} \hookrightarrow \mathbb{C}^{d+1}$ est l'inclusion et $\Omega := \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ est la forme symplectique standard de $\mathbb{C}^{d+1} \simeq \mathbb{R}^{2(d+1)}$. En 1985, Fortune-Weinstein [**35**] ont montré que tout difféomorphisme hamiltonien de $\mathbb{C}\mathrm{P}^d$ possède au moins $d+1$ points fixes, comme Arnol'd l'avait conjecturé. Les travaux de Givental [**43**] et Théret [**75**] des années 90, que nous prolongerons dans cette partie de la thèse, fournirent une preuve alternative de ce résultat. Étant donné un difféomorphisme φ, un point k-périodique de φ est par définition un point fixe de la k-ième itérée φ^k. Sur une variété close et symplectiquement asphérique (*e.g.* le tore $\mathbb{T}^{2d}$) tout difféomorphisme hamiltonien a une infinité de points périodiques. Ce résultat fut conjecturé par Conley, démontré par Hingston [**52**] dans le cas du tore et généralisé par Ginzburg au cas général [**39**] après des décennies de grandes avancées : Conley-Zehnder démontrèrent le cas non dégénéré pour les tores [**31**], Salamon-Zehnder démontrèrent le cas non dégénéré pour les variétés asphériques [**67**], Franks-Handel et Le Calvez démontrèrent la conjecture pour les surfaces [**38, 55**].

À l'opposé des variétés symplectiquement asphériques, la conjecture de Conley n'est pas valable sur $\mathbb{C}\mathrm{P}^d$: il existe des difféomorphismes hamiltoniens avec un nombre fini de points périodiques. Un contre-exemple simple est donné par le difféomorphisme

$$[z_1 : z_2 : \cdots : z_{d+1}] \mapsto \left[e^{2i\pi a_1} z_1 : e^{2i\pi a_2} z_2 : \cdots : e^{2i\pi a_{d+1}} z_{d+1}\right],$$

pour des réels $a_1, \ldots, a_{d+1} \in \mathbb{R}$ rationnellement indépendants. En effet, c'est un difféomorphisme hamiltonien dont les seuls points périodiques sont les points fixes : la projection de la base canonique de $\mathbb{C}^{d+1}$. On remarque que ce difféomorphisme hamiltonien a le nombre minimum de points périodiques, à savoir le nombre minimum de points fixes conjecturé par Arnol'd. Un difféomorphisme hamiltonien de $\mathbb{C}\mathrm{P}^d$ ayant exactement $d+1$ points périodiques est appelé une *pseudo-rotation* de $\mathbb{C}\mathrm{P}^d$.

Dans le cas où $d = 1$, $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{S}^2$ et les difféomorphismes hamiltoniens sont les difféomorphismes préservant l'aire isotopes à l'identité. Franks [**36, 37**] démontra que de tels difféomorphismes (ou plus généralement homéomorphismes) ont ou bien 2 ou bien une infinité de points périodiques. En 1994, Hofer-Zehnder [**53**, p.263] conjecturèrent une généralisation en grande dimension de ce résultat : tout difféomorphisme hamiltonien de $\mathbb{C}\mathrm{P}^d$ a ou bien $d+1$ points périodiques ou bien une infinité (cette conjecture était énoncée pour une classe de variétés symplectiques plus générale). Dans cette direction, une preuve symplectique du théorème de Franks fut proposée par Collier *et al.* [**30**] dans le cas lisse.

Dans cette thèse, nous nous intéresserons en particulier au résultat de Ginzburg-Gürel suivant.

Theorème A. *Tout difféomorphisme hamiltonien de $\mathbb{C}\mathrm{P}^d$ ayant un point fixe hyperbolique a une infinité de points périodiques.*

Ce théorème fut démontré dans [**41**] dans un cadre plus général incluant certaines grassmanniennes complexes, $\mathbb{C}\mathrm{P}^d \times P^{2k}$ avec P symplectiquement asphérique et $k \leq d$, les produits monotones $\mathbb{C}\mathrm{P}^d \times \mathbb{C}\mathrm{P}^d$. Nous remarquons que le cas $\mathbb{C}\mathrm{P}^d \times \mathbb{T}^{2k}$ avec $k \leq d$ peut aisément être déduit avec nos outils.

Les méthodes employées dans la preuve permettent aussi de montrer une sorte de contraposée vérifiée par les pseudo-rotations, comme Ginzburg-Gürel le remarquèrent dans [**42**].

Theorème B. *Les points fixes d'une pseudo-rotation de $\mathbb{C}\mathrm{P}^d$ sont isolés comme ensembles invariants.*

En 2019, Shelukhin démontra une version « homologique » de la conjecture de Hofer-Zehnder qui peut être ainsi énoncée dans le cas particulier de $\mathbb{C}\mathrm{P}^d$ [**69**].

Theorème C. *Tout difféomorphisme hamiltonien φ de $\mathbb{C}\mathrm{P}^d$ tel que $N(\varphi;\mathbb{F}) > d+1$ pour un corps $\mathbb{F}$ a une infinité de points périodiques. De plus, si φ a un nombre fini de points fixes, dans le cas où la caractéristique de $\mathbb{F}$ est nulle, il existe $A \in \mathbb{N}$ tel que, pour tout nombre premier $p \geq A$, φ a un point p-périodique qui n'est pas un point fixe ; dans le cas où la caractéristique de $\mathbb{F}$ est $p \neq 0$, φ a une infinité de points périodiques dont la période appartient à $\{p^k \mid k \in \mathbb{N}\}$.*

L'entier $N(\varphi;\mathbb{F}) \in \mathbb{N}$ est un « compte homologique » du nombre de points fixes de φ, qui est égal au nombre de points fixes lorsque φ est non dégénéré (*i.e.* lorsque le graphe de φ est transverse au graphe de l'identité). En particulier, cela démontre la conjecture de Hofer-Zehnder dans le cas des difféomorphismes hamiltoniens non dégénérés. Comme nous le verrons, la preuve de Shelukhin est basée sur la théorie des code-barres en topologie symplectique introduite par Polterovich-Shelukhin dans [**65**].

Dans la première partie de cette thèse, nous nous appuyons sur les idées de Givental [**43**] et Théret [**75**] pour construire un analogue à l'homologie de Floer des difféomorphismes hamiltoniens de $\mathbb{C}\mathrm{P}^d$ reposant sur la théorie de Morse classique et les fonctions génératrices. Avec cet outil, nous obtenons des preuves des Théorèmes A, B et C « qui auraient pu être données dans les années 90 ».

Organisation de la Partie 1. Dans le Chapitre 1, nous présentons les outils classiques qui nous seront utiles. Dans le Chapitre 2, nous étudions l'homologie des joints projectifs de sous-ensembles de l'espace projectif. Cette discussion générale sera essentielle à l'étude homologique des sous-niveaux des fonctions génératrices de difféomorphismes hamiltoniens de $\mathbb{C}\mathrm{P}^d$. Dans le Chapitre 3, nous développons la théorie de l'homologie des fonctions génératrices pour les difféomorphismes hamiltoniens de $\mathbb{C}\mathrm{P}^d$. Ce chapitre est le cœur de cette partie de la thèse. Dans le Chapitre 4, nous démontrons les Théorèmes A et B (dénommés Corollaires 4.2 et 4.3 dans le corps du texte). Dans le Chapitre 5, nous démontrons le Théorème C (dénommé Théorème 5.1 dans le corps du texte).

Partie 2 : chemins géodésiques et géodésiques fermées

Ici, nous nous cantonnerons à une dynamique hamiltonienne particulière : le flot géodésique sur le fibré tangent d'une variété riemannienne (M, g) ou, plus généralement, une variété Finsler (M, F). Cette dynamique se décrit directement sur la variété M : il s'agit des trajectoires que peut décrire un point mobile de cette variété se déplaçant « tout droit ». Ainsi les droites dans un espace euclidien ou les grands cercles sur la sphère ronde. De telles trajectoires sont appelées géodésiques et nous nommerons « géodésiques fermées » les trajectoires périodiques de cette dynamique (suivant cette terminologie, les droites d'un espace euclidien ne sont pas des géodésiques fermées).

Géodésiques fermées d'une surface complète. Les résultats énoncés dans cette section sont issus d'une collaboration avec Tobias Soethe. Le problème de l'existence et de la multiplicité des géodésiques fermées joue un rôle important en géométrie et dynamique riemannienne. Remontant aux travaux de Hadamard et Poincaré [**49, 64**], il est toujours ouvert pour de nombreuses variétés riemanniennes. Étant donnée une variété riemannienne complète (M, g), une question célèbre est de savoir si celle-ci possède une géodésique fermée quelque soit la métrique g. La réponse est toujours positive pour les M closes [**19, 57, 33**]. On peut alors se demander si le nombre de géodésiques fermées est ou non infini. Ce nombre est toujours infini dans le cas des surfaces closes [**36, 10, 51**]. Cependant, la question est toujours ouverte pour les sphères de dimension supérieure. Ici, nous nous intéressons aux surfaces complètes et non compactes pour lesquelles l'existence même d'une géodésique fermée n'est pas toujours garantie : les plans et les cylindres (nous étudions aussi le ruban de Möbius qui peut ne contenir qu'une seule géodésique fermée). Par exemple, le plan euclidien n'a aucune géodésique fermée. Néanmoins, sous certaines conditions géométriques, des résultats intéressants peuvent être énoncés. En 1980, Bangert montra que tout cylindre, plan ou ruban de Möbius complet d'aire finie a une infinité de géodésiques fermées [**9**]. Dans le cas des plans et cylindres, il obtint les mêmes résultats sous l'hypothèse affaiblie de l'existence de voisinages convexes de l'infini. Notre but est de donner des conditions simples sous lesquelles l'existence d'une ou deux géodésiques fermées distinctes implique l'existence d'une infinité de géodésiques fermées.

Soit $M \simeq S^1 \times \mathbb{R}$ un cylindre riemannien complet et ΛM son espace des lacets. Deux lacets $\alpha, \beta \in \Lambda M$ sont dits géométriquement distincts si leurs images sont distinctes : $\alpha(S^1) \neq \beta(S^1)$. Étant donné un anneau R, une géodésique fermée $\gamma \in \Lambda M$ est dite homologiquement visible sur R si l'homologie locale à coefficients dans R de son cercle critique $S^1 \cdot \gamma \subset \Lambda M$ est non nulle pour la fonctionnelle énergie (voir la Section 2 du Chapitre 6 pour les définitions précises). À l'exception de ce qui concerne le ruban de Möbius, les résultats suivants sont vrais sur tout anneau R (une fois fixé), nous ne mentionnerons donc pas R explicitement.

THEORÈME D. *Soit un cylindre riemannien complet M dont les géodésiques fermées sont isolées vérifiant l'une des conditions suivantes :*

1. il existe une géodésique fermée contractile,
2. il existe une géodésique fermée s'intersectant avec elle-même,
3. il existe deux géodésiques fermées distinctes s'intersectant,
4. il existe une géodésique fermée d'indice moyen non nul,
5. il existe deux géodésiques fermées homologiquement visibles.

Il existe alors un compact $K \subset M$ tel que M a une infinité de géodésiques homologiquement visibles intersectant K, l'une au moins ne s'intersectant pas avec elle-même.

Cela démontre, en particulier, la conjecture d'Abbondandolo suivante.

Corollaire. *Tout cylindre riemannien complet dont les géodésiques fermées sont isolées possède zéro, une ou une infinité de géodésiques fermées homologiquement visibles.*

En prenant le revêtement connexe double, on déduit l'analogue suivant du Théorème D lorsque M est un ruban de Möbius complet (voir la Section 7 du Chapitre 6).

Theorème E. *Soit un ruban de Möbius riemannien complet M dont les géodésiques fermées sont isolées vérifiant l'une des conditions suivantes :*

1. il existe une géodésique fermée contractile,
2. il existe une géodésique fermée s'intersectant avec elle-même,
3. il existe deux géodésiques fermées distinctes s'intersectant,
4. il existe une géodésique fermée d'indice moyen non nul,
5. il existe deux géodésiques fermées homologiquement visibles sur $\mathbb{F}_2$.

Il existe alors un compact $K \subset M$ tel que M a une infinité de géodésiques homologiquement visibles intersectant K, l'une au moins ne s'intersectant pas avec elle-même.

Selon Thorbergsson [**76**, Théorème 3.2], tout ruban de Möbius complet contient au moins une géodésique fermée homologiquement visible ne s'intersectant pas avec elle-même.

Corollaire. *Tout ruban de Möbius riemannien complet dont les géodésiques fermées sont isolées possède une seule ou une infinité de géodésiques fermées homologiquement visibles sur $\mathbb{F}_2$.*

Des résultats similaires sont obtenus pour les plans complets $M \simeq \mathbb{R}^2$.

Theorème F. *Soit un plan riemannien complet M avec dont les géodésiques fermées sont isolées vérifiant l'une des conditions suivantes :*

1. il existe une géodésique fermée s'intersectant avec elle-même,
2. il existe deux géodésiques fermées distinctes s'intersectant,
3. il existe une géodésique fermée d'indice moyen non nul,
4. il existe deux géodésiques fermées homologiquement visibles.

Il existe alors un compact $K \subset M$ tel que M a une infinité de géodésiques homologiquement visibles intersectant K, l'une au moins ne s'intersectant pas avec elle-même.

Corollaire. *Tout plan riemannien complet dont les géodésiques fermées sont isolées possède zéro ou une infinité de géodésiques fermées homologiquement visibles.*

Afin de mettre ces résultats en perspective, rappelons quelques résultats connus sur l'existence de géodésiques fermées dans les variétés riemanniennes complètes non compactes. En 1978, Thorbergsson montra l'existence de géodésiques fermées dans les variétés riemanniennes complètes contenant un compact convexe non trivial homologiquement ou ayant une courbure sectionnelle positive hors d'un compact [**76**]. Dans les années 90, Benci et Giannoni montrèrent que toute variété riemannienne complète d-dimensionnelle dont la limite supérieure de la courbure sectionnelle est négative à l'infini et dont l'homologie de l'espace des lacets libres est non triviale en un degré supérieur à $2d$ admet une géodésique fermée [**16, 17**]. En 2017, Asselle et Mazzucchelli montrèrent l'existence d'une infinité de géodésiques fermées pour les variétés riemanniennes complètes d-dimensionnelles n'ayant pas de points conjugués à l'infini et dont la suite des nombres de Betti de son espace des lacets libres prise à partir du degré $2d$ n'est pas bornée [**5**]. Ils améliorèrent aussi le résultat de Benci et Giannoni en remplaçant l'hypothèse concernant la courbure asymptotique en l'hypothèse concernant les points conjugués à l'infini et en améliorant la borne sur l'homologie de l'espace des lacets libres. Cependant, l'existence d'une géodésique fermée dans n'importe quelle variété riemannienne de volume fini est toujours un problème ouvert (consulter par exemple l'état de l'art suivant [**22**]).

Croissance du nombre de géodésiques entre deux points fixés. Ici, nous supposerons toujours nos variétés M connexes. Étant donnés deux points p, q d'une variété de Finsler complète (M, F), le plus court chemin joignant p à q nous donne un segment géodésique entre p et q. Nous nous intéressons ici à des conditions topologiques sur M impliquant l'existence d'une infinité de géodésiques géométriquement distinctes reliant les points p et q (pour cette question spécifique, l'irréversibilité de la métrique F n'amène pas de complication). Pour $\ell > 0$, notons $n(\ell; p, q)$ le nombre de géodésiques géométriquement distinctes reliant p à q de longueur $\leq \ell$. Il est bien connu que lorsque $\pi_1(M)$ est « suffisamment grand », $n(\ell; p, q)$ tend vers l'infini sans condition additionnelle. En effet, si la croissance du groupe $\pi_1(M)$ est sur-linéaire, $n(\ell; p, q)$ tend vers l'infini. Nous nous intéressons donc au cas où la croissance de $\pi_1(M)$ est linéaire. En nous inspirant d'un travail célèbre de Bangert-Hingston [**11**], nous démontrons qu'il suffit de demander que $\pi_1(M)$ soit infini dans la mesure où M n'est pas homotopiquement équivalente au cercle S^1.

Theorème G. *Soit une variété (connexe) M de groupe fondamental infini et n'étant pas homotopiquement équivalente au cercle. Alors, pour toute métrique Finsler complète en avant,*

$$n(\ell; p, q) \to +\infty \quad \textit{lorsque} \quad \ell \to +\infty, \quad \forall p, q \in M.$$

Bien entendu, la conclusion du Théorème G n'est pas vérifiée pour les cylindres plats $S^1 \times \mathbb{R}^n$, lesquels sont homotopiquement équivalents au cercle. Dans sa thèse de doctorat, Mentges démontra le Théorème G dans le cas particulier où $p = q$ et M n'est pas contractile [**60**, *Satz* 2.2.1.]. On peut être plus précis lorsque le rang de $H_1(M; \mathbb{Z})$ est non nul.

Theorème H. *Soit une variété close M n'étant pas homotopiquement équivalente au cercle (c'est-à-dire une variété close de dimension ≥ 2) et de premier nombre de Betti $\beta_1(M; \mathbb{Z}) \geq 1$. Alors, quelle que soit la métrique de Finsler sur M, il existe $a > 0$ et $b \in \mathbb{R}$ tels que*

$$n(\ell; p, q) \geq a \log \ell + b, \quad \forall \ell > 0, \forall p, q \in M.$$

Theorème I. *Soit une variété M n'étant pas homotopiquement équivalente au cercle et de premier nombre de Betti $\beta_1(M; \mathbb{Z}) \geq 1$. Alors, pour toute métrique de Finsler complète en avant sur M, il existe une fonction continue $b : M \to \mathbb{R}$ telle que*

$$n(\ell; p, q) \geq \frac{\log(\log \ell)}{2 \log 2} + b(q), \quad \forall \ell > 0, \forall p, q \in M.$$

Lorsque le revêtement universel de M n'est pas contractile (autrement dit lorsque M n'est pas un espace de Eilenberg-MacLane), Les Théorèmes G, H et I se déduisent d'un argument de min-max inspiré de celui de Bangert-Hingston [**11**]. Lorsque le revêtement universel de M est contractile, les estimées sont meilleures encore car la croissance est au moins linéaire.

Lemme. *Soit une variété M n'étant pas homotopiquement équivalente au cercle et de revêtement universel contractile. Alors son groupe fondamental a une croissance au moins quadratique.*

Ces dernières hypothèses sont en particulier vérifiées par les variétés closes de dimension ≥ 2 et de revêtement universel non contractile.

L'étude des liens entre le nombre de géodésiques joignant deux points et la topologie de la variété remonte aux travaux fondateurs de Morse [**63, 62**], dans lesquels il démontra notamment que toute paire de points d'une variété riemannienne close M peut être jointe par un nombre infini de géodésiques à la condition que les groupes d'homologie de l'espace des lacets de M aient un rang non trivial pour un nombre infini de degrés. Serre montra que cette condition est toujours vérifiée pour les variétés M simplement connexes en étudiant la suite spectrale associée à la fibration $\mathrm{ev} : P \to M$, où P est l'espace des chemins $\gamma \in C^0([0, 1], M)$ de point base $\gamma(0) = p$ fixé et $\mathrm{ev}(\gamma) := \gamma(1)$ [**68**, Prop. IV.11]. Nous attirons l'attention du lecteur sur le fait que dans ce résultat les géodésiques ainsi dénombrées ne sont pas nécessairement géométriquement distinctes (au contraire de l'ensemble des énoncés que nous démontrons dans cette partie de la thèse). Montrer l'existence d'une infinité de géodésiques géométriquement distinctes à l'aide de la théorie de Morse nécessite une étude plus subtile. Inspiré par Gromoll-Meyer (voir ci-après), Tanaka [**72**, Problem C] demanda s'il était suffisant, pour une variété riemannienne simplement connexe (M, g), de supposer que la suite des nombres de Betti de l'espace des lacets $(\beta_i(\Omega M))$ sous un corps fixé était non bornée pour obtenir $n(\ell; p, q) \to \infty$ pour toute paire de points $p, q \in M$. Lorsque p et q ne sont pas conjugués, il esquissa la preuve et Caponio-Javaloyes [**23**] donnèrent une preuve détaillée du cas plus général où M est une variété de Finsler complète en avant et en arrière. Lorsque p et q sont conjugués, la question est toujours ouverte.

Concernant le problème analogue du nombre de géodésiques fermées d'une variété riemannienne, l'un des premiers résultats comparables est dû à Gromoll-Meyer [**47**] : si la suite des nombres de Betti de l'espace des lacets libres d'une variété riemannienne simplement connexe M n'est pas bornée, alors M possède une infinité de géodésiques fermées géométriquement distinctes. En ce qui concerne la vitesse de croissance du nombre de géodésiques fermées, Bangert et Hingston montrèrent qu'elle est au moins aussi grande que celle des nombres premiers ($i.e. \gtrsim \frac{\ell}{\log \ell}$) à constantes additive et multiplicative près, lorsque $\pi_1(M)$ est infini *abélien* [**11**] ou lorsque $M = \mathbb{S}^2$ [**51**]. L'obstruction à ce que nous ne trouvions pas d'aussi bonnes estimations semble provenir de l'absence d'application d'itération $\gamma \mapsto (t \mapsto \gamma(mt))$, $m \in \mathbb{N}^*$, sur l'espace des segments géodésiques reliant p à q. Ce résultat fut étendu par Taĭmanov à une large classe de groupes fondamentaux infinis non abéliens dans [**73**]. Cependant, la question de l'existence d'un nombre infini de géodésiques fermées géométriquement distinctes dans une variété riemannienne close de groupe fondamental infini quelconque demeure ouverte.

Organisation de la Partie 2. Dans le chapitre 6, nous prouvons les Théorèmes D et F (dénommés Théorèmes 6.1 et 6.5 dans le corps du texte). Dans le chapitre 7, nous prouvons les Théorèmes G, H et I (dénommés Théorèmes 7.2, 7.3 et 7.4 dans le corps du texte).

English introduction

The subject of this thesis is the study of periodic orbits in Hamiltonian dynamics. Let us briefly introduce these concepts. Hamiltonian dynamics is the field of mathematical physics that studies special kinds of trajectories, some of which would be the ones of conservative physical systems of Newton's theory. Let us assume that we are studying a physical system with n degrees of freedom $q_1, \dots, q_n \in \mathbb{R}$ (as a special case, one can think of $n/3$ massive particles which degrees of freedom are the tridimensional spatial coordinates). According to the formalism of Hamiltonian dynamics, its physical interactions can be described by means of a scalar map: the Hamiltonian map of the system $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, depending on time $t \in \mathbb{R}$, position $q := (q_1, \dots, q_n) \in \mathbb{R}^n$ and momentum $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ (which expression depends on the system). As a matter of fact, the evolution of the position q of the physical system with time is described by the non-linear differential system:

$$\dot{q} = \frac{\partial H_t}{\partial p}(q,p) \quad \text{and} \quad \dot{p} = -\frac{\partial H_t}{\partial q}(q,p),$$

where $\dot{q}$ and $\dot{p}$ denote the respective time derivatives of q and p. This equation is the famous Hamilton equation. As we are interested in the study of periodic solutions of such an equation, we will assume that the Hamiltonian map is 1-periodic in time: $H_{t+1} = H_t$ for all $t \in \mathbb{R}$ and we will see t as an element of $S^1 := \mathbb{R}/\mathbb{Z}$.

The Hamilton equation defined on $\mathbb{R}^{2n}$ can be generalized to symplectic manifolds (M, ω), which are manifolds M endowed with a non-degenerate 2-form ω called a symplectic form. Indeed, the former equation can be written as

$$X_t \lrcorner \Omega = \mathrm{d}H_t \quad \text{with} \quad \Omega := \sum_{j=1}^{n} \mathrm{d}q_j \wedge \mathrm{d}p_j \quad \text{and} \quad X_t := (\dot{q}, \dot{p}).$$

The definition of the Hamilton equation can thus be extended to Hamiltonian map $H : S^1 \times M \to \mathbb{R}$ by $X_t \lrcorner \omega = \mathrm{d}H_t$. The time-one map of the Hamilton equation flow will be one of our major subject of study: it is a diffeomorphism $\varphi : M \to M$. Such a diffeomorphism is referred to as a Hamiltonian diffeomorphism and we denote by $\mathrm{Ham}(M, \omega)$ the set of Hamiltonian diffeomorphisms of the symplectic manifold (M, ω). As a matter of fact, when H_t truly depends on t, the general study of the periodic orbits boils down to the study of periodic points of the associated Hamiltonian diffeomorphism.

In this thesis, we are studying dynamical and topological conditions that imply infinitely many periodic orbits for special kinds of Hamiltonian systems. In a first part, we extend the theory of Givental and Théret based on generating functions that allows to study the case of complex projective spaces $M = \mathbb{C}\mathrm{P}^d$; we provide new proofs of recent results without appealing to the theory of J-holomorphic curves. In a second part, we study the geodesic flow and we show new results that provide examples of dynamical or topological conditions imposing infinitely many orbits. Work of the second part is partially joint with Tobias Soethe. The results in this thesis also appear in some published articles or preprints: [**1**] and [**3**] for results of the first part, [**2**] and [**4**] for the second one.

Part 1: periodic points of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$ via generating functions

Let $\mathbb{C}\mathrm{P}^d$ be the complex d-dimensional projective space endowed with the Fubini-Study symplectic structure ω, that is $\pi^*\omega = i^*\Omega$ where $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ is the quotient map, $i : \mathbb{S}^{2d+1} \hookrightarrow \mathbb{C}^{d+1}$ is the inclusion map and $\Omega := \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ is the canonical symplectic form of $\mathbb{C}^{d+1} \simeq \mathbb{R}^{2(d+1)}$. In 1985, Fortune-Weinstein [**35**] proved that any Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has at least

$d+1$ fixed points, as was conjectured by Arnol'd. Given a diffeomorphism φ, a k-periodic point of φ is by definition a fixed point of the k-iterated map φ^k. On closed symplectically aspherical manifolds (*e.g.* on tori $\mathbb{T}^{2d}$), every Hamiltonian diffeomorphism has infinitely many periodic points. This result was conjectured by Conley, proven for the tori by Hingston [**52**] and generalized by Ginzburg [**39**] after decades of flourishing advances: Conley-Zehnder proved the non-degenerate case for tori [**31**], Salamon-Zehnder proved the non-degenerate case for aspherical manifolds [**67**], Franks-Handel and Le Calvez proved the conjecture for surfaces [**38, 55**]. Contrary to aspherical symplectic manifolds, the Conley conjecture does not hold in $\mathbb{C}\mathrm{P}^d$: there exist Hamiltonian diffeomorphisms with only finitely many periodic points. A simple counter-example is the diffeomorphism

$$[z_1 : z_2 : \cdots : z_{d+1}] \mapsto \left[e^{2i\pi a_1} z_1 : e^{2i\pi a_2} z_2 : \cdots : e^{2i\pi a_{d+1}} z_{d+1}\right],$$

with rationally independent coefficients $a_1, \dots, a_{d+1} \in \mathbb{R}$. This is indeed a Hamiltonian diffeomorphism whose only periodic points are its fixed points: the projection of the canonical base of $\mathbb{C}^{d+1}$. Notice that this Hamiltonian diffeomorphism has the minimal number of periodic points, that is the minimal number of fixed points conjectured by Arnol'd. A Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ which has exactly $d+1$ periodic points is called a *pseudo-rotation* of $\mathbb{C}\mathrm{P}^d$.

In the case $d = 1$, $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{S}^2$ and Hamiltonian diffeomorphisms are the area preserving diffeomorphisms. Franks [**36, 37**] proved that every area preserving homeomorphism has either 2 or infinitely many periodic points. In 1994, Hofer-Zehnder [**53**, p. 263] conjectured a higher-dimensional generalization of this result: every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has either $d+1$ or infinitely many periodic points (it was stated for more general symplectic manifolds). In this direction, a symplectic proof of Franks' result was provided by Collier *et al.* [**30**] in the smooth setting.

We are interested in the following theorem of Ginzburg-Gürel.

Theorem A. *Every Hamiltonian diffeomorphism of* $\mathbb{C}\mathrm{P}^d$ *with a hyperbolic fixed point has infinitely many periodic points.*

This theorem was proven in [**41**] in a more general setting, including some complex Grassmannians, $\mathbb{C}\mathrm{P}^d \times P^{2k}$ where P is symplectically aspherical and $k \leq d$, monotone products $\mathbb{C}\mathrm{P}^d \times \mathbb{C}\mathrm{P}^d$. We mention that the case of $\mathbb{C}\mathrm{P}^d \times \mathbb{T}^{2k}$, when $k \leq d$, can be deduced as well from our techniques.

The machinery involved in the proof can also be used in order to prove this counterpart for pseudo-rotations, as Ginzburg and Gürel pointed out in [**42**].

Theorem B. *Each fixed point of a pseudo-rotation of* $\mathbb{C}\mathrm{P}^d$ *is not isolated as an invariant set.*

In 2019, Shelukhin proved a version of the Hofer-Zehnder conjecture that can be expressed in the following way for the special case of $\mathbb{C}\mathrm{P}^d$ [**69**].

Theorem C. *Every Hamiltonian diffeomorphism* φ *of* $\mathbb{C}\mathrm{P}^d$ *such that* $N(\varphi; \mathbb{F}) > d+1$ *for some field* $\mathbb{F}$ *has infinitely many periodic points. Moreover, when* φ *has finitely many fixed points, if* $\mathbb{F}$ *has characteristic* 0 *in the former assumption, there exists* $A \in \mathbb{N}$ *such that, for all prime* $p \geq A$, φ *has a* p*-periodic point that is not a fixed point; if* $\mathbb{F}$ *has characteristic* $p \neq 0$, φ *has infinitely many periodic points with period in* $\{p^k \mid k \in \mathbb{N}\}$.

The number $N(\varphi; \mathbb{F}) \in \mathbb{N}$ is a "homology count" of the number of fixed points of φ that is equal to the number of fixed points when φ is non-degenerate (*i.e.* the graph of φ is transverse to the graph of the identity). In particular, it solves the Hofer-Zehnder conjecture for non-degenerate Hamiltonian diffeomorphisms. As we will see, his proof is based on the theory of barcodes in symplectic topology introduced by Polterovich-Shelukhin in [**65**].

In the first part of this thesis, we elaborate on the ideas of Givental [**43**] and Théret [**75**] to build an analogue of the Floer homology of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$ with classical Morse theory and generating functions. With this tool, we then deduce a proof of Theorems A, B and C "that could have been given in the 90s".

Organization of Part 1. In Chapter 1, we introduce the well-known tools that will be needed. In Chapter 2, we study the homology of projective join of subsets of the projective space. This general discussion will be key to the homology study of the sublevel sets of generating functions of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$. In Chapter 3, we develop the theory of generating function homology for Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$. This chapter is the core of this part of the thesis. In Chapter 4, we prove Theorems A and B (referred as Corollary 4.2 and 4.3 in main text). In Chapter 5, we prove Theorem C (referred as Theorem 5.1 in the main text).

Part 2: closed geodesics and geodesic chords

Here we restrict ourselves to a special kind of dynamics: the geodesic flow on the tangent bundle of a Riemannian manifold (M, g) or, more generally, a Finsler manifold (M, F). This dynamic can be directly described on the manifold M: its trajectories are the ones that a point of M draws by "going straightforward". For instance, right lines of a Euclidean space or great circles of a round sphere. Such trajectories are called geodesics and periodic ones are called "closed geodesics".

Closed geodesics on complete surfaces. The problem of the existence and multiplicity of closed geodesics plays an important role in both Riemannian geometry and dynamics. Going back to Hadamard and Poincaré [**49, 64**], it is still open for a large class of Riemannian manifolds. Given a complete Riemannian manifold (M, g), a famous question is whether it possesses a closed geodesic for every Riemannian metric g. This is always true if M is closed [**19, 57, 33**]. We can then ask whether the number of closed geodesics is infinite or not. It is known that every closed surface has infinitely many geometrically distinct closed geodesics [**36, 10, 51**]. However, this question is still open for spheres of higher dimension. Here, we are interested in non-compact complete Riemannian surfaces for which even the existence of one closed geodesic fails in general: planes and cylinders (we also study the Möbius band that can have only one closed geodesic). For instance, the Euclidean plane does not possess any closed geodesic. Nevertheless, under specific geometric conditions, interesting results can be stated. In 1980, Bangert proved that any complete Riemannian cylinder, plane or Möbius band of finite area has infinitely many closed geodesics [**9**]. For the plane and the cylinder he proved the same result even under the weaker assumption of just the existence of a convex neighborhood of infinity. We will discuss this result in greater depth as it is used extensively in our proofs. Our purpose is to give simple conditions under which the existence of one or two distinct closed geodesics implies that a complete Riemannian cylinder, Möbius band or plane contains infinitely many geometrically distinct closed geodesics.

Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $M \simeq S^1 \times \mathbb{R}$ be a complete Riemannian cylinder. Let ΛM be its loop space. Two loops $\alpha, \beta \in \Lambda M$ are said to be geometrically distinct if their images are distinct: $\alpha(S^1) \neq \beta(S^1)$. Throughout this section, by writing that two closed geodesics are distinct we will always mean that they are geometrically distinct. Given a ring R, a closed geodesic $\gamma \in \Lambda M$ is said to be homologically visible over R if the local homology of the critical circle $S^1 \cdot \gamma \subset \Lambda M$ of the energy functional is non-zero over the coefficient ring R. With the exception of the Möbius band, every result are true over any coefficient ring R (once fixed) so the ring R will not be mentioned explicitly.

THEOREM D. *Let M be a complete Riemannian cylinder where all closed geodesics are isolated and assume one of the following hypothesis:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics.*

Then M contains infinitely many homologically visible closed geodesics intersecting some common compact set $K \subset M$ and at least one without self-intersection.

The fact that hypothesis 5 implies that there exist infinitely many homologically visible closed geodesics proves a conjecture of Abbondandolo:

COROLLARY. *Any complete Riemannian cylinder where all closed geodesics are isolated has zero, one or infinitely many homologically visible closed geodesics.*

By essentially taking the double cover, one can thus deduce the following counter-part of Theorem D when M is a complete Möbius band (see Section 7 of Chapter 6).

COROLLARY. *Let M be a complete Riemannian Möbius band where all closed geodesics are isolated and assume one of the following hypothesis:*

1. there exists a contractible closed geodesic,
2. there exists a self-intersecting closed geodesic,
3. there exist two distinct closed geodesics that intersect,
4. there exists a closed geodesic of non-zero average index,
5. there exist two homologically visible closed geodesics over $\mathbb{F}_2$.

Then M *contains infinitely many closed geodesics intersecting some common compact set* $K \subset M$ *that are homologically visible over* $\mathbb{F}_2$.

According to Thorbergsson [**76**, Theorem 3.2], any complete Möbius band has at least one homologically visible closed geodesic without self-intersection.

Corollary. *Any complete Riemannian Möbius band where all closed geodesics are isolated has one or infinitely many homologically visible closed geodesics over* $\mathbb{F}_2$.

Similar results can also be obtained when $M \simeq \mathbb{R}^2$ is a complete plane:

Theorem E. *Let* M *be a complete Riemannian plane where all closed geodesics are isolated and assume one of the following hypothesis:*

1. there exists a self-intersecting closed geodesic,
2. there exist two distinct closed geodesics that intersect,
3. there exists a closed geodesic of non-zero average index,
4. there exists a homologically visible closed geodesic.

Then M *contains infinitely many homologically visible closed geodesics intersecting some common compact set* $K \subset M$ *and at least one without self-intersection.*

Corollary. *Any complete Riemannian plane where all closed geodesics are isolated has zero or infinitely many homologically visible closed geodesics.*

In order to put these results in perspective, we recall some known results concerning existence of closed geodesics on complete non-compact Riemannian manifolds. In 1978, Thorbergsson proved the existence of closed geodesics on a complete Riemannian manifold M if it contains a convex compact set which is not homotopically trivial or if M has a non-negative sectional curvature outside some compact set [**76**]. In the 1990s, Benci and Giannoni proved that any complete d-dimensional Riemannian manifold such that the limit superior of its sectional curvature at infinity is non-positive and the homology of its free loop space is non-trivial in some degree larger than $2d$ possesses a closed geodesic [**16, 17**]. In 2017, Asselle and Mazzucchelli showed the existence of infinitely many closed geodesics for complete d-dimensional Riemannian manifolds which have no close conjugate points at infinity and a free loop space with unbounded Betti numbers in degrees larger than d [**5**]. They also improved the result of Benci and Giannoni by replacing the asymptotic curvature assumption by an assumption on the conjugated points at infinity and by improving the bound on the homology of the free loop space. The existence of one closed geodesic in any complete Riemannian manifold of finite volume is a hard open problem (see for instance the following recent review of the subject [**22**]).

Growth rate of geodesic chords. Given a pair of points p, q in a complete Finsler manifold (M, F), the smallest path joining p to q gives one geodesic path between p and q. We are interested in topological conditions on M that imply the existence of infinitely many geometrically distinct such geodesic paths, p and q being fixed once for all (for this special concern, the irreversibility of the metric F does not seem to add any qualitative difference). For $\ell > 0$, we denote by $n(\ell; p, q)$ the number of geometrically distinct geodesics between p and q of length $\leq \ell$. It is well known that for $\pi_1(M)$ "large enough", $n(\ell; p, q)$ tends to infinity without any further assumption. Indeed, if the growth rate of $\pi_1(M)$ is surlinear, $n(\ell; p, q)$ tends to $+\infty$. Therefore, we are interested in the remaining case in which the growth rate of $\pi_1(M)$ is linear. Inspired by a famous work of Bangert-Hingston [**11**], we prove that it is enough to ask that $\pi_1(M)$ is infinite as long as M is not homotopy-equivalent to the circle S^1.

Theorem F. *Let* M *be a manifold of infinite fundamental group* $\pi_1(M)$ *and not homotopy-equivalent to* S^1. *Then, given any forward complete Finsler metric on* M,

$$n(\ell; p, q) \to +\infty \quad as \quad \ell \to +\infty, \quad \forall p, q \in M.$$

Of course, the assertion of Theorem F does not hold for the flat cylinders $S^1 \times \mathbb{R}^n$, which are homotopy-equivalent to S^1. In his Ph.D. thesis, Mentges proved Theorem F in the case where $p = q$ and the universal cover of M is not contractible [**60**, *Satz* 2.2.1.]. We can be more specific when $H_1(M;\mathbb{Z})$ has non-zero rank:

Theorem G. *Let M be a closed manifold not homotopy-equivalent to S^1 (that is any closed M of dimension ≥ 2) and with first Betti number $\beta_1(M;\mathbb{Z}) \geq 1$. Then, given any Finsler metric on M, there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$n(\ell;p,q) \geq a \log \ell + b, \quad \forall \ell > 0, \forall p,q \in M.$$

Theorem H. *Let M be a manifold not homotopy-equivalent to S^1 and with first Betti number $\beta_1(M;\mathbb{Z}) \geq 1$. Then, given any forward complete Finsler metric on M, there exists a continuous function $b : M \to \mathbb{R}$ such that*

$$n(\ell;p,q) \geq \frac{\log(\log \ell)}{2\log 2} + b(q), \quad \forall \ell > 0, \forall p,q \in M.$$

When the universal cover of M is not contractible (that is M is not an Eilenberg-MacLane space), Theorems F, G and H are deduced from a min-max argument inspired by Bangert-Hingston [**11**]. When M has a contractible universal cover, the estimate is even stronger, since the growth is at least linear:

Lemma. *Let M be a manifold not homotopy-equivalent to S^1 and with a contractible universal cover. Then, $\pi_1(M)$ has at least a quadratic growth rate.*

We notice that any closed manifold of dimension ≥ 2 with a contractible universal cover satisfies the above condition.

Investigations on the links between the number of geodesics joining two points and the manifold topology go back to Morse seminal works [**63, 62**], where he proved that any couple of points of a closed Riemannian manifold M can be joined by infinitely many geodesics provided that the homology groups of the loop space of M have a non trivial rank in infinitely many degrees. Serre proved that this assumption is always satisfied for simply connected M by studying the spectral sequence associated with the fibration $\mathrm{ev} : P \to M$, where P is the space of paths $\gamma \in C^0([0,1],M)$ such that $\gamma(0) = p$ is a fixed base point and $\mathrm{ev}(\gamma) := \gamma(1)$ [**68**, Prop. IV.11]. In the above result, geodesics are not necessarily geometrically distinct. Proving that there are infinitely many geometrically distinct geodesics with Morse theory is more subtle. Inspired by Gromoll-Meyer (see below), Tanaka [**72**, Problem C] asked if it is enough for a simply connected Riemannian manifold (M,g) to assume that the sequence of Betti numbers of the loop space $(\beta_i(\Omega M))$ on some field is unbounded to get that $n(\ell;p,q) \to \infty$ for any pair of points $p,q \in M$. When p and q are non-conjugate, he sketched the proof and Caponio-Javaloyes [**23**] gave a detailed proof in the more general case of a connected, forward and backward complete Finsler manifold. When p and q are conjugate, it is still an open problem.

For the related problem of closed geodesics in Riemannian manifold, one of the first results in that direction was due to Gromoll-Meyer [**47**]: if the sequence of the Betti numbers of the free loop space of a simply connected Riemannian manifold M is unbounded, then there are infinitely many geometrically distinct closed geodesics on M. As for the growth rate, Bangert and Hingston proved that it is at least like the one of prime numbers (*i.e.* $\gtrsim \frac{\ell}{\log \ell}$) up to a multiplicative and an additive constant when $\pi_1(M)$ is infinite *abelian* [**11**], or when $M = \mathbb{S}^2$ [**51**]. The obstruction for us to find such a better growth seems to come from the lack of iteration map $\gamma \mapsto (t \mapsto \gamma(mt))$, $m \in \mathbb{N}^*$, in the space of geodesic chords joining p and q. This result was extended by Taĭmanov to a large class of infinite non-abelian fundamental groups in [**73**]. Nevertheless, the existence of infinitely many geometrically distinct closed geodesics in closed Riemannian manifolds with a general infinite fundamental group is still an open problem.

Organization of part 2. In Chapter 6, we prove Theorems D and E (referred as Theorems 6.1 and 6.5 in the main text). In Chapter 7, we prove Theorems F, G and H (referred as Theorems 7.2, 7.3 and 7.4 in the main text).

Part 1

Periodic points of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$ via generating functions

CHAPTER 1

Preliminaries

In this chapter, we provide the background in the literature needed to develop the theory of generating function homology of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$.

1. Persistence modules and barcodes

Let us fix a field $\mathbb{F}$. A persistence module (V, π) over the field $\mathbb{F}$ is an $\mathbb{R}$-family of $\mathbb{F}$-vector spaces $(V^t)_{t\in\mathbb{R}}$ with a collection of morphisms $\pi_s^t : V^s \to V^t$ for $s \leq t$ such that $\pi_t^t = \mathrm{id}_{V^t}$ and $\pi_s^t \circ \pi_r^s = \pi_r^t$ whenever $r \leq s \leq t$ that one can call persistence morphisms. We extend this definition to families $(V^t)_{t\in\mathbb{R}\setminus S}$ satisfying the same axioms with the exception that $t \in \mathbb{R} \setminus S$ where $S \subset \mathbb{R}$ is discrete by identifying (V^t) with the persistence module $\left(\overline{V}^t\right)$ defined by taking the direct limit

$$\overline{V}^t := \varinjlim_{s<t} V^s, \quad \forall t \in S,$$

with the obvious extension of the morphisms π_s^t. Given two persistence modules (V, π) and (V', π'), one defines the direct sum of them in an obvious way $(V \oplus V', \pi \oplus \pi')$ which is a persistence module. A morphism of persistence module $f : (V, \pi) \to (V', \pi')$ is a family of morphisms $f_t : V^t \to V'^t$ commuting with the persistence morphisms.

A persistence module (V^t) is of finite type if $V^t = 0$ for t sufficiently close to $-\infty$, every V^t has a finite dimension and there exists a finite set $S \subset \mathbb{R}$ such that π_s^t is an isomorphism whenever s and t belong to the same connected component of $\mathbb{R} \setminus S$. The fundamental example is given by $V^t := H_*(\{f \leq t\}; \mathbb{F})$, where $f : M \to \mathbb{R}$ is a smooth function on a compact manifold M with finitely many critical points (if M is non-compact or f has infinitely many critical points, we only have a general persistence module). Given an interval $I = (a, b]$ or $I = (a, +\infty)$, we define the persistence module $\mathbb{F}(I)$ by $V^t = \mathbb{F}$ for $t \in I$ and $V^t = 0$ otherwise, $\pi_s^t = \mathrm{id}$ when t and s belong to the same connected component of $\mathbb{R} \setminus \{a, b\}$ or $\mathbb{R} \setminus \{a\}$ respectively and $\pi_s^t = 0$ otherwise. It is a persistence module of finite type. We think of it as representing a class that appears at $t = a$ and persists until $t = b$ or indefinitely if $I = (a, +\infty)$. Graphically, we represent $\mathbb{F}(I)$ by drawing a horizontal bar from $t = a$ to $t = b$ or without right endpoint if $I = (a, +\infty)$. In order to state the normal form theorem properly, we always assume that our persistence modules (V^t) satisfy the following left-continuity property: for all $t \in \mathbb{R}$, π_s^t is an isomorphism for those $s \leq t$ that are close to t. The normal form theorem asserts that for every persistence module V of finite type, there exists a unique finite collection of couples (I_k, m_k), $I_k \subset \mathbb{R}$ being a bar as above and $m_k \in \mathbb{N}^*$, so that there is an isomorphism of persistence modules

$$V \simeq \bigoplus_k \mathbb{F}(I_k)^{\oplus m_k},$$

(see for instance [**13, 83**]). The collection $\mathcal{B}(V) := \{(I_k, m_k)\}$ is called the barcode of V; it is graphically represented by drawing each horizontal bars of the I_k's with multiplicity in the same figure (see Figure 2 for an example).

Although we will not need it in its full strength, we recall the isometry theorem between the bottleneck distance between barcodes and the interleaving distance between persistence modules (of finite type). Given $\delta, \delta' \in \mathbb{R}$ with $\delta + \delta' \geq 0$, a (δ, δ')-interleaving between persistence modules (V, π) and (W, κ) is a couple of morphisms of persistence modules $f : (V^t) \to (W^{t+\delta})$ and $g : (W^t) \to (V^{t+\delta'})$ such that $g_{t+\delta} \circ f_t = \pi_t^{t+\delta+\delta'}$ and $f_{t+\delta'} \circ g_t = \kappa_t^{t+\delta+\delta'}$ for all $t \in \mathbb{R}$. When such an interleaving exists, it is said that V and W are (δ, δ')-interleaved. Given $\delta \geq 0$, a δ-interleaving is by definition a (δ, δ)-interleaving. For instance, if (V^t) and (W^t) are (δ, δ')-interleaved, then

(V^t) and $(W^{t+\delta_-})$ are δ_+-interleaved, where $\delta_- = \frac{\delta-\delta'}{2}$ and $\delta_+ = \frac{\delta+\delta'}{2}$. The interleaving distance between two persistence modules V and W is defined by

$$d_{\mathrm{int}}(V,W) := \inf\{\delta \geq 0 \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\}.$$

This is a true distance between persistence modules up to isomorphisms taking values in $[0,+\infty]$.

Let $\mathcal{B} := \{(I_k, m_k)\}$ and $\mathcal{B}' := \{(J_l, m'_l)\}$ be two barcodes that we see as multisets of intervals. Given an interval $I = (a,b]$ or $(a,+\infty)$, we set $I^\delta := (a-\delta, b+\delta]$ or $(a-\delta,+\infty)$ respectively. Given $\delta \geq 0$, a δ-matching between the barcodes $\mathcal{B}$ and $\mathcal{B}'$ is a bijection of multisets $\mu : \mathcal{B}_0 \to \mathcal{B}'_0$ where $\mathcal{B}_0$ and $\mathcal{B}'_0$ are some submultisets of $\mathcal{B}$ and $\mathcal{B}'$ containing (at least) every interval of length $\geq 2\delta$ and such that $\mu(I) \subset I^\delta$ and $I \subset \mu(I)^\delta$ for every $I \in \mathcal{B}_0$. When such a δ-matching exists, it is said that $\mathcal{B}$ and $\mathcal{B}'$ are δ-matched. The bottleneck distance between two barcodes $\mathcal{B}$ and $\mathcal{B}'$ is defined by

$$d_{\mathrm{bottleneck}}(\mathcal{B},\mathcal{B}') := \inf\{\delta \geq 0 \mid \mathcal{B} \text{ and } \mathcal{B}' \text{ are } \delta\text{-matched}\}.$$

This is a true distance taking values in $[0,+\infty]$.

The isometry theorem between the bottleneck distance and the interleaving distance states that given any persistence modules of finite type V and W,

$$d_{\mathrm{int}}(V,W) = d_{\mathrm{bottleneck}}(\mathcal{B}(V),\mathcal{B}(W)),$$

(see for instance [**15**]).

2. Morse theory

Let M be a closed manifold and $f : M \to \mathbb{R}$ be a C^1-map that is at least C^2 in the neighborhood of its critical points. In this section we will briefly recall the fundamental results in the study of the homology of the sublevel sets of f (these results also hold for non compact manifolds when f satisfies the Palais-Smale condition). As a general reference, one can see [**25**].

Throughout this whole part of the thesis, $H_*(X)$ and $H^*(X)$ denote respectively the singular homology and the singular cohomology of a topological space or pair X over an indeterminate ring R. If one needs to specify the ring R, one writes $H_*(X;R)$ and $H^*(X;R)$ instead. We only state the results for homology groups for the sake of being brief but we will use their cohomology counterparts as well in the thesis.

Since we are interested in the sublevel sets, let us denote $f^{\leq\lambda} := \{f \leq \lambda\}$ and $f^{<\lambda} := \{f < \lambda\}$ in this specific section, where

$$\{f \leq \lambda\} := \{x \in M \mid f(x) \leq \lambda\}, \text{ etc.}$$

Let us first recall the Morse deformation lemma: if b is a critical value of f and $c \geq b$ is such that $(b,c]$ is an interval of regular values, then the inclusion gives an isomorphism

$$H_*\left(f^{\leq b}, f^{<a}\right) \xrightarrow{\simeq} H_*\left(f^{\leq c}, f^{<a}\right),$$

for all $a \leq b$. Indeed, the topological pair $(f^{\leq c}, f^{<a})$ retracts on $(f^{\leq b}, f^{<a})$. This retraction is essentially given by pushing points along the flow of a reversed pseudo-gradient (in the case where c is replaced by $c+\varepsilon > c$ in the above isomorphism, the retraction is given by such a flow taken at a sufficiently large time whereas the time must depend on the point in the sharp case). In the reversed direction, if b is a critical value of f, $c \geq b$ and $a \leq b$ such that $[a,b)$ is an interval of regular values, the inclusion induces the isomorphism

$$H_*\left(f^{\leq c}, f^{<a}\right) \xrightarrow{\simeq} H_*\left(f^{\leq c}, f^{<b}\right).$$

Therefore, if $[a,c]$ has only one critical value $b \in (a,c)$,

$$H_*\left(f^{\leq c}, f^{\leq a}\right) \simeq H_*\left(f^{\leq b}, f^{<b}\right) \simeq H_*\left(f^{\leq b+\varepsilon}, f^{\leq b-\varepsilon}\right),$$

for $\varepsilon > 0$ small enough. Hence, in order to study the homology of sublevel sets of f, it is essentially enough to study the homology of sublevel sets "in the neighborhood of each critical value".

Given an isolated critical point $x \in M$ of f, its local homology group is defined by

$$\mathrm{C}_*(f;x) := H_*\left(f^{\leq c}, f^{\leq c} \setminus x\right), \tag{1.1}$$

where $c := f(x)$. It is local in the sense that for all neighborhood U of x, the inclusion induces an isomorphism

$$H_*\left(f^{\leq c} \cap U, (f^{\leq c} \setminus x) \cap U\right) \xrightarrow{\simeq} \mathrm{C}_*(f;x),$$

by excision. One also has an isomorphism induced by inclusion

$$H_*\left(f^{<c}\cup\{x\}, f^{<c}\right) \xrightarrow{\simeq} \mathrm{C}_*(f;x),$$

that gives another way to define the local homology. Let us assume that f has a finite number of critical points $x_1,\dots,x_n$ with value c. By applying this last isomorphism with an excision, one gets a natural isomorphism

$$H_*\left(f^{\leq c}, f^{<c}\right) \simeq \bigoplus_j \mathrm{C}_*(f;x_j).$$

Therefore, if f has only finitely many fixed points, the evolution of the homology of the sublevel sets $f^{\leq\lambda}$ is essentially dictated by the local homologies.

We want to describe the above natural isomorphism when it is extended to a small window of regular values around the critical value c. According to the Morse deformation lemma, if U is a small neighborhood of x that is invariant under a reversed pseudo-gradient flow of f, the inclusion induces an isomorphism

$$\mathrm{C}_*(f;x) \simeq H_*\left(f^{\leq c+\varepsilon}\cap U, f^{\leq c-\varepsilon}\cap U\right). \tag{1.2}$$

In the same way as above, an excision gives the natural isomorphism in a small window of values around c

$$\bigoplus_j H_*\left(f^{\leq c+\varepsilon}\cap U_j, f^{\leq c-\varepsilon}\cap U_j\right) \xrightarrow{\simeq} H_*\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right), \tag{1.3}$$

if the U_j's are "small" neighborhoods of the x_j's that are invariant under a reversed pseudo-gradient flow of f (by "small" we mean that $U_i\cap U_j\subset f^{<c-\varepsilon}$ for $i\neq j$). The composition of this isomorphism with (1.2) gives us the natural isomorphism

$$\bigoplus_j \mathrm{C}_*(f;x_j) \simeq H_*\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right). \tag{1.4}$$

The support of a graded group C_* is defined by

$$\operatorname{supp} C_* := \{k\in\mathbb{Z} \mid C_k\neq 0\}\subset\mathbb{Z}.$$

A classical result due to Gromoll-Meyer [**46**, remark following Lemma 1] implies that for any isolated critical point $x\in M$ of f, the local homology is finitely generated and

$$\operatorname{supp}\mathrm{C}_*(f;x)\subset\left[\operatorname{ind}(x,f), \operatorname{ind}(x,f)+\dim\ker \mathrm{d}^2 f(x)\right]. \tag{1.5}$$

Let us finally recall the Lyusternik-Schnirelmann theorem. Here we do not assume that f has finitely many critical points. For all non-zero classes $\alpha\in H_*(f^{\leq b}, f^{<a})$ one can define a min-max value $c(f;\alpha)\in[a,b]$ by

$$c(f;\alpha) := \inf\left\{t\in[a,b] \mid \alpha\in\operatorname{im}\left(H_*(f^{\leq t}, f^{<a})\to H_*(f^{\leq b}, f^{<a})\right)\right\}.$$

According to the Morse deformation lemma, this is a critical value of f. Let us assume that there exist a non-zero class $\alpha\in H_*(f^{\leq b}, f^{<a})$ and a non-zero cohomology class $v\in H^*(f^{\leq b})$ such that

$$c(f;\alpha\frown v) = c(f;\alpha)\in(a,b),$$

where $\alpha\frown v\in H_*(f^{\leq b}, f^{<a})$ denotes the cap-product of α by v (we remark that the inequality $c(f;\alpha\frown v)\leq c(f;\alpha)$ is always satisfied by naturality of the cap-product). Under this hypothesis, Lyusternik and Schnirelmann assert that for all neighborhood $U\subset f^{\leq b}$ of the set of critical points of f with value $c(f;\alpha)$, the restriction of v to U is a non-zero class. In particular, if $\deg v\geq 1$, then f has infinitely many critical points with value $c(f;\alpha)$. In particular, it implies that every C^1 map $f : M\to\mathbb{R}$ must have at least $CL(M)+1$ critical points, where the so-called cup-length $CL(M)\in\mathbb{N}$ is the maximum number of cohomology classes in $H^*(M)$ of degree ≥ 1 that when multiplied give a non-zero result.

3. Generating functions

Generating functions are a standard tool of Hamiltonian dynamics and symplectic topology. The modern theory particularly benefits from the seminal works of Sikorav [**70**], Viterbo [**79**] and Chekanov [**28**]. Here, we are closely following Théret [**75**]. A generating function for a Lagrangian submanifold of $T^*\mathbb{C}^n$ is a smooth function $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ such that 0 is a regular value of the $\mathbb{C}^k$-fiber derivative $\frac{\partial F}{\partial \xi}$. The space

$$\Sigma_F := \left\{ (q;\xi) \in \mathbb{C}^n \times \mathbb{C}^k \mid \frac{\partial F}{\partial \xi}(q;\xi) = 0 \right\} \tag{1.6}$$

is a smooth submanifold with dimension $2n$. Let $\iota_F : \Sigma_F \to T^*\mathbb{C}^n$ denote the map $\iota_F(q;\xi) := (q, \partial_q F(q;\xi))$. Then ι_F is a Lagrangian immersion and we say that F generates the immersed Lagrangian submanifold $L := \iota_F(\Sigma_F)$.

A conical generating function of $\mathbb{C}^{2n} \simeq T^*\mathbb{C}^n$ is a C^1 map $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ such that

1. F is S^1-invariant and 2-homogeneous, that is
$$F(\lambda\zeta) = |\lambda|^2 F(\zeta), \quad \forall \lambda \in \mathbb{C}, \forall \zeta \in \mathbb{C}^n \times \mathbb{C}^k,$$
2. F is smooth in the neighborhood of $\Sigma_F \setminus 0$ where the subset $\Sigma_F \subset \mathbb{C}^n \times \mathbb{C}^k$ is still defined by (1.6)
3. 0 is a regular value of the fiber derivative $\partial_\xi F$ on $\mathbb{C}^n \times \mathbb{C}^k \setminus 0$.

The set Σ_F is $\mathbb{C}$-invariant and so is $\widetilde{L} := \iota_F(\Sigma_F)$. If $\pi : \mathbb{C}^{2n} \setminus 0 \to \mathbb{C}\mathrm{P}^{2n-1}$ denotes the quotient map, then $L := \pi(\widetilde{L})$ is a smooth immersed Lagrangian of $\mathbb{C}\mathrm{P}^{2n-1}$. We will say that $\widetilde{L}$ is a conical immersed Lagrangian.

A quadratic generating function $Q : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{R}$ is a generating function which is also a quadratic form. In this case, the induced Lagrangian $\iota_Q(\Sigma_Q)$ is a linear Lagrangian subspace of $T^*\mathbb{C}^n$. Notice that if $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$ is a generating function of the Lagrangian $L \subset \mathbb{C}^{2n}$, then the quadratic form $\mathrm{d}^2F(x) : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{R}$, for $x \in \Sigma_F$, is a quadratic generating function of the tangent space $T_{\iota_F(x)}L \subset \mathbb{C}^{2n}$. The same is true if F is conical and $x \in \Sigma_F \setminus 0$. Moreover, $\mathbb{C}x \subset \ker \mathrm{d}^2F(x)$ in this case.

The existence of generating functions is well known for Lagrangians which are isotopic to the 0-section $\mathbb{C}^n \times \{0\}$ with a "suitably controlled" behavior at infinity (*e.g.* for a compactly supported isotopy or for a linear isotopy). In fact, we usually find a generating family of a whole isotopy $(L_t) := (\Phi_t(\mathbb{C}^n \times \{0\}))$, where (Φ_t) is a Hamiltonian flow, that is a continuous family (F_t) of generating functions with F_t generating L_t for all $t \in [0,1]$. In Section 2, we give a construction of generating families for Hamiltonian flows.

Let $\mathrm{Ham}(\mathbb{C}^d)$ be the set of Hamiltonian diffeomorphisms of $\mathbb{C}^d \simeq T^*\mathbb{R}^d$. The map

$$\tau : \overline{\mathbb{C}^d} \times \mathbb{C}^d \to \mathbb{C}^{2d}, \quad \tau(z,Z) = \left(\frac{z+Z}{2}, i(z-Z) \right), \tag{1.7}$$

is a $\mathbb{C}$-linear symplectomorphism sending the diagonal $\{(z,z) \mid z \in \mathbb{C}^d\}$ to the 0-section of $\mathbb{C}^{2d}$. Let $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$, the image of the graph $z \mapsto (z, \Phi(z))$ of Φ under τ is then a Lagrangian submanifold $L_\Phi \subset \mathbb{C}^{2d}$. A generating function of the Hamiltonian diffeomorphism Φ is a generating function of L_Φ. In other words, if the generating function F is generating the Hamiltonian diffeomorphism Φ of $\mathbb{C}^{d+1}$ then

$$\forall z \in \mathbb{C}^{d+1}, \exists!(x;\xi) \in \Sigma_F, \quad x = \frac{z + \Phi(z)}{2} \quad \text{and} \quad \partial_x F(x;\xi) = i(z - \Phi(z)).$$

The critical points of a generating function of Φ are in one-to-one correspondence with the fixed points of Φ, the bijection being $(x;\xi) \mapsto x$. Given generating functions $F : \mathbb{C}^{d+1} \times \mathbb{C}^k \to \mathbb{R}$ and $G : \mathbb{C}^{d+1} \times \mathbb{C}^l \to \mathbb{R}$ of Φ and Ψ respectively, the fiberwise sum of F and G denotes the map

$$(F+G)(x;\xi,\eta) := F(x;\xi) + G(x;\eta). \tag{1.8}$$

Although this is not a generating function of $\Phi \circ \Psi$, the critical points of $F + G$ are also in bijection with the fixed points of $\Phi \circ \Psi$ *via* $(x;\xi,\eta) \mapsto x - i\partial_x G(x;\eta)/2$ (these statements are easy consequences of the definitions). A generating family of a Hamiltonian flow (Φ_t) is a generating family of (L_{Φ_t}).

Let Φ be a conical Hamiltonian diffeomorphism of $\mathbb{C}^d$, that is a homeomorphism $\Phi : \mathbb{C}^d \to \mathbb{C}^d$ with $\Phi|_{\mathbb{C}^d\setminus 0} \in \mathrm{Ham}(\mathbb{C}^d \setminus 0)$ and which is $\mathbb{C}$-equivariant:

$$\Phi(\lambda z) = \lambda\Phi(z), \quad \forall\lambda \in \mathbb{C}, \forall z \in \mathbb{C}^d.$$

To simplify notation, we will write $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^d)$ and say that Φ is a $\mathbb{C}$-equivariant Hamiltonian diffeomorphism. The last definition extends to Hamiltonian flows in an obvious way. Then the induced subset $L_\Phi \subset \mathbb{C}^{2d}$ is a conical Lagrangian. A conical generating function of Φ (or simply a generating function of Φ) is a conical generating function of L_Φ. It extends to conical flows in an obvious way. As a consequence of the general case, if F is a generating function of $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$ and $(z;\xi) \neq 0$ is a critical point of F then z is a fixed point of Φ and $\mathrm{d}^2F(z;\xi)$ is a quadratic generating function of $\mathrm{d}\Phi(z)$. Moreover,

$$\dim\ker \mathrm{d}^2F(z;\xi) = \dim\ker(\mathrm{d}\Phi(z) - \mathrm{id}).$$

4. Maslov Index

4.1. Maslov index of a path in Sp$(2d)$. Let $\Gamma = (\Gamma_t) : [0,1] \to \mathrm{Sp}(2d)$ be a continuous path in the space of symplectic matrices $\mathrm{Sp}(2d)$ of $\mathbb{R}^{2d} \simeq \mathbb{C}^d$. Then there exists a continuous family (Q_t) of quadratic generating functions such that, for $t \in [0,1]$, $Q_t : \mathbb{C}^N \to \mathbb{R}$ is generating Γ_t. The variation of index $\mathrm{ind}(Q_1) - \mathrm{ind}(Q_0) \in \mathbb{Z}$ is independent of the choice of (Q_t) and is called the Maslov index of Γ denoted

$$\mathrm{mas}((\Gamma_t)) := \mathrm{ind}(Q_1) - \mathrm{ind}(Q_0) \in \mathbb{Z}.$$

Other equivalent definitions of the Maslov index (which is sometimes also called Conley-Zehnder index) are available in the literature, see [**67**], [**56**] and references therein.

In order to state the general properties of mas, following Théret, in this section we will denote by $R \bullet S$ the concatenation of two paths $R = (R_t)$ and $S = (S_t)$ in $\mathrm{Sp}(2d)$ satisfying $R_1 = S_0$, that is $(R \bullet S)_t = R_{2t}$ for $t \in [0,1/2]$ and $(R \bullet S)_t = S_{2t-1}$ for $t \in [1/2,1]$. The path RS stands for the pointwise matrix product of two paths in $\mathrm{Sp}(2d)$ that is $(RS)_t = R_tS_t$ for all t. Given a path $R = (R_t)$ in $\mathrm{Sp}(2d)$, the path $R^{(-1)}$ will stand for the reverse path (R_{1-t}), whereas R^{-1} will stand for the path of inverses (R_t^{-1}). Identifying matrices with their canonical linear maps, for two square matrices A and B, $A \oplus B$ will stand for the square matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and given two paths $R = (R_t)$ and $S = (S_t)$ in $\mathrm{Sp}(2n)$ and $\mathrm{Sp}(2m)$ respectively, $(R\oplus S)_t := (R_t\oplus S_t)$ as a path in $\mathrm{Sp}(2(n+m))$. We recall the basic properties of the Maslov index (see for instance [**74**, Prop. 39 and 58]).

Proposition 1.1. *Let R be a path in* $\mathrm{Sp}(2n)$*,*

(1) if S is a path in $\mathrm{Sp}(2n)$ with $S_0 = R_1$, then $\mathrm{mas}(R \bullet S) = \mathrm{mas}(R) + \mathrm{mas}(S)$,
(2) the Maslov index of the reverse path is $\mathrm{mas}(R^{(-1)}) = -\mathrm{mas}(R)$,
(3) if S is a path in $\mathrm{Sp}(2m)$, then $\mathrm{mas}(R \oplus S) = \mathrm{mas}(R) + \mathrm{mas}(S)$,
(4) if $A \in \mathrm{Sp}(2d)$, then $\mathrm{mas}(ARA^{-1}) = \mathrm{mas}(R)$.
(5) if S is a path homotopic to R relative to endpoints, that is there exists a continuous family $s \mapsto R^s$ of paths in $\mathrm{Sp}(2n)$ with $R^0 = R$ and $R^1 = S$ such that $R^s_0 \equiv R^0_0$ and $R^s_1 \equiv R^0_1$, then $\mathrm{mas}(S) = \mathrm{mas}(R)$,
(6) if $S_t := \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{bmatrix} \in \mathrm{Sp}(2)$, then $\mathrm{mas}(S_t, t \in [0,s]) = -2\lfloor s \rfloor$ for $s \geq 0$.

Let (Φ_t) be a Hamiltonian flow on $\mathbb{C}^d$ starting at $\Phi_0 = \mathrm{id}$. If $z \in \mathbb{C}^d$ is a fixed point of Φ_1, the Maslov index of z is set to be the Maslov index of the path $t \mapsto \mathrm{d}\Phi_t(z)$ in $\mathrm{Sp}(2d)$, that is

$$\mathrm{mas}(z, (\Phi_t)) := \mathrm{mas}((\mathrm{d}\Phi_t(z))).$$

Suppose that $F_t : \mathbb{C}^N \to \mathbb{C}$, $t \in [0,1]$, defines a continuous family of generating functions of (Φ_t). Let $\zeta_t \in \Sigma_{F_t} \subset \mathbb{C}^N$ be a continuous family associated with $\Phi_t(z)$. Then the continuous family of Hessians $Q_t := \mathrm{d}^2F_t(\zeta_t)$ is a continuous family of quadratic generating functions of $\mathrm{d}\Phi_t(z)$, thus

$$\mathrm{mas}(z, (\Phi_t)) = \mathrm{ind}(\zeta_1, F_1) - \mathrm{ind}(\zeta_0, F_0), \tag{1.9}$$

where $\mathrm{ind}(\zeta, F) := \mathrm{ind}(\mathrm{d}^2F(\zeta)) \in \mathbb{N}$ denotes the Morse index of F at the critical point ζ.

This definition is extended to every symplectic manifold M^{2d} as follows. Let (φ_t) be a Hamiltonian flow on M^{2d} starting at $\varphi_0 = \mathrm{id}$ and let $z \in M$ be a fixed point of φ_1 such that the loop $t \mapsto \varphi_t(z)$ is contractible. Let $D^2 := \{w \in \mathbb{C} \mid |w| \leq 1\}$ be the closed unit disk of $\mathbb{C}$. Since the loop is contractible, there exists a smooth map $u : D^2 \to M$ such that $u(e^{2i\pi t}) = \varphi_t(z)$. Then there exists a trivialization $D^2 \times \mathbb{C}^d \to u^*TM$, $(w, \zeta) \mapsto \xi(w)\zeta$ so that, for all $w \in D^2$, $\xi(w) : \mathbb{C}^d \to T_{u(w)}M$ is a symplectic map. Moreover, if we endow M with an almost complex structure, the trivialization can be made $\mathbb{C}$-linear. The set of every such trivialization is contractible, for a fixed choice of u (see [**67**, Lemma 5.1] for instance). Then $\gamma_t := \xi(e^{2i\pi t})^{-1}\mathrm{d}\varphi_t(z)\xi(1)$, $t \in [0,1]$, is a symplectic path in $\mathrm{Sp}(2d)$ and the Maslov index of z with respect to the *capping* u is set to be

$$\mathrm{mas}(z, u) := \mathrm{mas}((\gamma_t)).$$

It does not depend on the specific choice of trivialization, in fact it only depends on the homotopy class of u relative to the boundary ∂D^2. Thus, if $\pi_2(M) = 0$ any choice of u gives the same index.

4.2. Maslov index of a $\mathbb{C}$-equivariant Hamiltonian diffeomorphism. Let (Φ_t) be a $\mathbb{C}$-equivariant Hamiltonian flow on $\mathbb{C}^{d+1}$ lifting a Hamiltonian flow (φ_t) on $\mathbb{C}\mathrm{P}^d$. Let $Z_0 \in \mathbb{S}^{2d+1}$ be a fixed point of Φ_1 and let us denote by $Z_t := \Phi_t(Z_0)$, $t \in [0,1]$, the associated loop in $\mathbb{S}^{2d+1}$. Let $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ be the quotient map. Let $z_t := \pi(Z_t)$ be the associated loop in $\mathbb{C}\mathrm{P}^d$ so that $z_t = \varphi_t(z_0)$. Let $U : D^2 \to \mathbb{S}^{2d+1}$ be any smooth capping of (Z_t), *i.e.* $Z_t = U(e^{2i\pi t})$. All such cappings are homotopic since $\pi_2(\mathbb{S}^{2d+1}) = 0$. We set $u := \pi \circ U$.

Proposition 1.2. *With the above notation,*

$$\mathrm{mas}(Z_0, (\Phi_t)) = \mathrm{mas}(z_0, u).$$

Proof. For all $t \in [0,1]$, let $\gamma_t := \mathrm{d}\varphi_t(z_0) : T_{z_0}\mathbb{C}\mathrm{P}^d \to T_{z_t}\mathbb{C}P^d$ and $\Gamma_t := \mathrm{d}\Phi_t(Z_0)$ which is a path in $\mathrm{Sp}(2(d+1))$. For all $w \in D^2$, let $\xi(w) : \mathbb{C}^d \to T_{u(w)}\mathbb{C}\mathrm{P}^d$ be a smooth family of $\mathbb{C}$-linear symplectic maps induced by u as explained above. Throughout the proof, if f denotes a map whose domain is D^2, then, for $t \in [0,1]$, $f_t := f(e^{2i\pi t})$. For all $t \in [0,1]$ let $\xi_t := \xi(e^{2i\pi t})$ and $\gamma'_t := \xi_t^{-1}\gamma_t\xi_0 \in \mathrm{Sp}(2d)$ so that

$$\mathrm{mas}(Z_0, (\Phi_t)) = \mathrm{mas}((\Gamma_t)) \quad \text{and} \quad \mathrm{mas}(z_0, u) = \mathrm{mas}((\gamma'_t)).$$

Notice that, for all $Z \in \mathbb{S}^{2d+1}$, the tangent space $T_{\pi(Z)}\mathbb{C}\mathrm{P}^d \simeq \mathbb{C}^{d+1}/\mathbb{C}Z$ is canonically isomorphic to $(\mathbb{C}Z)^\perp$ (given a $\mathbb{C}$-subspace $E \subset \mathbb{C}^{d+1}$, $E^\perp$ denotes its hermitian orthogonal subspace, which is also its Euclidean orthogonal subspace or its symplectic orthogonal subspace). Let $L(w) := (\mathbb{C}U(w))^\perp \to T_{u(w)}\mathbb{C}\mathrm{P}^d$, $w \in D^2$, be the induced continuous family of $\mathbb{C}$-linear symplectic maps. Let us define the following continuous family of endomorphisms of $\mathbb{C}^{d+1}$ indexed by $w \in D^2$,

$$A(w) : \mathbb{C} \times \mathbb{C}^d \to \mathbb{C}U(w) \oplus (\mathbb{C}U(w))^\perp, \quad A(w)(\lambda, \zeta) = \lambda U(w) + L(w)^{-1}\xi(w)\zeta.$$

Since the linear maps $\lambda \mapsto \lambda U(w)$ and $L(w)^{-1}\xi(w)$ are symplectic maps and since both direct sums $\mathbb{C} \times \mathbb{C}^d$ and $\mathbb{C}U(w) \oplus (\mathbb{C}U(w))^\perp$ are symplectic-orthogonal sums, $A(w) \in \mathrm{Sp}(2(d+1))$.

Since Φ_t is a $\mathbb{C}$-equivariant diffeomorphism, the symplectic map $\mathrm{d}\Phi_t(Z_0) = \Gamma_t$ sends the orthogonal subspaces $\mathbb{C}Z_0$ and $(\mathbb{C}Z_0)^\perp$ respectively on $\mathbb{C}Z_t$ and $(\mathbb{C}Z_t)^\perp$ with

$$\Gamma_t(\lambda Z_0 + \zeta) = \lambda Z_t + L_t^{-1}\gamma_t L_0\zeta, \quad \forall \lambda \in \mathbb{C}, \forall \zeta \in (\mathbb{C}Z_0)^\perp,$$

where $L_t := L(e^{2i\pi t}) : (\mathbb{C}Z_t)^\perp \to T_{z_t}\mathbb{C}\mathrm{P}^d$. Thus $\Gamma'_t := A_t^{-1}\Gamma_t A_0$ is the symplectic path $\Gamma'_t = I_2 \oplus \gamma'_t$, so Proposition 1.1 (3) implies $\mathrm{mas}((\Gamma'_t)) = \mathrm{mas}((\gamma'_t))$. Since $A_t = A(e^{2i\pi t})$ with $A : D^2 \to \mathrm{Sp}(2(d+1))$ continuous, (Γ'_t) is homotopic to $(A_0^{-1}\Gamma_t A_0)$ relative to endpoints, thus $\mathrm{mas}((\Gamma'_t)) = \mathrm{mas}((A_0^{-1}\Gamma_t A_0)) = \mathrm{mas}((\Gamma_t))$, according to Proposition 1.1 (5) and (4). ☐

4.3. Bott's iteration inequalities. Let (Φ_t) be a Hamiltonian flow on $\mathbb{C}^d$ starting at $\Phi_0 = \mathrm{id}$ and let $z \in \mathbb{C}^d$ be a fixed point. Even though $(\Phi_t(z))$ is a loop in $\mathbb{C}^d$, $\Gamma_t := \mathrm{d}\Phi_t(z)$, $t \in \mathbb{R}_+$, defines only a path in $\mathrm{Sp}(2d)$, so that in general $\mathrm{mas}(\Gamma_{kt}, t \in [0,1]) \neq k\,\mathrm{mas}(\Gamma_t, t \in [0,1])$. Notice that the path $(\Gamma_t)_{t\in\mathbb{R}}$ only depends on $(\Gamma_t)_{t\in[0,1]}$ since $\Gamma_{t+k} = \Gamma_t\Gamma_1^k$ for $k \in \mathbb{N}$ and $t \geq 0$.

Theorem 1.3. *Let* $\Gamma := (\Gamma_t)_{t\geq 0}$ *be a continuous path in* $\mathrm{Sp}(2d)$ *such that* $\Gamma_0 = I_{2d}$ *and* $\Gamma_{t+k} = \Gamma_t\Gamma_1^k$ *for all* $k \in \mathbb{N}$ *and* $t > 0$. *Then the* average Maslov index

$$\overline{\mathrm{mas}}(\Gamma) := \lim_{k\to\infty} \frac{\mathrm{mas}(\Gamma_{kt}, t \in [0,1])}{k} \in \mathbb{R}$$

is a well-defined real number and we have the iteration inequalities

$$k\,\overline{\mathrm{mas}}(\Gamma) - d \leq \mathrm{mas}(\Gamma_{kt}, t \in [0,1]),$$
$$\mathrm{mas}(\Gamma_{kt}, t \in [0,1]) + \dim\ker(\Gamma_1^k - I_{2d}) \leq k\,\overline{\mathrm{mas}}(\Gamma) + d.$$

We refer to [**59**, Theorem 3.6] for a more precise statement and a proof. Notice that, by definition, the average Maslov index is homogeneous:

$$\overline{\mathrm{mas}}((\Gamma_{kt})) = k\,\overline{\mathrm{mas}}((\Gamma_t)).$$

Let us denote by $\overline{\mathrm{mas}}(z,(\Phi_t)) \in \mathbb{R}$ the average Maslov index of the fixed point z, that is

$$\overline{\mathrm{mas}}(z,(\Phi_t)) := \overline{\mathrm{mas}}(\mathrm{d}\Phi_t(z), t \geq 0).$$

So that Theorem 1.3 gives for all $k \in \mathbb{N}$,

$$k\,\overline{\mathrm{mas}}(z,(\Phi_t)) - d \leq \mathrm{mas}(z,(\Phi_{kt})),$$
$$\mathrm{mas}(z,(\Phi_{kt})) + \dim\ker(\mathrm{d}\Phi_k(z) - \mathrm{id}) \leq k\,\overline{\mathrm{mas}}(z,(\Phi_t)) + d.$$

This inequality can be extended to every symplectic manifold M^{2d} as follows. Let (φ_t) be a Hamiltonian flow on M^{2d} starting at $\varphi_0 = \mathrm{id}$ and let $z \in M$ be a fixed point of φ_1 such that $(\varphi_t(z))$ is contractible. Let $u : D^2 \to M$ be a capping of z and $\xi(w) : \mathbb{C}^d \to T_{u(w)}M$, $w \in D^2$, be an induced trivialization. For $k \in \mathbb{N}^*$, let $u^k : D^2 \to M$ be the smooth map $u^k(w) := u(w^k)$, $w \in D^2$. This map is the natural capping of z as a fixed point of the time-one map of the Hamiltonian flow (φ_{kt}) induced by (z,u). If $\bar{z} := (z,u)$, it is often denoted by $\bar{z}^k = (z,u^k)$. An induced trivialization is $\xi^k(w) := \xi(w^k)$, so that $\gamma_t^{(k)} = \gamma_{kt}$, where $\gamma_t^{(k)} := \xi^k(e^{2i\pi t})^{-1}\mathrm{d}\varphi_{kt}(z)\xi^k(1)$, and $\gamma_t := \xi(e^{2i\pi t})^{-1}\mathrm{d}\varphi_t(z)\xi(1)$, $t \geq 1$. Since $\mathrm{mas}(\bar{z}^k) := \mathrm{mas}(\gamma_t^{(k)}, t \in [0,1])$ with $\gamma_{t+k} = \gamma_t\gamma_1^k$ for all $k \in \mathbb{N}$ and $t \geq 0$, Theorem 1.3 gives for all $k \in \mathbb{N}$,

$$\begin{aligned} &k\,\overline{\mathrm{mas}}(\bar{z}) - d \leq \mathrm{mas}(\bar{z}^k), \\ &\mathrm{mas}(\bar{z}^k) + \dim\ker(\mathrm{d}\varphi_k(z) - \mathrm{id}) \leq k\,\overline{\mathrm{mas}}(\bar{z}) + d. \end{aligned} \tag{1.10}$$

where $\overline{\mathrm{mas}}(\bar{z}) := \overline{\mathrm{mas}}(\gamma_t, t \geq 0)$ is the average Maslov index of the capped fixed point $\bar{z} = (z,u)$. Let (Φ_t) be a $\mathbb{C}$-equivariant Hamiltonian flow of $\mathbb{C}^{d+1}$ with $\Phi_0 = \mathrm{id}$ which is the lift of a Hamiltonian flow (φ_t) of $\mathbb{C}\mathrm{P}^d$ with $\varphi_0 = \mathrm{id}$. Let $Z \in \mathbb{S}^{2d+1}$ be a fixed point of Φ_1 and $\bar{z} = (\pi(Z),u)$ be the capped fixed point of φ_1 associated with it, then

$$\mathrm{mas}(Z,(\Phi_{kt})) = \mathrm{mas}(\bar{z}^k), \quad \forall k \in \mathbb{N}.$$

Indeed, if $U : D^2 \to \mathbb{S}^{2d+1}$ is a capping of Z so that $u = \pi \circ U$, then U^k is a capping of Z relative to $(\Phi_{kt})_{t\in[0,1]}$ and $u^k = \pi \circ U^k$ (recall that $\mathrm{mas}(Z,(\Phi_{kt}))$ does not depend on the choice of capping since $\pi_2(\mathbb{S}^{2d+1}) = 0$). So, according to equation (1.10), for every fixed point $Z \in \mathbb{S}^{2d+1}$ of any $\mathbb{C}$-equivariant Hamiltonian flow (Φ_t) of $\mathbb{C}^{d+1}$ which is the lift of some Hamiltonian flow (φ_t) of $\mathbb{C}\mathrm{P}^d$, for all $k \in \mathbb{N}$,

$$\begin{aligned} &k\,\overline{\mathrm{mas}}(Z,(\Phi_t)) - d \leq \mathrm{mas}(Z,(\Phi_{kt})), \\ &\mathrm{mas}(Z,(\Phi_{kt})) + \dim\ker(\mathrm{d}\varphi_k(z) - \mathrm{id}) \leq k\,\overline{\mathrm{mas}}(Z,(\Phi_t)) + d, \end{aligned} \tag{1.11}$$

where $z := \pi(Z)$.

CHAPTER 2

Projective join

In [**43**, Appendix], Givental studied the cohomology of projective joins by using S^1-equivariant cohomology.

In this chapter, we study the homology properties of the projective join of subsets of the complex projective space (although this study could easily be extended to projective spaces over other algebras). These properties will be key to the study of the sublevel sets of generating functions associated with Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$.

Let $m, n \in \mathbb{N}$ and let $\pi : \mathbb{C}^{m+n+2} \setminus 0 \to \mathbb{C}\mathrm{P}^{m+n+1}$ be the quotient projection. We projectively embed $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ by identifying $\mathbb{C}\mathrm{P}^m$ with $\pi((\mathbb{C}^{m+1} \times 0) \setminus 0)$ and $\mathbb{C}\mathrm{P}^n$ with $\pi((0 \times \mathbb{C}^{n+1}) \setminus 0)$ so that $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^m$ do not intersect. This is equivalent to considering two projective subspaces of respective $\mathbb{C}$-dimensions m and n in general position. Let $A \subset \mathbb{C}\mathrm{P}^m$ and $B \subset \mathbb{C}\mathrm{P}^n$. The *projective join* $A * B \subset \mathbb{C}\mathrm{P}^{m+n+1}$ is the union of every projective lines intersecting A and B. In other words, $A * B = A \cup B \cup \pi(\widetilde{A} \times \widetilde{B})$ where $\widetilde{A}$ and $\widetilde{B}$ are the lifts of A and B to $\mathbb{C}^{m+1} \setminus 0$ and $\mathbb{C}^{n+1} \setminus 0$ respectively.

In Section 1, we study the homology of a projective stabilisation of A — that is a projective join of the form $A * \mathbb{C}\mathrm{P}^n$ — and its relationship with the homology of A. This elementary study essentially boils down to the Thom isomorphism. In Section 2, we introduce a homology product that mimics the projective join at the level of homology classes. This study is a bit more involved and allows us to extend some properties of the stabilisation of a subset to the join of two arbitrary subsets.

1. Homology of a projective stabilisation

Let us notice that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$ and that if $[a:b] \in \mathbb{C}\mathrm{P}^{m+n+1}$, with $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$, does not belong to $\mathbb{C}\mathrm{P}^m$ nor to $\mathbb{C}\mathrm{P}^n$, then only one projective line intersecting $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ contains $[a:b]$, namely the line joining $\alpha := [a:0]$ to $\beta := [0:b]$ denoted by $(\alpha\beta)$. Given $A \subset \mathbb{C}\mathrm{P}^m$, we denote by $p_A : A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n \to A$ the projection $[a:b] \mapsto [a:0]$.

Given $A \subset \mathbb{C}\mathrm{P}^m$, let $T \subset A * \mathbb{C}\mathrm{P}^n$ be a tubular neighborhood of $\mathbb{C}\mathrm{P}^m$ such that $(A * \mathbb{C}\mathrm{P}^n, T)$ retracts on $(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. By excision $H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \simeq H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, T \setminus \mathbb{C}\mathrm{P}^n)$. Using this identification, we define the cup-product $H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ by the following commutative diagram

$$\begin{tikzcd}
H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \arrow[r, "\smile"] \arrow[d, "\simeq"] & H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \arrow[d, "\simeq"] \\
H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, T) & H^*(A * \mathbb{C}\mathrm{P}^n, T) \\
H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, T \setminus \mathbb{C}\mathrm{P}^n) \arrow[r, "\smile"] \arrow[u, "\simeq"'] & H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, T \setminus \mathbb{C}\mathrm{P}^n) \arrow[u, "\simeq"']
\end{tikzcd},$$

where the vertical arrows are induced by inclusions and the bottom arrow is the usual cup-product. This cup-product makes the following diagram commute:

$$\begin{tikzcd}
H^*(A * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \arrow[r, "\smile"] & H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \\
H^*(A * \mathbb{C}\mathrm{P}^n) \otimes H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \arrow[u] \arrow[ur, "\smile"'] &
\end{tikzcd}, \tag{2.1}$$

where the vertical arrow is induced by inclusion and the diagonal arrow is the usual cup-product. In a dual way, we define a cap-product making the following diagram commute:

$$\begin{tikzcd} H_*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\times H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) \arrow[r, "\frown"] \arrow[dr, "\frown"'] & H_*(A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n) \arrow[d] \\ & H_*(A*\mathbb{C}\mathrm{P}^n) \end{tikzcd},$$

where the vertical arrow is induced by inclusion and the diagonal arrow is the usual cap-product.

According to the long exact sequence of the couple $(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$, the morphism

$$H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)\to H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1})$$

induced by the inclusion is an isomorphism (the dimension of $\mathbb{C}\mathrm{P}^n$ being $2n<2n+1$) so that we can see the class $u^{n+1}\in H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1})$ in $H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$ *via* this identification. Given $A\subset\mathbb{C}\mathrm{P}^m$, let $t_A\in H^{2(n+1)}(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$ be the image of $u^{n+1}\in H^{2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1},\mathbb{C}\mathrm{P}^n)$ induced by the inclusion and let $f_A : H^*(A)\to H^{*+2(n+1)}(A*\mathbb{C}\mathrm{P}^n)$ be the morphism given by $f_A(v):=p_A^*(v)\smile t_A$. We define dually $f^A : H_*(A*\mathbb{C}\mathrm{P}^n)\to H_{*-2(n+1)}$ by $f^A(\alpha):=(p_A)_*(\alpha\frown t_A)$.

Proposition 2.1. *Let $A\subset\mathbb{C}\mathrm{P}^m$. One has the following isomorphisms:*

$$H^k(A*\mathbb{C}\mathrm{P}^n)\simeq\begin{cases}H^k(\mathbb{C}\mathrm{P}^n) & \text{for } k\le 2n+1,\\ H^{k-2(n+1)}(A) & \text{for } k>2n+1,\end{cases}$$

where the isomorphisms $H^k(A*\mathbb{C}\mathrm{P}^n)\to H^k(\mathbb{C}\mathrm{P}^n)$ are induced by the inclusion and the isomorphisms $H^{k-2(n+1)}(A)\to H^k(A*\mathbb{C}\mathrm{P}^n)$ are given by f_A. The dual statement for homology is also true.

Proof. Let us consider the long exact sequence of the couple $(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$:

$$\cdots\to H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\xrightarrow{j^*}H^*(A*\mathbb{C}\mathrm{P}^n)\xrightarrow{i^*}H^*(\mathbb{C}\mathrm{P}^n)\to\cdots \tag{2.2}$$

The inclusions of $A*\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ give the following commutative diagram:

$$\begin{tikzcd} H^*(A*\mathbb{C}\mathrm{P}^n) \arrow[r, "i^*"] & H^*(\mathbb{C}\mathrm{P}^n) \\ H^*(\mathbb{C}\mathrm{P}^{m+n+1}) \arrow[u] \arrow[ur] & \end{tikzcd}$$

where the diagonal arrow is onto (we recall that $\mathbb{C}\mathrm{P}^n$ is projectively embedded inside $\mathbb{C}\mathrm{P}^{m+n+1}$), thus i^* is onto. Hence the long exact sequence (2.2) can be reduced to the short exact sequence

$$0\to H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\xrightarrow{j^*}H^*(A*\mathbb{C}\mathrm{P}^n)\xrightarrow{i^*}H^*(\mathbb{C}\mathrm{P}^n)\to 0. \tag{2.3}$$

Let us consider $p_A : A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n\to A$. This projection defines a complex vector bundle of dimension $n+1$. Indeed, let $E_A:=A*\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^n$ and $U_i\subset\mathbb{C}\mathrm{P}^{m+n+1}$ be the affine chart $\{[a_0:\cdots:a_m:z_0:\cdots:z_n]\mid a_i\neq 0\}$. Since the intersection of a projective line with the projective hyperplane $\mathbb{C}\mathrm{P}^{m+n+1}\setminus U_i$ is either a point or the projective line itself, we see that $p_A^{-1}(A\cap U_i)=E_A\cap U_i$. We then have the trivialization $E_A\cap U_i\simeq A\cap U_i\times\mathbb{C}^{n+1}$ given by $[a:z]\mapsto([a],z/a_i)$. Thus E_A is a fiber bundle, moreover this is the restriction of $E_{\mathbb{C}\mathrm{P}^m}$ to A. We can even say that $E_{\mathbb{C}\mathrm{P}^m}\simeq(\gamma_m^1)^{\oplus(n+1)}$ where γ_m^1 is the tautological fiber bundle of $\mathbb{C}\mathrm{P}^m$, by looking at the transition maps of the above trivialization charts (but this will not be relevant for us). Let us endow $\mathbb{C}\mathrm{P}^{m+n+1}$ with the Riemannian metric induced by the round metric of $\mathbb{S}^{2(m+n)+3}$ and let $T\subset A*\mathbb{C}\mathrm{P}^n$ be the tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ defined as the set of points at distance less than $r\in(0,\pi/2)$ of $\mathbb{C}\mathrm{P}^n$. Then the topological pair $(A*\mathbb{C}\mathrm{P}^n,T)$ retracts on $(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$ so that the inclusion map induces an isomorphism $H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\simeq H^*(A*\mathbb{C}\mathrm{P}^n,T)$ in cohomology. Since the compact $\mathbb{C}\mathrm{P}^n$ is included in the interior of T, by excision $H^*(A*\mathbb{C}\mathrm{P}^n,T)\simeq H^*(E_A,T\cap E_A)$. In the trivialization charts, each fibers of $E_A\setminus T$ is a round ball of $\mathbb{C}^{n+1}$ so that $(E_A,E_A\setminus A)$ retracts on $(E_A,T\cap E_A)$. According to Thom isomorphism theorem,

$$H^{*-2(n+1)}(A)\simeq H^*(E_A,E_A\setminus A)\simeq H^*(A*\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n),$$

where the isomorphism $H^{*-2(n+1)}(A) \to H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is given by the cup-product of the pull-back of the class by p_A with the Thom class $t'_A \in H^{2(n+1)}(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. Furthermore, since $H^k(\mathbb{C}\mathrm{P}^n)$ is zero when $k > 2n$ and $H^k(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is zero when $k < 2(n+1)$, the short exact sequence (2.3) obviously decomposes: $H^*(A * \mathbb{C}\mathrm{P}^n) \simeq H^*(A * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \oplus H^*(\mathbb{C}\mathrm{P}^n)$.

Since E_A is the restriction of $E_{\mathbb{C}\mathrm{P}^m}$, the Thom class t'_A is the image of the Thom class $t'_{\mathbb{C}\mathrm{P}^m}$ under the morphism induced by inclusion. Since j^* must be an isomorphism in degree $2(n+1)$ in the exact sequence (2.3) for $A = \mathbb{C}\mathrm{P}^m$, we must have $t'_{\mathbb{C}\mathrm{P}^m} = \pm u^{n+1}$ (recall that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$). In fact $t'_{\mathbb{C}\mathrm{P}^m} = u^{n+1} = t_{\mathbb{C}\mathrm{P}^m}$ as the orientation of a complex fiber $\simeq \mathbb{C}^{n+1}$ coincides with the orientation of a projective subspace of $\mathbb{C}$-dimension $n+1$ (they all come from the complex structure of $\mathbb{C}\mathrm{P}^{m+n+1}$). $\square$

Given a topological pair $A = (A_1, A_0)$ included in $\mathbb{C}\mathrm{P}^m$, one can extend the above extensions of the cup-product and cap-product to maps:

$$H^*(A_1 * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n) \otimes H^*(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n) \xrightarrow{\smile} H^*(A_1 * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n),$$
$$H_*(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n) \times H^*(A_1 * \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \xrightarrow{\frown} H_*(A_1 * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^n).$$

In order to avoid such lengthy notation, we denote $A * \mathbb{C}\mathrm{P}^n := (A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n)$ so that these new maps satisfy the same formal properties as the former. We also extend the definition of $t_A \in H^{2(n+1)}(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n)$, $f_A : H^*(A) \to H^{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n)$ and f^A. Let us remark that the compatibility of the cup-products (2.1) implies that the following diagram commutes:

$$\begin{tikzcd} H^*(A) \arrow[r, "f_A"] & H^{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \\ & H^*(A * \mathbb{C}\mathrm{P}^n) \arrow[ul] \arrow[u, "\cdot \smile u^{n+1}"'] \end{tikzcd},$$

where the diagonal arrow is induced by inclusion. Dually,

(2.4)
$$\begin{tikzcd} H_{*-2(n+1)}(A) \arrow[dr] & H_*(A * \mathbb{C}\mathrm{P}^n) \arrow[l, "f^A"'] \arrow[d, "\cdot \frown u^{n+1}"] \\ & H_{*-2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \end{tikzcd}.$$

An application of the five lemma gives the following corollary to Proposition 2.1.

Corollary 2.2. *Let (A_1, A_0) be a topological pair included in $\mathbb{C}\mathrm{P}^m$ with $A_0 \neq \varnothing$, the map $f_A : v \mapsto p^*_{A_1}(v) \smile t_A$ gives an isomorphism*
$$H^*(A_1, A_0) \to H^{*+2(n+1)}(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n).$$
Dually, the map $\alpha \mapsto (p_{A_1})_*(\alpha \frown t_A)$ gives an isomorphism
$$H_*(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n) \to H_{*-2(n+1)}(A_1, A_0).$$

Following Givental, we define $\ell(A) \in \mathbb{N}$ for $A \subset \mathbb{C}\mathrm{P}^N$ as the rank of the morphism $H^*(\mathbb{C}\mathrm{P}^N) \to H^*(A)$ induced by the inclusion (*e.g.* $\ell(\mathbb{C}\mathrm{P}^n) = n+1$). The integer $\ell(A)$ is also the maximal $k \in \mathbb{N}$ such that the restriction of u^{k-1} to $H^*(A)$ is non-zero. This definition coincides with the equivariant cohomological index defined by Fadell and Rabinowitz [**32**] (in the special case of the free action of S^1 on $\mathbb{S}^{2N+1}$).

Corollary 2.3. *Let $A \subset \mathbb{C}\mathrm{P}^m$, then $\ell(A * \mathbb{C}\mathrm{P}^n) = \ell(A) + n + 1$.*

Proof. Since $f_{\mathbb{C}\mathrm{P}^m}(u^k) = u^{n+1} \smile u^k$ for $0 \leq k \leq m$, we have the following commutative diagram:

$$\begin{tikzcd} H^*(\mathbb{C}\mathrm{P}^{m+n+1}) \arrow[r, "u^{n+1} \smile \cdot"] \arrow[d] & H^{*+2(n+1)}(\mathbb{C}\mathrm{P}^{m+n+1}) \arrow[d] \\ H^*(A) \arrow[r, "f_A"] & H^{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \end{tikzcd}$$

where the vertical arrows are induced by inclusions. For the grading $* = 2(\ell(A * \mathbb{C}\mathrm{P}^n) - n - 1)$, the map $u^{n+1} \smile \cdot$ is onto, so $\ell(A * \mathbb{C}\mathrm{P}^n) \leq \ell(A) + n + 1$. According to Proposition 2.1, the map f_A is an injection for the grading $* = 2\ell(A)$, so $\ell(A * \mathbb{C}\mathrm{P}^n) \geq \ell(A) + n + 1$. □

2. Homology projective join

We describe an operation on the homology of subsets of a projective space relating the homology of two subsets A and B to the homology of their projective join $A * B$. The analogous operation for the topological join was already defined by Whitehead in [**81**] and used by Granja-Karshon-Pabiniak-Sandon in [**44**] for a purpose similar to ours. However, the projective join of two simplices is not a simplex and one cannot extend this construction so easily.

Since the proof of some fundamental properties of this operation are only technical and does not shed much light on their applications, we have put these proofs in a specific section. More precisely, proofs of Proposition 2.6 and 2.8 are postponed to Section 2.2

2.1. Definition and properties. Let $m, n \in \mathbb{N}$ and let $\pi : \mathbb{C}^{m+n+2} \setminus 0 \to \mathbb{C}\mathrm{P}^{m+n+1}$ be the quotient map. We projectively endow $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ by identifying $\mathbb{C}\mathrm{P}^m$ with $\pi(\mathbb{C}^{m+1} \times 0 \setminus 0)$ and $\mathbb{C}\mathrm{P}^n$ with $\pi(0 \times \mathbb{C}^{n+1} \setminus 0)$ so that $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ do not intersect. This is equivalent to considering two projective subspaces of respective $\mathbb{C}$-dimension m and n in general position. Let $A \subset \mathbb{C}\mathrm{P}^m$ and $B \subset \mathbb{C}\mathrm{P}^n$ be non-empty sets. Then the projective join $A * B \subset \mathbb{C}\mathrm{P}^{m+n+1}$ is the union of every projective line intersecting A and B. In other words, $A * B = A \cup B \cup \pi(\widetilde{A} \times \widetilde{B})$ where $\widetilde{A}$ and $\widetilde{B}$ are the lifts of A and B to $\mathbb{C}^{n+1} \setminus 0$ and $\mathbb{C}^{m+1} \setminus 0$ respectively. One can remark that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$ and that if $[a : b] \in \mathbb{C}\mathrm{P}^{m+n+1}$, with $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$, does not belong to $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$, then only one projective line intersecting these two subspaces contains $[a : b]$, namely the line joining $\alpha := [a : 0]$ to $\beta := [0 : b]$ denoted by $(\alpha\beta)$.

We need to define a projective join in the level of homology which would be a map $\mathrm{pj}_* : H_*(A \times B) \to H_{*+2}(A * B)$, or dually in the level of cohomology $\mathrm{pj}^* : H^*(A * B) \to H^{*+2}(A \times B)$. One can perhaps proceed directly by defining a projective join at the level of chains, sending two chains $\alpha \in C_i(A)$ and $\beta \in C_j(B)$ to a chain $\alpha * \beta \in C_{i+j+2}(A * B)$ that triangulates the projective join of their images. We will proceed in an indirect way by remaining in the level of homology.

Let $E_{A,B} \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^{m+n+1}$ be the set

$$E_{A,B} := \{(a, b, c) \in A \times B \times (A * B) \mid c \in (ab)\}.$$

Let p_1 and p_2 be the canonical projection of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^{m+n+1}$ on the factor $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^{m+n+1}$ respectively. Then $p_1|_{E_{A,B}}$ defines a $\mathbb{C}\mathrm{P}^1$-fiber bundle on $A \times B$, the fiber of any $(a, b) \in A \times B$ being $a \times b \times (ab) \simeq (ab)$. As $\mathbb{C}\mathrm{P}^1$ can be identified with the 2-sphere $\mathbb{S}^2$, the Gysin long exact sequence holds:

$$\cdots \xrightarrow{\cdot \smile e} H^*(A \times B) \xrightarrow{(p_1)^*} H^*(E_{A,B}) \xrightarrow{(p_1)_*} H^{*-2}(A \times B) \xrightarrow{\cdot \smile e} \cdots, \tag{2.5}$$

where $e \in H^3(A \times B)$ denotes the Euler class of the $\mathbb{S}^2$-bundle $E_{A,B}$.

Définition 2.4. The cohomology projective join $\mathrm{pj}^* : H^*(A * B) \to H^{*-2}(A \times B)$ denotes the map $\mathrm{pj}^* := (p_1)_* \circ (p_2)^*$, where $(p_1)_*$ is defined by (2.5) and $(p_2)^*$ is induced by $p_2 : E_{A,B} \to A * B$. The homology projective join $\mathrm{pj}_* = (p_2)_* \circ (p_1)^* : H_*(A \times B) \to H_{*+2}(A * B)$ is defined dually. Given $\alpha \in H_*(A)$ and $\beta \in H_*(B)$ we denote $\alpha * \beta := \mathrm{pj}_*(\alpha \times \beta)$.

We extend this definition to topological pairs $(A, B) \subset \mathbb{C}\mathrm{P}^m$, $(C, D) \subset \mathbb{C}\mathrm{P}^n$ the following way. Let $(A, B) * (C, D) := (A * C, A * D \cup B * C)$, the map p_1 defines a relative $\mathbb{C}\mathrm{P}^1$-fiber bundle $(E_{A,C}, E_{A,D} \cup E_{B,C})$ on $(A, B) \times (C, D)$ while p_2 maps this bundle on $(A, B) * (C, D)$. Hence, one can set $\mathrm{pj}^* := (p_1)_* \circ (p_2)^*$ and $\mathrm{pj}_* := (p_2)_* \circ (p_1)^*$ as before. By naturality of the maps induced by p_1 and p_2, this extension is natural: projective join commutes with long exact sequences of topological pairs or triples.

These maps are also natural in the following way: let $A, C \subset \mathbb{C}\mathrm{P}^m$ and $B, D \subset \mathbb{C}\mathrm{P}^n$ and assume that $f : A * B \to C * D$ is the restriction of a projective map satisfying $f|_A : A \to C$ and $f|_B : B \to D$, then the $\mathbb{C}\mathrm{P}^1$-fiber bundle $E_{A,B}$ is the pull-back of $E_{C,D}$ by $f|_A \times f|_B$, so that the following diagram commutes:

$$\begin{tikzcd} H_*(A\times B) \arrow[r, "\mathrm{pj}_*"] \arrow[d, "(f|_A\times f|_B)_*"] & H_{*+2}(A*B) \arrow[d, "f_*"] \\ H_*(C\times D) \arrow[r, "\mathrm{pj}_*"] & H_{*+2}(C*D) \end{tikzcd} \quad . \tag{2.6}$$

This statement extends to topological pairs in an obvious way.

PROPOSITION 2.5. *The homology projective join is associative: given A, B and C included in $\mathbb{C}\mathrm{P}^n$,*

$$\forall(\alpha,\beta,\gamma)\in H_*(A)\times H_*(B)\times H_*(C),\quad \mathrm{pj}_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma)=\mathrm{pj}_*(\alpha\times\mathrm{pj}_*(\beta\times\gamma)).$$

As R-algebras, one has

$$H^*(\mathbb{C}\mathrm{P}^{m+n+1})=R[u]/u^{m+n+2}\quad\text{and}\quad H^*(\mathbb{C}\mathrm{P}^m\times\mathbb{C}\mathrm{P}^n)=R[u_1,u_2]/(u_1^{m+1},u_2^{n+1})$$

where u, u_1 and u_2 restrict to orientation classes of $\mathbb{C}\mathrm{P}^1$ (with $\mathbb{C}\mathrm{P}^1\subset\mathbb{C}\mathrm{P}^m$ for u_1 and $\mathbb{C}\mathrm{P}^1\subset\mathbb{C}\mathrm{P}^n$ for u_2).

PROPOSITION 2.6. *Let pj^* be the cohomology projective join on $\mathbb{C}\mathrm{P}^m\times\mathbb{C}\mathrm{P}^n$, with the above notation one has*

$$\mathrm{pj}^*u^k=\sum_{i+j=k-1}u_1^iu_2^j,\quad\forall k\in\mathbb{N}^*.$$

Dually, the homology projective join pj_* on $\mathbb{C}\mathrm{P}^m\times\mathbb{C}\mathrm{P}^n$ satisfies

$$\mathrm{pj}_*\left([\mathbb{C}\mathrm{P}^i]\times[\mathbb{C}\mathrm{P}^j]\right)=[\mathbb{C}\mathrm{P}^{i+j+1}],\quad\forall i\in\{0,\dots,m\},\forall j\in\{0,\dots,n\}.$$

We recall that the cohomological length $\ell(A)$ of a subspace $A\subset\mathbb{C}\mathrm{P}^N$ is the rank of the morphism $H^*(\mathbb{C}\mathrm{P}^N;\mathbb{Z})\to H^*(A;\mathbb{Z})$ induced by the inclusion (*e.g.* $\ell(\mathbb{C}\mathrm{P}^n)=n+1$). This is also the rank of the morphism $H_*(A;\mathbb{Z})\to H_*(\mathbb{C}\mathrm{P}^N;\mathbb{Z})$.

Given two subsets $A\neq\varnothing$ and B as above, let $\ell:=\ell(B)$. The restriction of $u^\ell\in H^*(\mathbb{C}\mathrm{P}^{m+n+1})$ to $H^*(B)$ is zero and is non-zero in $H^*(A*B)$. Let $v_B\in H^*(A*B,B)$ be one of its inverse image. As we have seen in the special case $B=\mathbb{C}\mathrm{P}^n$, there is a well-defined cap-product

$$H_k(A*B,B)\times H^l(A*B,B)\xrightarrow{\frown}H_{k-l}(A*B\setminus B)$$

which is defined by the following commutative diagram:

$$\begin{tikzcd} H_*(A*B,B)\times H^*(A*B,B) \arrow[d, "\simeq"] \arrow[drr, "\frown"] & & \\ H_*(A*B,T)\times H^*(A*B,T) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & & H_*(A*B\setminus B) \\ H_*(A*B\setminus B,T\setminus B)\times H^*(A*B\setminus B,T\setminus B) \arrow[u, "\simeq"] \arrow[urr, "\frown"'] & & \end{tikzcd} ,$$

where $T\subset A*B$ is the restriction of a tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ to $A*B$, the bottom diagonal arrow is the usual cap-product and vertical maps are isomorphisms induced by inclusion maps (the isomorphisms come from retractions at the top and from excision at the bottom). Let $p_A:A*B\setminus B\to A$ be the map $p_A[a:b]:=[a:0]$. Let $f:H_*(A*B)\to H_{*-2\ell}(A)$ be the map $f(\alpha):=(p_A)_*(\alpha\frown v_B)$. These definitions extend to the case where A is a topological pair (A_1,A_0) with $A_1\neq A_0$ by taking $v_B\in H^{\ell(B)}(A_1*B,B)$ and by using the cap-product

$$H_k(A_1*B,A_0*B)\times H^l(A_1*B,B)\xrightarrow{\frown}H_{k-l}(A_1*B\setminus B,A_0*B\setminus B),$$

defined the same way as above.

COROLLARY 2.7 ([**43**, Corollary A.2]). *For all non-empty subsets $A\subset\mathbb{C}\mathrm{P}^m$ and $B\subset\mathbb{C}\mathrm{P}^n$, one has*

$$\ell(A*B)=\ell(A)+\ell(B).$$

Proof. Let $\alpha \in H_{2\ell(A)-2}(A)$ and $\beta \in H_{2\ell(B)-2}(B)$ be classes that are sent to the class $[\mathbb{C}\mathrm{P}^{\ell(A)-1}] \in H_*(\mathbb{C}\mathrm{P}^m)$ and $[\mathbb{C}\mathrm{P}^{\ell(B)-1}] \in H_*(\mathbb{C}\mathrm{P}^n)$ respectively. According to Proposition 2.6 and naturality (2.6), $\mathrm{pj}_*(\alpha \times \beta)$ is sent to $[\mathbb{C}\mathrm{P}^{\ell(A)+\ell(B)-1}]$ in $H_*(\mathbb{C}\mathrm{P}^{m+n+1})$. Hence $\ell(A * B) \geq \ell(A) + \ell(B)$. The converse inequality comes from the commutativity of the following diagram:

$$\begin{tikzcd}
H_*(A * B) \arrow[r, "f"] \arrow[d] & H_{*-2\ell(B)}(A) \arrow[d] \\
H_*(\mathbb{C}\mathrm{P}^{m+n+1}) \arrow[r, "\cdot\frown u^{\ell(B)}"] & H_{*-2\ell(B)}(\mathbb{C}\mathrm{P}^{m+n+1})
\end{tikzcd}.$$

□

Proposition 2.8. *Let $A \subset \mathbb{C}\mathrm{P}^m$, $B \subset \mathbb{C}\mathrm{P}^n$ be non-empty sets and $\ell := \ell(B)$. Let $\beta \in H_*(B)$ be a class that is sent to $[\mathbb{C}\mathrm{P}^{\ell-1}] \in H_*(\mathbb{C}\mathrm{P}^n)$. The following diagram commutes:*

$$\begin{tikzcd}
H_{*+2\ell-2}(A \times B) \arrow[r, "\mathrm{pj}_*"] & H_{*+2\ell}(A * B) \arrow[d, "f"] \\
H_*(A) \arrow[u, "\cdot\times\beta"] \arrow[r, "\mathrm{id}", "="'] & H_*(A)
\end{tikzcd},$$

where $f(\alpha) := (p_A)_*(\alpha \frown v_B)$ as defined above. This result also holds when A is a topological pair (A_1, A_0) with $A_1 \neq A_0$.

By applying Corollary 2.2 and the commutativity of (2.4), one gets that the following diagram commutes:

$$\begin{tikzcd}
H_*(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^n]", "\simeq"'] \arrow[dr] & H_{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \arrow[d, "\cdot\frown u^{n+1}"] \\
& H_*(A * \mathbb{C}\mathrm{P}^n)
\end{tikzcd}, \tag{2.7}$$

where $A = (A_1, A_0)$ with $A_0 \neq \varnothing$ and the diagonal arrow is the inclusion.

Lemma 2.9. *Let $A = (A_1, A_0) \subset \mathbb{C}\mathrm{P}^m$ with $A_0 \neq \varnothing$ and let $\ell \in \mathbb{N}$. Let us assume that there exist a topological pair B so that $B * \mathbb{C}\mathrm{P}^\ell \subset \mathbb{C}\mathrm{P}^m$ and a continuous map $A \to B * \mathbb{C}\mathrm{P}^\ell$ or $B * \mathbb{C}\mathrm{P}^\ell \to A$ that induces an isomorphism in homology and that admits a conical lift on $\mathbb{C}^{m+1} \setminus 0$. Then the following diagram commutes*

$$\begin{tikzcd}
H_*(A) \arrow[d, "\cdot\frown u^{\ell+1}"] \arrow[dr] & \\
H_{*-2(\ell+1)}(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^\ell]", "\simeq"'] & H_*(A * \mathbb{C}\mathrm{P}^\ell)
\end{tikzcd}.$$

Proof. The hypothesis on the homology isomorphism $H_*(A) \simeq H_*(B * \mathbb{C}\mathrm{P}^\ell)$ allows us to assume that $A = B * \mathbb{C}\mathrm{P}^\ell$. Indeed, suppose for instance $f : A \to B * \mathbb{C}\mathrm{P}^\ell$ is the continuous map inducing the isomorphism f_* and denote by $\tilde{f} : \pi^{-1}(A) \to \pi^{-1}(B * \mathbb{C}\mathrm{P}^\ell)$ one of its conical lift. Then one can define the map $f * \mathrm{id} : A * \mathbb{C}\mathrm{P}^\ell \to B * \mathbb{C}\mathrm{P}^{2\ell+1}$ by $(f * \mathrm{id})([a : b]) := [\tilde{f}(a) : b]$ and apply the naturality of the join (2.6) to show the equivalence between both statements.

Let us consider the following diagram:

$$\begin{tikzcd}
H_{*-2(\ell+1)}(B) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^\ell]", "\simeq"'] \arrow[dr] & H_*(B * \mathbb{C}\mathrm{P}^\ell) \arrow[d, "\cdot\frown u^{\ell+1}"] \arrow[dr] & \\
& H_{*+2(\ell+1)}(B * \mathbb{C}\mathrm{P}^\ell) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^\ell]", "\simeq"'] & H_*(B * \mathbb{C}\mathrm{P}^{2\ell+1})
\end{tikzcd}.$$

By naturality of the join (2.6), the parallelogram commutes while the left triangle commutes by (2.7). Hence the right triangle commutes. □

Corollary 2.10. *Let $A \subset \mathbb{C}\mathrm{P}^m$ and $\ell \leq n$ be following the assumption of Lemma 2.9. Then the following diagram commutes*

$$\begin{tikzcd} H_*(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^n]"] \arrow[d, "\cdot\frown u^{\ell+1}"] & H_{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \arrow[d, "\cdot\frown u^{\ell+1}"] \\ H_{*-2(\ell+1)}(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^n]"] & H_{*+2(n-\ell)}(A * \mathbb{C}\mathrm{P}^n) \end{tikzcd}$$

Proof. The case $n = \ell$ is true by commutativity of (2.7) and Lemma 2.9. In the case $n > \ell$, one can split the diagram of the statement into the following diagram:

$$\begin{tikzcd} H_*(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^{n-\ell-1}]"] \arrow[d, "\cdot\frown u^{\ell+1}"] \arrow[dr] & H_{*+2(n-\ell)}(A * \mathbb{C}\mathrm{P}^{n-\ell-1}) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^{\ell}]"] \arrow[dr] & H_{*+2(n+1)}(A * \mathbb{C}\mathrm{P}^n) \arrow[d, "\cdot\frown u^{\ell+1}"] \\ H_{*-2(\ell+1)}(A) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^{\ell}]"] & H_*(A * \mathbb{C}\mathrm{P}^{\ell}) \arrow[r, "\cdot*[\mathbb{C}\mathrm{P}^{n-\ell-1}]"] & H_{*+2(n-\ell)}(A * \mathbb{C}\mathrm{P}^n) \end{tikzcd}.$$

The left triangle commutes by Lemma 2.9, the parallelogram commutes by naturality (2.6) and the right triangle commutes according to (2.7). □

2.2. Technical proofs. We will denote $E_{\mathbb{C}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n}$ by $E_{m,n}$. The bundle $E_{A,B}$ is the restriction of the bundle $E_{m,n}$ to $A \times B$, hence e is the pullback of the Euler class of $E_{m,n}$ which lies in $H^3(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) = 0$. Therefore $e = 0$ and (2.5) reduces to the short exact sequence

$$0 \to H^*(A \times B) \xrightarrow{(p_1)^*} H^*(E_{A,B}) \xrightarrow{(p_1)_*} H^{*-2}(A \times B) \to 0. \tag{2.8}$$

Proof of Proposition 2.5. Let us first define a projective join with 3 entries $\mathrm{pj}^3_* : H_*(A \times B \times C) \to H_{*+4}(A * B * C)$ then prove that

$$\mathrm{pj}_*(\mathrm{pj}_*(\alpha \times \beta) \times \gamma) = \mathrm{pj}^3_*(\alpha \times \beta \times \gamma) = \mathrm{pj}_*(\alpha \times \mathrm{pj}_*(\beta \times \gamma)). \tag{2.9}$$

Given three points a, b and c of some projective space $\mathbb{C}\mathrm{P}^N$ that are projectively independent, we use the classical notation $(abc) \subset \mathbb{C}\mathrm{P}^N$ to denote the complex projective plane Let $E_{A,B,C} \subset (\mathbb{C}\mathrm{P}^n)^3 \times \mathbb{C}\mathrm{P}^{3n+2}$ be the set

$$E_{A,B,C} := \{(a, b, c, z) \in A \times B \times C \times (A * B * C) \mid z \in (abc)\}$$

and let $P_1 : E_{A,B,C} \to A \times B \times C$ and $P_2 : E_{A,B,C} \to A * B * C$ be the associated projection maps. The map P_1 defines a $\mathbb{C}\mathrm{P}^2$-fiber bundle which is the restriction of the fiber bundle $E_{\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n} \to (\mathbb{C}\mathrm{P}^n)^3$, so the action of $\pi_1(A \times B \times C)$ on the homology group $H_*(\mathbb{C}\mathrm{P}^2)$ of a fiber is the restriction of the action of $\pi_1((\mathbb{C}\mathrm{P}^n)^3) = 0$ *i.e.* trivial. We define pj^3_* by $\mathrm{pj}^3_* := (P_2)_* \circ (P_1)^*$ where $(P_1)^* : H_*(A \times B \times C) \to H_{*+4}(E_{A,B,C})$ denotes the morphism dual to the integration along the fiber of the fibration P_1 (the complex structure of $\mathbb{C}\mathrm{P}^2$ gives a natural identification $H_4(\mathbb{C}\mathrm{P}^2) \simeq R$). We refer to Appendix A for the definition and properties of this morphism.

In order to prove (2.9), let us introduce the set

$$E'_{A,B,C} := \{(a, b, c, x, z) \in A \times B \times C \times (A * B) \times (A * B * C) \mid x \in (ab) \text{ and } z \in (xc)\}$$

with the projection maps $P'_2 : E'_{A,B,C} \to A * B * C$, $P'_1 : E'_{A,B,C} \to E_{A,B} \times C$ sending (a, b, c, x, z) to $(a, b, x; c)$, $\tilde{f} : E'_{A,B,C} \to E_{A*B,C}$ sending (a, b, c, x, z) to (x, c, z) and $g : E'_{A,B,C} \to E_{A,B,C}$ sending (a, b, c, x, z) to (a, b, c, z). The map P'_1 is a $\mathbb{C}\mathrm{P}^1$-fiber bundle, in fact $\tilde{f}$ is a morphism of fiber bundle with base-space morphism $f := p_2 \times \mathrm{id}_C$. In order to summarize the situation, we have the following commutative diagram:

$$
\begin{tikzcd}
 & A * B * C & \\
E_{A,B,C} \arrow[ur, "P_2"] \arrow[ddr, "P_1"'] & E'_{A,B,C} \arrow[l, "g"] \arrow[u, "P'_2"] \arrow[r, "\tilde{f}"] \arrow[d, "P'_1"] & E_{A*B,C} \arrow[ul, "p'_2"'] \arrow[d, "p'_1"] \\
 & E_{A,B} \times C \arrow[r, "f"] \arrow[d, "p_1 \times \mathrm{id}_C"] & (A * B) \times C \\
 & A \times B \times C &
\end{tikzcd} \tag{2.10}
$$

According to the naturality of the integration along the fiber (A.1), it follows that $(p'_2)_*(p'_1)^* f_*(p_1 \times \mathrm{id}_C)^* = (P'_2)_*(P'_1)^*(p_1 \times \mathrm{id}_C)^*$, which means that

$$
\mathrm{pj}_*(\mathrm{pj}_*(\alpha \times \beta) \times \gamma) = (P'_2)_*(P'_1)^*(p_1 \times \mathrm{id}_C)^*(\alpha \times \beta \times \gamma). \tag{2.11}
$$

The map g commutes with the Serre fibration $q := (p_1 \times \mathrm{id}_C) \circ P'_1$ and the fiber bundle P_1. Let us fix a base-point $(a, b, c) \in A \times B \times C$, of fiber $F = a \times b \times c \times (abc) \simeq (abc)$ for P_1 and of fiber $F' \simeq \{(x, z) \mid x \in (ab) \text{ and } z \in (xc)\}$ for q. According to Proposition A.1 and the remark after it, in order to show that the following diagram commutes:

$$
\begin{tikzcd}
H_{*+4}(E_{A,B,C}) & H_{*+4}(E'_{A,B,C}) \arrow[l, "g_*"] \\
 & H_*(A \times B \times C) \arrow[ul, "P_1^*"] \arrow[u, "q^*"']
\end{tikzcd} \tag{2.12}
$$

(with coefficients of every H_* in the same ring R), one must prove that $g_* : H_4(F') \to H_4(F)$ commutes with the identity of R under the isomorphisms $H_4(F) \simeq R$ and $H_4(F') \simeq R$ given by the local complex orientation. This comes from the fact that the quotient space $F'/((ab) \times c)$ is canonically homeomorphic to $(abc) \simeq F$ (in particular, preserving the orientation), the homeomorphism being induced by $g|_{F'}$. The long exact sequence of the couple $(F', (ab) \times c)$ concludes. Therefore, diagram (2.12) commutes. Thus, according to the left hand side of the diagram (2.10) together with the composition property $q^* = (P'_1)^*(p_1 \times \mathrm{id}_C)^*$, one has $(P'_2)_*(P'_1)^*(p_1 \times \mathrm{id}_C)^* = (P_2)_*(P_1)^*$ and (2.11) gives the first equality of (2.9).

The second equality is proven in a symmetric way. □

Proof of Proposition 2.6. For now, let us work on $E_{m,n}$. By the universal coefficient theorem, it is enough to prove this proposition for the cohomology projective join over $\mathbb{Z}$. First, let us see that $\mathrm{pj}^* u = 1$. By naturality (2.6), it boils down to showing that $\mathrm{pj}^* : H^2((ab)) \to H^0(a \times b)$ maps the restriction of u to (ab) to $1 \in H^0(a \times b)$ for all $(a, b) \in \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$. Now $E_{a,b} = a \times b \times (ab)$ so that $(p_2)^*$ is an isomorphism sending the orientation class of (ab) to the orientation class of $E_{a,b}$. According to (2.8), $(p_1)_*$ is also an isomorphism (preserving the orientation), hence the result.

Let $u_0 := (p_2)^* u \in H^2(E_{m,n})$. We must now study $(p_1)_* u_0^k$ for $k \in \mathbb{N}^*$.

Let $T \subset \mathbb{C}\mathrm{P}^{m+n+1}$ be a tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ that is a deformation retract. Its pullback $p_2^{-1}(T)$ is a deformation retract of $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$, hence

$$
H^*\left(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) \simeq H^*\left(E_{m,n}, p_2^{-1}(T)\right) \simeq H^*\left(E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n), p_2^{-1}(T \setminus \mathbb{C}\mathrm{P}^n)\right), \tag{2.13}
$$

where the isomorphisms are induced by inclusion, the second one coming from excision. The space $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ is the $\mathbb{C}\mathrm{P}^1$-fiber bundle $E_{m,n}$ with one global section taken away, so it is a $\mathbb{C}$-fiber bundle. Let $t \in H^2(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ be its Thom class (under the natural identification given by (2.13)). Therefore, according to the Thom isomorphism theorem, the map

$$
H^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) \to H^{*+2}(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)), \quad \alpha \mapsto t \smile (p_1)^* \alpha \tag{2.14}
$$

is an isomorphism. By looking at restrictions to $E_{a,b}$'s, we see that t is non-zero on $H^*(E_{m,n})$ and is sent to $1 \in H^0(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n)$ by $(p_1)_*$. According to (2.8), on $H^*(E_{m,n})$ we have $u_0 - t = (p_1)^* v$

for some $v \in H^2(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n)$. In order to find v, we consider the following commutative diagram:

$$
\begin{tikzcd}
 & & H^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) \arrow[dl, "(p_1)^*"'] \arrow[d, "(p_1)^*"] \\
H^*\left(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) \arrow[r] & H^*(E_{m,n}) \arrow[r] & H^*\left(p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) \\
H^*(\mathbb{C}\mathrm{P}^{m+n+1}, \mathbb{C}\mathrm{P}^n) \arrow[r] \arrow[u, "(p_2)^*"] & H^*(\mathbb{C}\mathrm{P}^{m+n+1}) \arrow[r] \arrow[u, "(p_2)^*"] & H^*(\mathbb{C}\mathrm{P}^n) \arrow[u, "(p_2)^*"']
\end{tikzcd}
\tag{2.15}
$$

where horizontal arrows are induced by inclusion and form exact sequences. The restriction of p_1 to $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ induces a homeomorphism $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n \simeq p_2^{-1}(\mathbb{C}\mathrm{P}^n)$. Under this identification, the restriction of p_2 to $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ is the projection onto the second factor $\mathbb{C}\mathrm{P}^n$. Hence, the right-hand side vertical arrow $(p_2)^*$ of (2.15) sends u^k to $(p_1)^*u_2^k$. In particular, by commutativity of (2.15), $u_0 \in H^2(E_{m,n})$ is sent to $(p_1)^*u_2$. Since $t \in H^2(E_{m,n})$ is in the image of the top left arrow, it is sent to 0 in $H^2(p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ by exactness. Thus $u_0 - t \in H^2(E_{m,n})$ is sent to $(p_1)^*u_2$ whereas $(p_1)^*v \in H^2(E_{m,n})$ is sent to $(p_1)^*v \in H^2(p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ by commutativity of the up right triangle (with a slight abuse of notation). Therefore $v = u_2$.

In order to study the powers of u_0, we now study the powers of $t \in H^2(E_{m,n})$. Seen in $H^4(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n))$, $t^2 = t \smile (p_1)^*(\lambda u_1 + \mu u_2)$ for some $\lambda, \mu \in \mathbb{Z}$ according to Thom isomorphism (2.14). Let us first find the value of λ by restricting the complex line bundle $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ to the base space $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^0$. This complex line bundle is $E_{m,0} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^0)$ and its Thom class t' is the restriction of t to

$$H^2\left(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0)\right) \simeq H^2\left(E_{m,0}/p_2^{-1}(\mathbb{C}\mathrm{P}^0)\right),$$

so that $t'^2 = \lambda t' \smile (p_1)^*u_1$. Since $\mathbb{C}\mathrm{P}^0$ is just a point, p_2 factors in a homeomorphism between $E_{m,0}/p_2^{-1}(\mathbb{C}\mathrm{P}^0)$ and $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0$. Thus p_2 induces an isomorphism of $\mathbb{Z}$-algebras

$$H^*(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0) \xrightarrow{\simeq} H^*(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0)). \tag{2.16}$$

According to the long exact sequence of the couple $(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0)$, $H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0) \simeq H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0)$ so that the generator $u \in H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0)$ can naturally be seen in $H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0)$. Under the isomorphism (2.16), u is mapped to t' so that u^2 is mapped to t'^2. Thus t'^2 must be a generator of $H^4(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0))$, hence $\lambda = \pm 1$. By applying the orientation preserving morphism $(p_1)_*$, we see that $\lambda = 1$.

Now, since $u_0 = t + (p_1)^*u_2$, one has

$$u_0^2 = t \smile (p_1)^*(u_1 + (\mu+2)u_2) + (p_1)^*u_2^2$$

hence $(p_1)_*u_0^2 = u_1 + (\mu+2)u_2$. By symmetry of $(p_1)_*u_0^2$ in u_1 and u_2, μ must be -1. Indeed, the above identity is still true by restricting ourselves to $E_{1,1}$ and must be invariant under the map induced by $(a, b, c) \mapsto (b, a, c)$ that swaps u_1 and u_2. Since $u_0 = t + (p_1)^*u_2$, one has that

$$u_0^k = \sum_{i+j=k} \binom{k}{i} t^i \smile (p_1)^*u_2^j.$$

Using $t^i = t \smile (p_1)^*(u_1 - u_2)^{i-1}$ for $i \in \mathbb{N}$ and $(p_1)_*(t \smile (p_1)^*w) = w$, one finally gets

$$(p_1)_*u_0^k = \sum_{i+j=k} \binom{k}{i} (u_1 - u_2)^{i-1} u_2^j = \sum_{i+j=k-1} u_1^i u_2^j, \tag{2.17}$$

the last equality can be obtained by identification of coefficients of the polynomial expression in u_1 and u_2. $\square$

Proof of Proposition 2.8. We first remark that there is a well-defined cap-product

$$H_*(E_{A,B}) \times H^*(E_{A,B}, p_2^{-1}(B)) \to H_*(E_{A,B} \setminus p_2^{-1}(B))$$

compatible with the one defines in $A * B$ through the map p_2 and defined the same way. This is summed up by saying that the left hand side of the following diagram is "commutative":

$$
\begin{tikzcd}
H_*(E_{A,B}) \times H^*(E_{A,B}, p_2^{-1}(B)) \arrow[r, "\frown"] \arrow[d, "(p_2)_*"] & H_*(E_{A,B} \setminus p_2^{-1}(B)) \arrow[r, "(p_1)_*", "\simeq"'] \arrow[d, "(p_2)_*"] & H_*(A \times B) \arrow[d, "(\mathrm{pr}_1)_*"] \\
H_*(A * B) \times H^*(A * B, B) \arrow[r, "\frown"] \arrow[u, shift right=8, "(p_2)^*"'] & H_*(A * B \setminus B) \arrow[r, "(p_A)_*", "\simeq"'] & H_*(A)
\end{tikzcd} \quad . \tag{2.18}
$$

In this diagram, $\mathrm{pr}_1 : A \times B \to A$ is the projection onto the first factor. The commutativity of the right hand side of the diagram comes from the obvious commutativity of the associated continuous maps: $p_A \circ p_2 = \mathrm{pr}_1 \circ p_1$ on $E_{A,B} \setminus p_2^{-1}(B)$. Let $v'_B := (p_2)^* v_B \in H^{2\ell}(E_{A,B}, p_2^{-1}(B))$. The space $E_{A,B} \setminus p_2^{-1}(B)$ is the restriction to $A \times B$ of the $\mathbb{C}$-fiber bundle $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$, so that the following map is an isomorphism for the same reason the map (2.14) was:

$$H^*(A \times B) \to H^{*+2}(E_{A,B}, p_2^{-1}(B)), \quad w \mapsto t' \smile (p_1)^* w,$$

where $t' \in H^2(E_{A,B}, p_2^{-1}(B))$ is the restriction of the class t in (2.14). Therefore, $v'_B = t' \smile (p_1)^* w$ for some $w \in H^{2\ell-2}(A \times B)$ satisfying $w = (p_1)_*(v'_B)$ (we recall that $(p_1)_* t' = 1$). Seen in $H^{2\ell}(A * B)$, the class v_B is the restriction of u^ℓ, so that, seen in $H^{2\ell}(E_{A,B})$, v'_B is the restriction of u_0^ℓ. According to the identity (2.17), one has

$$(p_1)_* v'_B = w = \sum_{i+j=\ell-1} u_1^i u_2^j, \tag{2.19}$$

identifying u_1 and u_2 with their restrictions to $H^*(A \times B)$ by a slight abuse of notation. We can now compute, for all $\alpha \in H_*(A)$,

$$
\begin{aligned}
(p_A)_*(\mathrm{pj}_*(\alpha \times \beta) \frown v_B) &= (p_A)_* \circ (p_2)_*((p_1)^*(\alpha \times \beta) \frown v'_B) \\
&= (\mathrm{pr}_1)_* \circ (p_1)_*((p_1)^*(\alpha \times \beta) \frown v'_B) \\
&= (\mathrm{pr}_1)_* \left((\alpha \times \beta) \frown \sum_{i+j=\ell-1} u_1^i u_2^j \right) \\
&= \sum_{i+j=\ell-1} (\mathrm{pr}_1)_* \left((\alpha \frown u^i) \times (\beta \frown u^j) \right) \\
&= \left\langle u^{\ell-1}, \beta \right\rangle \alpha \frown u^0 = \alpha.
\end{aligned}
$$

The second equality follows from commutativity of the diagram (2.18), the third uses (2.19) together with the projection formula $p_*(p^*\gamma \frown w) = \gamma \frown p_* w$ where p is a sphere bundle. By grading issues, only the indices $(i, j) = (0, \ell - 1)$ contribute to the sum and, by definition of β, $\left\langle u^{\ell-1}, \beta \right\rangle = 1$. The result of this computation is the statement we wanted to prove. $\square$

CHAPTER 3

Generating function homology

In this core chapter, we develop the theory of generating function homology of Hamiltonian diffeomorphisms of $\mathbb{CP}^d$.

The study of the homology of sublevel sets of generating functions was introduced by Viterbo [**79**] who introduced spectral invariants of Hamiltonian diffeomorphisms of $\mathbb{R}^{2d}$ with compact support. This work led to the definition of homology groups of these diffeomorphisms by Traynor [**77**] (which are in fact isomorphic to their Floer theoretic analogue [**80**]). Here, we show how to define similar homology groups for Hamiltonian diffeomorphisms of $\mathbb{CP}^d$ by elaborating on works of Givental [**43**] and Théret [**75**].

1. Outline of the chapter

Let $(h_s) : [0,1] \times \mathbb{CP}^d \to \mathbb{R}$ be a smooth periodic Hamiltonian map and let (φ_s) be the associated Hamiltonian flow on $\mathbb{CP}^d$. This Hamiltonian map defines a unique Hamiltonian map $(H_s) : [0,1] \times \mathbb{C}^{d+1} \to \mathbb{R}$ that is 2-homogeneous, invariant under the diagonal action of S^1 on $\mathbb{C}^{d+1}$ given by $\lambda \cdot (z_0, \ldots, z_d) := (\lambda z_0, \ldots, \lambda z_d)$ (we will simply write "S^1-invariant") so that its restriction on the unit sphere $\mathbb{S}^{2d+1} \subset \mathbb{C}^{d+1}$ is a lift of (h_s) under the quotient map $\mathbb{S}^{2d+1} \to \mathbb{CP}^d$. Let (Φ_s) be the $\mathbb{C}$-equivariant Hamiltonian flow associated with (H_s). We will say that (H_s) and (Φ_s) are the lifted Hamiltonian map and Hamiltonian flow of (h_s). In Section 2, we study decompositions of (Φ_s) into small Hamiltonian diffeomorphisms that we usually write $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ so that $\sigma_k \circ \cdots \circ \sigma_1 = \Phi_{t_k}$, for some $0 \leq t_1 \leq \cdots \leq t_n = 1$. For such a decomposition $\boldsymbol{\sigma}$, homology groups $G_*^{(a,b)}(\boldsymbol{\sigma})$ are defined and studied for almost all $-\infty \leq a < b \leq +\infty$ in Sections 3 and 4. In the end of Subsection 4.3, we prove that these homology groups and their natural morphisms do not depend on the choice of the decomposition $\boldsymbol{\sigma}$ of (Φ_s) (up to isomorphism) so that we can write

$$G_*^{(a,b)}(h_s) := G_*^{(a,b)}(\boldsymbol{\sigma}),$$

fixing a decomposition $\boldsymbol{\sigma}$. We call these homology groups "Generating function homology groups of (h_s)" or simply "GF-homology of (h_s)".

Generating function homology groups of (h_s) satisfy the same key properties as the Floer homology groups of (h_s) and one can hope that $G_*^{(a,b)}(h_s)$ is isomorphic to $HF_*^{(\pi a, \pi b)}(h_s)$ with commuting inclusion and boundary morphisms (the π factor is due to our normalisation, see (3.1) below). These groups are homology groups defined over any chosen ring R (in fact over any group G). Given a fixed point $z \in \mathbb{CP}^d$ of φ_1 and a capping $u : \mathbb{D}^2 \to \mathbb{CP}^d$, that is a smooth map from the unit 2-disk of $\mathbb{C}$ to $\mathbb{CP}^d$ so that $u(e^{2i\pi s}) = \varphi_s(z)$, one can define the action $a(\bar{z}) \in \mathbb{R}$ of the capped orbit $\bar{z} := (z, u)$ by

$$a(\bar{z}) = -\frac{1}{\pi}\left(\int_{\mathbb{D}^2} u^*\omega + \int_0^1 h_s \circ \varphi_s(z)\mathrm{d}s\right). \tag{3.1}$$

Recapping gives a $\mathbb{Z}$-orbit of action values: $a(A\#\bar{z}) = a(\bar{z}) + k$ where $A \in \pi_2(\mathbb{CP}^d)$ and $\pi k = -\langle[\omega], A\rangle$. On Floer homology, the recapping by the generator $A_0 \in \pi_2(\mathbb{CP}^d) \simeq \mathbb{Z}$ of symplectic area $\langle[\omega], A_0\rangle = -\pi$ induces the quantum operator

$$q_{HF} : HF_*^{(a,b)}(h_s) \xrightarrow{\simeq} HF_{*+2(d+1)}^{(a+\pi, b+\pi)}(h_s).$$

The analogue isomorphism is defined at (3.22).

Taking R to be a field $\mathbb{F}$, these homology groups are $\mathbb{F}$-vector spaces and the family

$$\left(G_*^{(-\infty,t)}(h_s;\mathbb{F})\right)_t$$

together with its inclusion morphisms define a persistent module that we call the persistence module associated with (h_s) over the field $\mathbb{F}$. Assuming that the Hamiltonian diffeomorphism φ_1 has finitely many fixed points, the persistence modules of fixed degree $(G_k^{(-\infty,t)}(h_s;\mathbb{F}))_t$, $k\in\mathbb{Z}$, satisfy suitable finiteness assumptions and one can define a finite barcode for each of them, giving a global countable (graded) barcode for the persistence module associated with (h_s). Let us describe this barcode (see Figure 2 for an example). The isomorphism of persistence modules $G_*^{(-\infty,t)}(h_s)\simeq G_{*+2(d+1)}^{(-\infty,t+1)}(h_s)$ induces a $\mathbb{Z}$-action on the bars of the barcode sending a bar (a,b) of degree k on a bar $(a+1,b+1)$ of degree $k+2(d+1)$. Therefore, it is enough to describe a set of representatives of bars under this action. In the case where φ_1 is non-degenerate, end-points of representative bars are in one-to-one correspondence with fixed points of $\mathbb{C}\mathrm{P}^d$, the value of an end-point being equal to the action of a capping of the associated fixed point. In general, a fixed point should be counted with multiplicity equal to its local homology, which gives a homology count $N((h_s);\mathbb{F})$ of the fixed points (see (3.13)). Among the representative bars, exactly $d+1$ are infinite. This is a consequence of Théret's proof of Fortune-Weinstein theorem [**75**], in fact the increasing sequence $(c_k(h_s))_{k\in\mathbb{Z}}$ of values of end-points of the infinite bars of the whole barcode corresponds to the sequence of spectral invariants of (h_s) (see Theorem 3.22).

2. Generating functions of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms

2.1. "Broken trajectories" and generating functions of $\mathbb{C}^d$. We follow the ideas of Chaperon [**26**] and Givental [**43**] to build and study generating functions associated with a decomposition of a Hamiltonian diffeomorphism.

Let $\Phi\in\mathrm{Ham}(\mathbb{C}^d)$ be a Hamiltonian diffeomorphism which can be decomposed as $\Phi=\sigma_n\circ\cdots\circ\sigma_1$ where every $\sigma_k\in\mathrm{Ham}(\mathbb{C}^d)$ is sufficiently C^1-close to id such that they admit generating functions $f_k:\mathbb{C}^d\to\mathbb{R}$ satisfying:

$$\forall z_k\in\mathbb{C}^d,\exists! w_k\in\mathbb{C}^d,\quad w_k=\frac{z_k+\sigma_k(z_k)}{2}\quad\text{and}\quad \nabla f_k(w_k)=i(z_k-\sigma_k(z_k)).\tag{3.2}$$

We call such generating functions without auxiliary variable elementary generating functions. We will say that the n-tuple $\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_n)$ is associated with the Hamiltonian flow (Φ_t) if there exist real numbers $0=t_0\leq t_1\leq\cdots\leq t_n=1$ such that $\sigma_k=\Phi_{t_k}\circ\Phi_{t_{k-1}}^{-1}$. A continuous family of such tuples $(\boldsymbol{\sigma}_s)$ will denote a family of tuples of the same size $n\geq 1$, $\boldsymbol{\sigma}_s=:(\sigma_{1,s},\dots,\sigma_{n,s})$ such that the maps $s\mapsto\sigma_{k,s}$ are C^1-continuous. Every compactly supported Hamiltonian flow and every $\mathbb{C}$-equivariant Hamiltonian flow $(\Phi_s)_{s\in[0,1]}$ admit a continuous family of associated tuples $(\boldsymbol{\sigma}_s)$ that is: $\boldsymbol{\sigma}_s$ is associated with Φ_s for all $s\in[0,1]$ (and the size can be taken as large as wanted). For all $k\in\mathbb{N}$, we denote $\boldsymbol{\varepsilon}^k$ the k-tuple

$$\boldsymbol{\varepsilon}^k:=(\mathrm{id},\dots,\mathrm{id}).$$

Let us denote by $F_{\boldsymbol{\sigma}}$ the following function $(\mathbb{C}^d)^n\to\mathbb{R}$:

$$F_{\boldsymbol{\sigma}}(v_1,\dots,v_n):=\sum_{k=1}^n f_k\left(\frac{v_k+v_{k+1}}{2}\right)+\frac{1}{2}\left\langle v_k,iv_{k+1}\right\rangle,\tag{3.3}$$

with convention $v_{n+1}=v_1$. Let $A_n:(\mathbb{C}^d)^n\to(\mathbb{C}^d)^n$ denotes the linear map such that, for $\mathbf{v}=(v_1,\dots,v_n)$, $A_n(\mathbf{v})=\mathbf{w}$ with $w_k=\frac{v_k+v_{k+1}}{2}$. Let $\psi:(\mathbb{C}^d)^n\to(\mathbb{C}^d)^n$ be the diffeomorphism $\psi(\mathbf{z})=\mathbf{w}$ defined by (3.2). The following proposition is a variation of ideas of Chaperon [**26**]; it is implicit in the work of Givental [**43**].

Proposition 3.1. *Under the above hypothesis, we have*

$$\forall k,\forall\mathbf{v}\in(\mathbb{C}^d)^n,\quad \partial_{v_k}F_{\boldsymbol{\sigma}}(v_1,\dots,v_n)=i(z_k-\sigma_{k-1}(z_{k-1})),$$

where $\mathbf{z}:=\psi^{-1}\circ A_n(\mathbf{v})$ and $z_0:=z_n$. Moreover, if n is odd, $F_{\boldsymbol{\sigma}}$ is a generating function of Φ with v_1 as main variable.

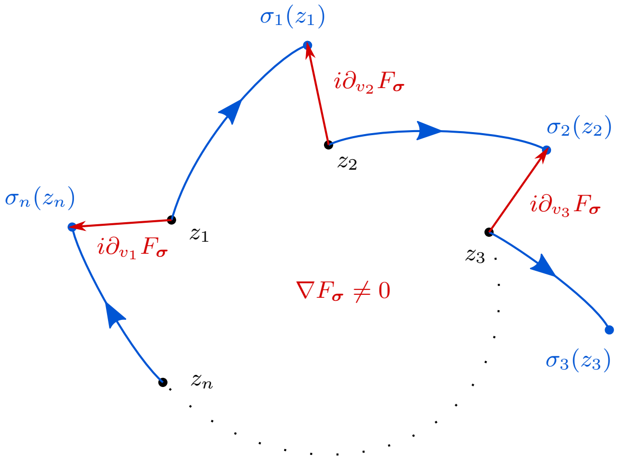

FIGURE 1. Geometric interpretation of the gradient of $F_{\boldsymbol{\sigma}}$

PROOF. Let $F := F_{\boldsymbol{\sigma}}$. Given any n-tuple $\mathbf{v} \in (\mathbb{C}^d)^n$, we associate n-tuples $\mathbf{w}$ and $\mathbf{z}$ in $(\mathbb{C}^d)^n$ given by $\mathbf{w} = A_n(\mathbf{v})$ and $\psi(\mathbf{z}) = \mathbf{w}$. Then

$$\begin{aligned}
\partial_{v_k}F(\mathbf{v}) &= \frac{1}{2}\left(\nabla f_{k-1}\left(\frac{v_{k-1}+v_k}{2}\right) + \nabla f_k\left(\frac{v_k+v_{k+1}}{2}\right) + i(v_{k+1}-v_{k-1})\right) \\
&= \frac{1}{2}\big(\nabla f_{k-1}(w_{k-1}) + \nabla f_k(w_k)\big) + i(w_k - w_{k-1}) \\
&= i(z_k - \sigma_{k-1}(z_{k-1})).
\end{aligned}$$

where indices are seen in $\mathbb{Z}/n\mathbb{Z}$. Now let us suppose n is odd, so that A_n is an isomorphism. If we denote by $\xi := (v_2, \dots, v_n)$ the auxiliary variables, we thus have $\partial_\xi F(\mathbf{v}) = 0$ if and only if $z_{k+1} = \sigma_k(z_k)$ for $1 \leq k \leq n-1$. Moreover, since $v_1 = \sum_k (-1)^{k+1} w_k$, if $\partial_\xi F(\mathbf{v}) = 0$ then

$$v_1 = \sum_{k=1}^{n} (-1)^{k+1} \frac{z_k + \sigma_k(z_k)}{2} = \frac{z_1 + \sigma_n(z_n)}{2},$$

as required (since $\sigma_n(z_n) = \Phi(z_1)$ recursively).

Finally we must show that $\partial_\xi F$ is transverse to 0. This is clear in the z-coordinates: the matrix

$$\mathrm{d}(\partial_\xi F)(\mathbf{v}) \cdot A_n^{-1} \cdot \mathrm{d}\psi(\mathbf{z}) = i \begin{bmatrix} -\mathrm{d}\sigma_1(z_1) & I_{2d} & & & \\ & -\mathrm{d}\sigma_2(z_2) & I_{2d} & & \\ & & \ddots & \ddots & \\ & & & -\mathrm{d}\sigma_n(z_n) & I_{2d} \end{bmatrix}$$

is invertible. □

This proposition provides a quantitative way to see how close a discrete trajectory $(z_1, \dots, z_n)$ given by $(v_1, \dots, v_n)$ is to a discrete trajectory of the dynamics $\sigma_n \circ \cdots \circ \sigma_1$ (see Figure 1).

Let $Q_n : (\mathbb{C}^{d+1})^n \to \mathbb{R}$ be the S^1-invariant quadratic form

$$Q_n(\mathbf{v}) := F_{\boldsymbol{\varepsilon}^n}(\mathbf{v}) = \frac{1}{2}\sum_{k=1}^{n} \langle v_k, i v_{k+1} \rangle = 2\sum_{k=1}^{n}\sum_{l=1}^{k-1} (-1)^{k+l} \langle w_k, i w_l \rangle .$$

The following proposition is a direct consequence of the fact that Q_n is both a quadratic form and a generating function.

PROPOSITION 3.2. *The quadratic form Q_n has nullity $2(d+1)$. Moreover*

$$Q_n(v_1, v_2, \dots, v_n) = -Q_n(v_1, v_n, v_{n-1}, \dots, v_2)$$

so that

$$\operatorname{ind} Q_n = \operatorname{coind} Q_n = (n-1)(d+1).$$

2.2. Generating family of the S^1-action. In this section we follow Théret [**75**] and study generating families of the unitary (Hamiltonian) flow (δ_t) of $\mathbb{C}^{d+1}$, $\delta_t(z) := e^{-2i\pi t}z$. Let us define a family of "good" tuples of small Hamiltonian diffeomorphisms generating (δ_t) in $t \in [-m, m]$ for $m \in \mathbb{N}^*$. For $|t| < 1/2$ the family (δ_t) is generated by the family of elementary generating functions

$$w \mapsto -\tan(\pi t)\|w\|^2, \quad \forall w \in \mathbb{C}^{d+1}.$$

Let us fix once for all an even number $n_0 \geq 4$ and let $(\boldsymbol{\delta}_t^{(1)})$ be the family of n_0-tuples

$$\left(\boldsymbol{\delta}_t^{(1)}\right) := \left(\delta_{t/n_0}, \dots, \delta_{t/n_0}\right)$$

generating $z \mapsto e^{-2i\pi t}z$ for $t \in (-2, 2)$. For all $m \in \mathbb{N}^*$, let $(\boldsymbol{\delta}_t^{(m)})$ be a family of mn_0-tuples generating $z \mapsto e^{-2i\pi t}z$ for $t \in (-m-1, m+1)$ and satisfying

$$\boldsymbol{\delta}_t^{(m+1)} = \left(\boldsymbol{\delta}_t^{(m)}, \boldsymbol{\varepsilon}^{n_0}\right), \quad \forall t \in [-m, m]. \tag{3.4}$$

More precisely, let $\chi : \mathbb{R} \to \mathbb{R}$ be an odd smooth non-decreasing map such that $\chi_m \equiv \mathrm{id}$ on $[-m - 1/4, m + 1/4]$ and $\chi_m \equiv m + 1/2$ on $[m + 3/4, +\infty)$. We set

$$\boldsymbol{\delta}_t^{(m+1)} = \left(\boldsymbol{\delta}_{\chi_m(t)}^{(m)}, \boldsymbol{\delta}_{t-\chi_m(t)}^{(1)}\right), \quad \forall t \in (-m-2, m+2).$$

LEMMA 3.3. *Let $m \in \mathbb{N}^*$ and $t \in (-m-1, m+1)$. With the above notation,*

$$\mathrm{ind}\left(F_{\boldsymbol{\delta}_t^{(m)}}\right) - \mathrm{ind}\left(F_{\boldsymbol{\delta}_0^{(m)}}\right) = 2(d+1)\lfloor t \rfloor.$$

PROOF. This is a direct application of Proposition 1.1 (6) to the path (δ_s) for s between 0 and t. $\square$

Given a tuple of small Hamiltonian diffeomorphisms $\boldsymbol{\sigma}$, we set

$$\boldsymbol{\sigma}_{m,t} := \left(\boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right), \quad \forall t \in [-m, m].$$

LEMMA 3.4 ([**75**, Lemma 4.4]). *Let $\boldsymbol{\sigma}$ be a m'-tuple, with m' odd, such that $F_t := F_{\boldsymbol{\sigma}_{m,t}} : (\mathbb{C}^{d+1})^{m'+mn_0} \to \mathbb{R}$ is a smooth family of conical generating functions. Then for $t \in [-m, m]$,*

(i) $\partial_t F_t(\mathbf{v}) \leq 0$, $\forall \mathbf{v} \in (\mathbb{C}^{d+1})^{m'+mn_0}$,
(ii) $\partial_t F_t(\mathbf{v}) < 0$, $\forall \mathbf{v} \in \Sigma_{F_t} \setminus 0$.

PROOF. The first property is a direct consequence of the definitions and the fact that the derivative $\partial_t(\tan(\pi t/m))$ is > 0. Let $\mathbf{v} = (v_1, \dots, v_{mn_0+m'}) \in (\mathbb{C}^{d+1})^{mn_0+m'}$ be such that $\partial_t F_t(\mathbf{v}) = 0$. Then, for $m' + 1 \leq k < m' + n_0$, $w_k := \frac{v_k + v_{k+1}}{2} = 0$ thus $z_k = 0$ where the family $\mathbf{z} = (z_k)$ is associated with the family $\mathbf{w} = (w_k)$ *via* (3.2) as usual. Thus if $\mathbf{v} \in \Sigma_{F_t}$, z_k must be 0 for all k for the sequence $(z_1, \dots, z_{m'+mn_0})$ to be the discrete dynamics of conical diffeomorphisms, hence $\mathbf{w} = 0$ and $\mathbf{v} = 0$. $\square$

2.3. A discrete variational principle for $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms. Let (φ_t) be the Hamiltonian flow of $\mathbb{CP}^d$ associated with the Hamiltonian map $h : [0,1] \times \mathbb{CP}^d \to \mathbb{R}$. Let $\tilde{h} : [0,1] \times \mathbb{S}^{2d+1} \to \mathbb{R}$ be the S^1-invariant lift of h defined by $\tilde{h}_t := h_t \circ \pi$ where $\pi : \mathbb{S}^{2d+1} \to \mathbb{CP}^d$ is the quotient map $\pi(z) := [z]$. Let $H : [0,1] \times \mathbb{C}^{d+1} \to \mathbb{R}$ be the 2-homogeneous Hamiltonian map such that $H_t(\lambda x) := \lambda^2 \tilde{h}_t(x)$ for all $x \in \mathbb{S}^{2d+1}$. It defines a $\mathbb{C}$-equivariant symplectic flow (Φ_t) stabilizing the Euclidean sphere $\mathbb{S}^{2d+1}$ and such that

$$\pi \circ \Phi_t|_{\mathbb{S}^{2d+1}} = \varphi_t \circ \pi, \quad \forall t \in [0,1]. \tag{3.5}$$

This flow (Φ_t) is uniquely defined by the choice of Hamiltonian map (h_t) of (φ_t). In fact, if (Φ'_t) is a $\mathbb{C}$-equivariant Hamiltonian flow stabilizing the sphere and such that (3.5), then $\Phi'_t = e^{i\theta(t)}\Phi_t$ with $\theta : [0,1] \to \mathbb{R}$ which boils down to a change of equivalent Hamiltonian map (h'_t) for (φ_t). We will usually write $\varphi := \varphi_1 \in \mathrm{Ham}(\mathbb{CP}^d)$ and $\Phi := \Phi_1 \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$. Given a choice of Hamiltonian map (h_t), one can define the action of capped fixed point $\bar{z}$ by (3.1). We denote by $a(z) \in \mathbb{R}/\mathbb{Z}$ the reduction of $a(\bar{z})$ modulo $\mathbb{Z}$ for any capping $\bar{z}$ of z, we call $a(z)$ the action of z. Fixed points $z \in \mathbb{CP}^d$ of action $a \in \mathbb{R}/\mathbb{Z}$ are in one-to-one correspondence with $\mathbb{C}$-lines $\mathbb{C}Z \subset \mathbb{C}^{d+1}$, $[Z] = z$, such that $\Phi_1(Z) = e^{2i\pi a}Z$, $Z \in \mathbb{C}^{d+1} \setminus 0$ (see [**75**, Prop. 5.8]). Since the action only depends on the choice of lift (Φ_t), when such a lift is given, we will simply call it the action of (φ_t) or the action of φ.

Following Théret, for all $m \in \mathbb{N}^*$, we now define a map $\mathcal{T}_m : M_m \to \mathbb{R}$ that provides a variational principle for capped fixed points of (φ_t) with action in $I_m := (-m-1, m+1)$. Let $(\boldsymbol{\sigma}_{m,t})$, $t \in I_m$, be the smooth family of tuples generating $e^{-2i\pi t}\Phi_1$ defined in Section 2.2. Then $F_{m,t} := F_{\boldsymbol{\sigma}_{m,t}} : \mathbb{C}^{N+1} \to \mathbb{R}$ gives us a family of conical functions generating $e^{-2i\pi t}\Phi_s$. Let $\tilde{f}_m : I_m \times \mathbb{S}^{2N+1} \to \mathbb{R}$ be the S^1-invariant function $\tilde{f}_m(t,\zeta) := F_{m,t}(\zeta)$ for $|\zeta| = 1$ and $f_m : I_m \times \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ be the induced function. Then there is a one-to-one correspondence between fixed points of φ of action $t \in I_m$ and critical points of $f_m(t,\cdot)$ with value 0.

According to property (ii) of Lemma 3.4, the differential $\mathrm{d}\tilde{f}_m = \partial_t(F_{m,t})\mathrm{d}t + \mathrm{d}F_{m,t}$ never vanishes on $\mathbb{C}^{N+1} \setminus 0$, so 0 is a regular value of f_m. Let us define the submanifold

$$M_m := \{(t,\zeta) \in I_m \times \mathbb{C}\mathrm{P}^N \mid f_m(t,\zeta) = 0\}$$

and let $\mathcal{T}_m : M_m \to I_m$ be the projection onto the first factor. Fixed points of action $t \in I_m$ are in one-to-one correspondence with critical points of $\mathcal{T}_m$ with critical value t: more precisely

$$\mathrm{d}\mathcal{T}_m(t,\zeta) = 0 \Leftrightarrow \mathrm{d}_\zeta f_m(t,\zeta) = 0.$$

Moreover, if $(t,\zeta) \in M_m$ is a critical point of $\mathcal{T}_m$, then the Hessian $\mathrm{d}^2\mathcal{T}_m(t,\zeta)$ is equivalent as a quadratic form to $\mathrm{d}^2_{\zeta,\zeta} f_m(t,\zeta)$ which is equivalent to $\mathrm{d}^2 F_{m,t}(\tilde{\zeta})$ restricted to a complement of the $\mathbb{C}$-line induced by $\tilde{\zeta} \in \mathbb{S}^{2N+1}$, where $\tilde{\zeta}$ is a lift of $\zeta \in \mathbb{C}\mathrm{P}^N$ (because $F_{m,t}$ is conical). Since this line $\mathbb{C}\tilde{\zeta}$ is included in $\ker \mathrm{d}^2 F_{m,t}(\tilde{\zeta})$, critical points $(t,\zeta) \in M_m$ and $\tilde{\zeta} \in \mathbb{C}^{N+1}$ share the same index:

$$\operatorname{ind}(\tilde{\zeta}, F_{m,t}) = \operatorname{ind}((t,\zeta), \mathcal{T}_m). \tag{3.6}$$

Moreover, if $z \in \mathbb{C}\mathrm{P}^d$ and $Z \in \mathbb{C}^{d+1}$ are fixed points associated with $\zeta \in \mathbb{C}\mathrm{P}^N$ and $\tilde{\zeta} \in \mathbb{C}^{N+1}$ respectively, since

$$\dim\ker \mathrm{d}^2 F_{m,t}(\tilde{\zeta}) = \dim\ker(e^{-2i\pi t}\mathrm{d}\Phi(Z) - \mathrm{id})$$

one has

$$\dim\ker \mathrm{d}^2_{\zeta,\zeta} f_m(t,\zeta) = \dim\ker \mathrm{d}^2\mathcal{T}_m(t,\zeta) = \dim\ker(\mathrm{d}\varphi(z) - \mathrm{id}) =: \nu(z). \tag{3.7}$$

3. Morse theory of conical generating functions

3.1. Convention and notation. We recall that in this part of the thesis, $H_*(X)$ and $H^*(X)$ denote respectively the singular homology and the singular cohomology of a topological space or pair X over an indeterminate ring R. If we need to specify the ring R, we write $H_*(X;R)$ and $H^*(X;R)$ instead. Let $\boldsymbol{\sigma}$ be an n-tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1}$. We denote by $Z(\boldsymbol{\sigma}) \subset \mathbb{C}\mathrm{P}^{n(d+1)-1}$ the sublevel set

$$Z(\boldsymbol{\sigma}) := \left\{\widehat{F}_{\boldsymbol{\sigma}} \leq 0\right\},$$

where $\widehat{F}_{\boldsymbol{\sigma}} : \mathbb{C}\mathrm{P}^{n(d+1)-1} \to \mathbb{R}$ is the C^1-map induced by $F_{\boldsymbol{\sigma}}|_{\mathbb{S}^{2n(d+1)-1}}$. We denote by $HZ_*(\boldsymbol{\sigma})$ the shifted homology group

$$HZ_*(\boldsymbol{\sigma}) := H_{*+(n-1)(d+1)}(Z(\boldsymbol{\sigma})),$$

and if $Z(\boldsymbol{\sigma}') \subset Z(\boldsymbol{\sigma})$, with $\boldsymbol{\sigma}'$ an n-tuple, we set

$$HZ_*(\boldsymbol{\sigma},\boldsymbol{\sigma}') := H_{*+(n-1)(d+1)}(Z(\boldsymbol{\sigma}), Z(\boldsymbol{\sigma}')).$$

For $m \in \mathbb{N}^*$ and $a \leq b$ in I_m, one has $F_{\boldsymbol{\sigma}_{m,b}} \leq F_{\boldsymbol{\sigma}_{m,a}}$ according to Lemma 3.4 so $Z(\boldsymbol{\sigma}_{m,a}) \subset Z(\boldsymbol{\sigma}_{m,b})$ and we can set

$$G_*^{(a,b)}(\boldsymbol{\sigma}, m) := HZ_*(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a}),$$

when a and b are not action values of $\boldsymbol{\sigma}$.

3.2. Cohomology of sublevel sets of $\mathcal{T}_m$. We take the notation $\boldsymbol{\sigma}$, $F_{m,t}$, M_m etc. of Section 2.3 of this chapter. Let $p : I_m \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ be the projection onto the second space and $i : M_m \hookrightarrow I_m \times \mathbb{C}\mathrm{P}^N$ be the inclusion map. Let $\widehat{F}_t : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ be the C^1 map induced by $F_{m,t}|_{\mathbb{S}^{2N+1}}$. According to Lemma 3.4, if $s \leq t$, then $\widehat{F}_t \leq \widehat{F}_s$ so that the subspace

$$A_t := \left\{(s,\zeta) \in (-m-1, t] \times \mathbb{C}\mathrm{P}^N \mid \widehat{F}_s(\zeta) \leq 0\right\}$$

retracts on $t \times \left\{\widehat{F}_t \leq 0\right\} = t \times Z(\boldsymbol{\sigma}_{m,t})$ by deformation, hence p induces an isomorphism

$$H_*(A_t) \to HZ_{*+(n-1)(d+1)}\left(\boldsymbol{\sigma}_{m,t}\right)$$

for all $t \in I_m$ and thus induces isomorphisms

$$p_* : H_*(A_b, A_a) \xrightarrow{\simeq} G_{*-(n-1)(d+1)}^{(a,b)}(\boldsymbol{\sigma}, m) \tag{3.8}$$

for $a \leq b$ in I_m. Let $a \leq b$ in I_m and $e > 0$ such that $a - e \in I_m$, the subspace A_b retracts by deformation on $\{\mathcal{T}_m \leq b\} \cup A_{a-e}$ by $(t, \zeta) \mapsto (s, \zeta)$ where s is the maximal $r \in (a-e, t]$ satisfying $\widehat{F}_r(\zeta) = 0$ or $s = a - e$ if such a maximum does not exist. By excision, we then have that i induces an isomorphism

$$i_* : H_*(\{\mathcal{T}_m \leq b\}, \{\mathcal{T}_m \leq a\}) \xrightarrow{\simeq} H_*(A_b, A_a), \tag{3.9}$$

for all $a \leq b$ in I. Putting (3.8) and (3.9) together, we get the following

LEMMA 3.5. *For all $a \leq b$ in I, the composition $p \circ i$ induces an isomorphism in homology*

$$G_{*-(n-1)(d+1)}^{(a,b)}(\boldsymbol{\sigma}, m) \simeq H_*(\{\mathcal{T}_m \leq b\}, \{\mathcal{T}_m \leq a\}).$$

This result extends to local homology, the precise statement being given in the next section. The dual cohomology statements are true for dual reasons.

3.3. Local cohomology of a fixed point. Let $\bar{z}$ be a capped fixed point of φ with action value $t \in (-m, m)$. We denote by $\mathrm{C}_*(\bar{z})$ the local homology of $\widehat{F}_{m,t}$ at the associated critical point $\zeta \in \mathbb{C}\mathrm{P}^N$ graded with the usual shift:

$$\mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}, m) := H_{*+(n-1)(d+1)}\left(\left\{\widehat{F}_{m,t} \leq 0\right\}, \left\{\widehat{F}_{m,t} \leq 0\right\} \setminus \zeta\right).$$

By an argument similar to the proof of Lemma 3.5, the map $p \circ i : M \to \mathbb{C}\mathrm{P}^N$ induces an isomorphism between the local homology of $\bar{z}$ and the local homology of (t, ζ) for $\mathcal{T}_m$:

$$\mathrm{C}_{*-(n-1)(d+1)}(\boldsymbol{\sigma}; \bar{z}, m) \simeq H_*(\{\mathcal{T}_m \leq t\}, \{\mathcal{T}_m \leq t\} \setminus (t, \zeta)) = \mathrm{C}_*(\mathcal{T}_m, (t, \zeta)). \tag{3.10}$$

Let us assume that there are finitely many capped fixed points $\bar{z}_1, \ldots, \bar{z}_q$ of action value t. For $\varepsilon > 0$ small enough, the natural isomorphism (1.4) applied to $\mathcal{T}_m$ composed with $(p \circ i)_*$ defines a natural isomorphism

$$\bigoplus_j \mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}_j, m) \simeq G_*^{(t-\varepsilon, t+\varepsilon)}(\boldsymbol{\sigma}, m). \tag{3.11}$$

According to Proposition 3.2,

$$\mathrm{ind}(\tilde{\zeta}, F_{m,t}) - \mathrm{ind}(Q_n) = \mathrm{ind}(\tilde{\zeta}, F_{m,t}) - (n-1)(d+1),$$

where $\tilde{\zeta} \in \mathbb{C}^{N+1}$ is a lift of ζ. Therefore, the relationship between Morse index and Maslov index, together with the index identity (3.6) gives

$$\mathrm{ind}(\tilde{\zeta}, F_{m,t}) - (n-1)(d+1) = \mathrm{mas}(Z, (e^{-2i\pi ts}\Phi_s)) = \mathrm{mas}(\bar{z}),$$

where the last equality comes from Proposition 1.2. According to (1.5) and (3.7), the support of the local homology of $\bar{z}$ then satisfies

$$\operatorname{supp} \mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}, m) \subset [\mathrm{mas}(\bar{z}), \mathrm{mas}(\bar{z}) + \nu(z)], \tag{3.12}$$

where we recall that $\nu(z) = \dim \ker(\mathrm{d}\varphi(z) - \mathrm{id})$ (it is independent of the capping of z).

The independence on m up to isomorphism of the definition of the local homology can easily be deduced from the isomorphism induced by θ_m^{m+1} (defined in Section 4.2) on the local homologies. Local homologies $\mathrm{C}_*(\boldsymbol{\sigma}; z, t)$ and $\mathrm{C}_*(\boldsymbol{\sigma}; z, t+1)$ are isomorphic up to a $2(d+1)$ shift in degree so we will not specify the choice of representative $t \in \mathbb{R}$ when the grading is irrelevant. One can prove it without using the isomorphism with local Floer homology by using the isomorphism induced by (3.22) in local homology. Following Shelukhin [**69**], in the case where φ has finitely many fixed points, we define the homology count of fixed point of $\boldsymbol{\sigma}$ generating φ over the field $\mathbb{F}$ by

$$N(\boldsymbol{\sigma}; \mathbb{F}) := \sum_{z \in \mathrm{Fix}(\varphi_1)} \dim \mathrm{C}_*(\boldsymbol{\sigma}; z; \mathbb{F}) \in \mathbb{N}. \tag{3.13}$$

We recall that an integer $k \in \mathbb{N}^*$ is said to be an admissible period of a fixed point z of φ if $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $\mathrm{d}\varphi(z)$. Until the end of the section, φ is associated with a tuple $\boldsymbol{\sigma}$ and the periodic points of φ are isolated in order to simplify the statements. The following proposition was proved by Ginzburg-Gürel in [**40**] for the local Floer homology.

PROPOSITION 3.6. *Let $k \in \mathbb{N}^*$ be an admissible period of the fixed point z of φ. Then as graded R-modules,*

$$\mathrm{C}_*(\boldsymbol{\sigma}^k; z) \simeq \mathrm{C}_{*-i_k}(\boldsymbol{\sigma}; z),$$

for some shift in degree $i_k \in \mathbb{Z}$.

In our setting, one can prove this result by directly applying the shifting theorem of Gromoll-Meyer [**46**, §3]. Let us shortly give the main steps of the proof when k is odd. One can assume that the fixed point z has action 0 so that $\mathrm{C}_*(\boldsymbol{\sigma}^k; z)$ is isomorphic to the local homology of $\widehat{F}_{\boldsymbol{\sigma}^k}$ at $[\mathbf{v}_0 : \cdots : \mathbf{v}_0]$, where $[\mathbf{v}_0] \in \mathbb{C}\mathrm{P}^N$ is the critical point of $\widehat{F}_{\boldsymbol{\sigma}}$ associated with z. Let $M \subset \mathbb{C}\mathrm{P}^N$ be a characteristic submanifold for $\widehat{F}_{\boldsymbol{\sigma}}$ at $[\mathbf{v}_0]$ (*i.e.* a submanifold containing $[\mathbf{v}_0]$ which is tangent to a pseudo-gradient of $\widehat{F}_{\boldsymbol{\sigma}}$ and whose tangent space at $[\mathbf{v}_0]$ coincides with the kernel of the Hessian of $\widehat{F}_{\boldsymbol{\sigma}}$). Then the image of M under the embedding $\iota : [\mathbf{v}] \mapsto [\mathbf{v} : \cdots : \mathbf{v}]$ is a characteristic submanifold for $\widehat{F}_{\boldsymbol{\sigma}^k}$ if and only if k is admissible (see Equation (3.7) and above). According to the shifting theorem, the local homology of a function at a given point is isomorphic to the local homology of the restriction of this function to a characteristic submanifold at the given point up to a shift in degree. Since $\widehat{F}_{\boldsymbol{\sigma}^k} \circ \iota = \widehat{F}_{\boldsymbol{\sigma}}$, the conclusion follows. This proof needs small changes when k is even (see Section 4.4 for an idea of the case $k = 2$), but it will not be needed to prove the following corollary.

COROLLARY 3.7. *For every fixed point z of φ, there exists $B > 0$ such that, for all prime p*

$$\dim \mathrm{C}_*(\boldsymbol{\sigma}^p; z; \mathbb{F}_p) < B.$$

PROOF. Applying Proposition 3.6 with $R := \mathbb{Z}$, the $\mathrm{C}_*(\boldsymbol{\sigma}^p; z; \mathbb{Z})$'s are isomorphic up to a shift in degree for sufficiently large prime numbers p. Since these $\mathbb{Z}$-modules are finitely generated (according to Gromoll-Meyer, see just above Equation (1.5)), the conclusion for an arbitrary finite field of coefficients is a straightforward application of the universal coefficient theorem. □

4. Generating function homology

4.1. Composition of sublevel sets of generating functions. Given an odd number $n \in \mathbb{N}$, let A_n be the linear automorphism of $(\mathbb{C}^{d+1})^n$ such that $\mathbf{w} = A_n \mathbf{v}$ with $w_k = \frac{v_k + v_{k+1}}{2}$. We will often omit A_n in our changes of variables and talking about w-variables.

We denote by $\widetilde{B}_{n,m} : (\mathbb{C}^{d+1})^n \times (\mathbb{C}^{d+1})^m \to (\mathbb{C}^{d+1})^{m+n+1}$ the $\mathbb{C}$-linear map

$$\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}') := \left(\mathbf{w}, \sum_{k=1}^{m} (-1)^{k+1} w'_k, \mathbf{w}' \right),$$

and we denote by $B_{n,m} : \mathbb{C}\mathrm{P}^{(d+1)n-1} * \mathbb{C}\mathrm{P}^{(d+1)m-1} \to \mathbb{C}\mathrm{P}^{(d+1)(n+m+1)-1}$ the associated projective map. A straightforward computation gives the following proposition.

PROPOSITION 3.8. *Given odd integers $n, m \in \mathbb{N}$, for all $\mathbf{w} \in (\mathbb{C}^{d+1})^n$ and $\mathbf{w}' \in (\mathbb{C}^{d+1})^m$, one has in w-variables,*

$$Q_{n+m+1}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) = Q_n(\mathbf{w}) + Q_m(\mathbf{w}').$$

COROLLARY 3.9. *Given tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ of odd respective sizes n and m, one has*

$$F_{(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) = F_{\boldsymbol{\sigma}}(\mathbf{w}) + F_{\boldsymbol{\sigma}'}(\mathbf{w}').$$

Therefore, $\widetilde{B}_{n,m}$ induces an injective map

$$\{F_{\boldsymbol{\sigma}} \leq a\} \times \{F_{\boldsymbol{\sigma}'} \leq b\} \to \left\{F_{(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')} \leq a + b\right\}$$

by restriction. If $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are tuples of C^1-small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, $B_{n,m}$ induces a map

$$Z(\boldsymbol{\sigma}) * Z(\boldsymbol{\sigma}') \to Z(\boldsymbol{\sigma}, \varepsilon, \boldsymbol{\sigma}').$$

PROOF. This is a direct consequence of the form that takes $F_{\boldsymbol{\sigma}}$ in w-variables:

$$F_{\boldsymbol{\sigma}}(\mathbf{w}) = \sum_{k=1}^{n} f_k(w_k) + Q_n(\mathbf{w}).$$

Therefore,

$$\begin{aligned} F_{(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')}\left(\widetilde{B}_{n,m}(\mathbf{w},\mathbf{w}')\right) &= \sum_{k=1}^{n} f_k(w_k) + \sum_{l=1}^{m} f'_l(w'_l) + Q_{n+m+1}\left(\widetilde{B}_{n,m}(\mathbf{w},\mathbf{w}')\right) \\ &= \sum_{k=1}^{n} f_k(w_k) + Q_n(\mathbf{w}) + \sum_{l=1}^{m} f'_l(w'_l) + Q_m(\mathbf{w}') \\ &= F_{\boldsymbol{\sigma}}(\mathbf{w}) + F_{\boldsymbol{\sigma}'}(\mathbf{w}'). \end{aligned}$$

□

By making use of the homology projective join defined in Chapter 2, we are now in position to define a special map in homology involving two different Hamiltonian flows and their composition. Let us fix 2 tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ of odd respective sizes n and n'. According to Corollary 3.9, the map $B_{n,n'}$ induces a natural morphism $H_*(Z(\boldsymbol{\sigma}) * Z(\boldsymbol{\sigma}')) \to H_*(Z(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}'))$. The composition of this morphism with the homology projective join defines a natural morphism

$$HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma}') \to HZ_{*-2d}((\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')).$$

It generalizes to the relative case $Z(\boldsymbol{\sigma}'') \subset Z(\boldsymbol{\sigma}')$:

$$HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma}',\boldsymbol{\sigma}'') \to HZ_{*-2d}((\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}'),(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}''));$$

the relative HZ_* could be in the left hand factor of the tensor product as well, as long as one of the two HZ_*'s is an absolute homology group. In symbols, we will write this map as $\alpha \otimes \beta \mapsto \alpha \circledast \beta$. The naturality of these morphisms under inclusion morphisms and boundary morphisms follows directly from the naturality of the homology projective join.

In particular, the following diagram commutes

$$\begin{tikzcd} HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma}') \arrow[r, "\circledast"] \arrow[d] & HZ_{*-2d}((\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')) \arrow[d] \\ H_{*+r}(\mathbb{C}\mathrm{P}^N) \otimes H_{*+r'}(\mathbb{C}\mathrm{P}^{N'}) \arrow[r, "\mathrm{pj}_*"] & H_{*+r+r'+2}(\mathbb{C}\mathrm{P}^{N''}) \end{tikzcd}, \tag{3.14}$$

where $r := (n-1)(d+1)$, $N := n(d+1)-1$ etc. $N'' := (n+n'+1)(d+1)-1$ and we see $\mathbb{C}\mathrm{P}^N$ and $\mathbb{C}\mathrm{P}^{N'}$ as the disjoint subspaces included in $\mathbb{C}\mathrm{P}^{N''}$ *via* $[\mathbf{w}] \mapsto [\mathbf{w}:0]$ and $[\mathbf{w}'] \mapsto [0:\mathbf{w}']$. The commutativity of this diagram follows from the naturality of pj_* and the fact that $B_{n,n'}$ is homotopic to $[\mathbf{w}:\mathbf{w}'] \mapsto [\mathbf{w}:0:\mathbf{w}']$ through maps $\mathbb{C}\mathrm{P}^{(d+1)n-1} * \mathbb{C}\mathrm{P}^{(d+1)n'-1} \to \mathbb{C}\mathrm{P}^{(d+1)(n+n'+1)-1}$.

PROPOSITION 3.10. *Composition morphisms are associative, that is the following diagram commutes:*

$$\begin{tikzcd} HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma}') \otimes HZ_*(\boldsymbol{\sigma}'') \arrow[r] \arrow[d] & HZ_*(\boldsymbol{\sigma}) \otimes HZ_{*-2d}((\boldsymbol{\sigma}',\boldsymbol{\varepsilon},\boldsymbol{\sigma}'')) \arrow[d] \\ HZ_{*-2d}((\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}')) \otimes HZ_*(\boldsymbol{\sigma}'') \arrow[r] & HZ_{*-4d}((\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\sigma}',\boldsymbol{\varepsilon},\boldsymbol{\sigma}'')) \end{tikzcd}.$$

In symbols, given $\alpha \in HZ_*(\boldsymbol{\sigma})$, $\beta \in HZ_*(\boldsymbol{\sigma}')$ *and* $\gamma \in HZ_*(\boldsymbol{\sigma}'')$,

$$(\alpha \circledast \beta) \circledast \gamma = \alpha \circledast (\beta \circledast \gamma).$$

*This is also true for the relative case where one of the initial groups (*e.g. $HZ_*(\boldsymbol{\sigma}'')$*) is replaced by a relative homology group (*e.g. $HZ_*(\boldsymbol{\sigma}'',\boldsymbol{\sigma}^{(3)})$*) while the other groups are still absolute homology groups.*

PROOF. We first remark that

$$\begin{aligned} \widetilde{B}_{n+n'+1,n''}\left(\widetilde{B}_{n,n'}(\mathbf{w},\mathbf{w}'),\mathbf{w}''\right) &= \left(\mathbf{w}, \sum_{k=1}^{n'}(-1)^{k+1}w'_k, \mathbf{w}', \sum_{l=1}^{n''}(-1)^{l+1}w''_l, \mathbf{w}''\right) \\ &= \widetilde{B}_{n,n'+n''+1}\left(\mathbf{w}, \widetilde{B}_{n',n''}(\mathbf{w}',\mathbf{w}'')\right), \quad \forall \mathbf{w},\mathbf{w}',\mathbf{w}'', \end{aligned}$$

so that $B_{n+n'+1,n''} \circ (B_{n,n'} * \mathrm{id}) = B_{n,n'+n''+1} \circ (\mathrm{id} * B_{n',n''})$. Here, we denote by $f * g$ the map $(f * g)[a : b] := [\tilde{f}(a) : \tilde{g}(b)]$ where $\tilde{f}$ and $\tilde{g}$ are $\mathbb{C}$-linear lifts of the respective projective maps f and g. Therefore, for all $\alpha \in HZ_*(\boldsymbol{\sigma})$, $\beta \in HZ_*(\boldsymbol{\sigma'})$ and $\gamma \in HZ_*(\boldsymbol{\sigma''})$ (or $\gamma \in HZ_*(\boldsymbol{\sigma''}, \boldsymbol{\sigma}^{(3)})$),

$$\begin{aligned}
(B_{n+n'+1,n''})_*\big((B_{n,n'})_*(\alpha * \beta) * \gamma\big) &= (B_{n+n'+1,n''})_*(B_{n,n'} * \mathrm{id})_*((\alpha * \beta) * \gamma) \\
&= (B_{n,n'+n''+1})_*(\mathrm{id} * B_{n',n''})_*((\alpha * \beta) * \gamma) \\
&= (B_{n,n'+n''+1})_*(\mathrm{id} * B_{n',n''})_*(\alpha * (\beta * \gamma)) \\
&= (B_{n,n'+n''+1})_*\big(\alpha * (B_{n',n''})_*(\beta * \gamma)\big)
\end{aligned}$$

where we use the naturality of the homology projective join (2.6) to get the first and last identity, the previous remark to get the second equality and the associativity of the homology projective join (Proposition 2.5) to get the third equality. □

4.2. The direct system of $G_*^{(a,b)}(\boldsymbol{\sigma})$. For a fixed $m \in \mathbb{N}$, the long exact sequence of triple induces inclusion and boundary morphisms fitting into a long exact sequence:

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\boldsymbol{\sigma}, m) \to G_*^{(a,c)}(\boldsymbol{\sigma}, m) \to G_*^{(b,c)}(\boldsymbol{\sigma}, m) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\boldsymbol{\sigma}, m) \to \cdots$$

where $-m-1 < a \leq b \leq c < m+1$. In order to precisely define these maps without reference of m anymore, we will define an isomorphism

$$\theta_m^{m+1} : G_*^{(a,b)}(\boldsymbol{\sigma}, m) \to G_*^{(a,b)}(\boldsymbol{\sigma}, m+1), \tag{3.15}$$

for $-m \leq a \leq b \leq m$, that commutes with the above mentioned inclusion and boundary morphisms and define $G_*^{(a,b)}(\boldsymbol{\sigma})$ as the direct limit of the direct system induced by $(\theta_m^{m+1})_m$:

$$G_*^{(a,b)}(\boldsymbol{\sigma}) := \varinjlim G_*^{(a,b)}(\boldsymbol{\sigma}, m).$$

We will then have inclusion morphisms

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\boldsymbol{\sigma}) \to G_*^{(a,c)}(\boldsymbol{\sigma}) \to G_*^{(b,c)}(\boldsymbol{\sigma}) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\boldsymbol{\sigma}) \to \cdots$$

for all $a \leq b \leq c$ that are not action values; one can thus set

$$G_*^{(-\infty,b)}(\boldsymbol{\sigma}) := \varprojlim G_*^{(a,b)}(\boldsymbol{\sigma}), \quad a \to -\infty,$$

and one can then define $G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$ by taking a direct limit in a similar way. The inclusion and boundary morphisms thus extend to the extended real number line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ with the convention that $\pm\infty$ are not action values.

In order to define the isomorphism (3.15), let us remark that for an odd $n \in \mathbb{N}$, the space $Z(\boldsymbol{\varepsilon}^n)$ retracts on the projectivization of the maximal non-positive linear subspace of Q_n which is a $\mathbb{C}\mathrm{P}^{N-1}$ with $N = (d+1)(n+1)/2$ according to Proposition 3.2. Therefore,

$$HZ_*(\boldsymbol{\varepsilon}^n) = \bigoplus_{k=-(d+1)(n-1)/2}^{d} R a_k^{(n)} \simeq H_{*+(n-1)(d+1)}\left(\mathbb{C}\mathrm{P}^{(d+1)(n+1)/2-1}\right),$$

where $a_k^{(n)}$ is the generator of degree $2k$ identified with the class $[\mathbb{C}\mathrm{P}^l]$ of appropriate degree $2l = 2k + (n-1)(d+1)$ under the isomorphism induced by the inclusion of a maximal complex projective subspace of $Z(\boldsymbol{\varepsilon}^n)$. With the help of the composition defined in the previous section, we now define (3.15) by

$$\theta_m^{m+1}(\alpha) := \alpha \circledast a_d^{(n_0-1)} \in G_*^{(a,b)}(\boldsymbol{\sigma}, m+1), \quad \forall \alpha \in G_*^{(a,b)}(\boldsymbol{\sigma}, m).$$

This is formally well-defined since

$$HZ_*((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n_0}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n_0})) = G_*^{(a,b)}(\boldsymbol{\sigma}, m+1),$$

for $-m \leq a \leq b \leq m$, according to (3.4).

Proposition 3.11. *For an odd $n \in \mathbb{N}$, the morphism*

$$\left\{ \begin{array}{ccc} HZ_*(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a}) & \to & HZ_*((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n+1}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n+1})) \\ \alpha & \mapsto & \alpha \circledast a_d^{(n)} \end{array} \right.$$

is an isomorphism (and the same is true for $\alpha \mapsto a_d^{(n)} \circledast \alpha$).

Corollary 3.12. *The morphism θ_m^{m+1} is an isomorphism.*

Proof. Let $A := Z(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a})$, $A' := Z((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n+1}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n+1}))$, $B := Z(\boldsymbol{\varepsilon}^n)$, n' be the size of $\boldsymbol{\sigma}_{m,b}$ and let us denote by θ the morphism in question. Up to a shift in degree, the morphism θ can be written explicitly as the composition

$$H_*(A) \xrightarrow{\cdot * a_d^{(n)}} H_{*+i_0}(A * B) \xrightarrow{(B_{n',n})_*} H_{*+i_0}(A'),$$

for some $i_0 \in \mathbb{N}$. According to Corollary 2.2, the first morphism is an isomorphism. It remains to show that $(B_{n',n})_*$ is also an isomorphism.

Let C be the following automorphism of $(\mathbb{C}^{d+1})^{n'+n+1}$:

$$C(\mathbf{v}, \mathbf{v}') := (\mathbf{v}, \mathbf{v}' + (v_1, v_{n'}, v_1, v_{n'}, \ldots, v_1, v_{n'})), \tag{3.16}$$

where $\mathbf{v} \in (\mathbb{C}^{d+1})^{n'}$ and $\mathbf{v}' \in (\mathbb{C}^{d+1})^{n+1}$. By a direct computation

$$F_{(\boldsymbol{\sigma}_{m,t}, \boldsymbol{\varepsilon}^{n+1})} \circ C(\mathbf{v}, \mathbf{v}') = F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v}) + Q_{n+2}(0, \mathbf{v}').$$

According to Givental [**43**, Proposition B.1], $\{\widehat{F}_{(\boldsymbol{\sigma}_{m,t}, \boldsymbol{\varepsilon}^{n+1})} \circ C \leq 0\}$ retracts on $\{\widehat{F}_{\boldsymbol{\sigma}_{m,t}} \leq 0\} * \{\widehat{Q}_{n+2}(0, \cdot) \leq 0\}$. The quadratic form $Q_{n+2}(0, \cdot)$ is non-degenerate with index $(n+1)(d+1)$, so that the sublevel set $\{\widehat{Q}_{n+2}(0, \cdot) \leq 0\}$ retracts on a complex projective space of $\mathbb{C}$-dimension $(d+1)(n+1)/2 - 1$.

The injective linear map $J : (\mathbb{C}^{d+1})^n \to (\mathbb{C}^{d+1})^{n+1}$, $J\mathbf{v} := (\mathbf{v}, v_1)$ satisfies

$$Q_{n+2}(0, J\mathbf{v}) = Q_n(\mathbf{v}).$$

Since the sum of the index and the nullity of Q_n is $(n+1)(d+1)$, $\{\widehat{Q}_n \leq 0\}$ retracts on a projective space of $\mathbb{C}$-dimension $(d+1)(n+1)/2-1$ in such a way that J induces an isomorphism in homology

$$J_* : H_*\left(\left\{\widehat{Q}_n \leq 0\right\}\right) \xrightarrow{\simeq} H_*\left(\left\{\widehat{Q}_{n+2}(0, \cdot) \leq 0\right\}\right).$$

Let $P : (\mathbb{C}^{d+1})^{n+1} \to (\mathbb{C}^{d+1})^n$ be the surjective linear map

$$P(v_1, \ldots, v_{n+1}) := (v_{n+1}, v_2, v_3, \ldots, v_n)$$

so that $PJ = \mathrm{id}$. Let $P' : (\mathbb{C}^{d+1})^{n+n'+1} \to (\mathbb{C}^{d+1})^{n+n'}$ be $P'(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, P\mathbf{v}')$ and let $J' : (\mathbb{C}^{d+1})^{n+n'} \to (\mathbb{C}^{d+1})^{n+n'+1}$ be $J'(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, J\mathbf{v}')$ so that $P'J' = \mathrm{id}$. In v-variables, $\widetilde{B}_{n',n}$ takes the form

$$(\mathbf{v}, \mathbf{v}') \mapsto (\mathbf{v}, v_1, \mathbf{v}' + (v_1' - v_1, v_1 - v_1', v_1' - v_1, \ldots, v_1 - v_1')).$$

A direct computation then shows that the endomorphism $\tilde{f} := P'C^{-1}\widetilde{B}_{n',n}$ is invertible. More precisely, $\tilde{f}(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, \tilde{g}(\mathbf{v}') + \tilde{h}(\mathbf{v}))$ where $\tilde{g}$ and $\tilde{h}$ are $\mathbb{C}$-linear and $\tilde{g}$ is invertible. Let $f : A*B \to A*B$ be the induced projective map.

We then have the following commutative diagram:

$$\begin{tikzcd}
H_*(A*B) \arrow[rr, "(B_{n',n})_*"] \arrow[d, "f_*", "\simeq"'] & & H_*(A') \\
H_*(A*B) \arrow[r, shift left, "J'_*"] \arrow[r, phantom, "\simeq"] & H_*\left(A * \left\{\widehat{Q}_{n+2}(0,\cdot) \leq 0\right\}\right) \arrow[l, shift left, "P'_*"] \arrow[ur, "C_*"', "\simeq"] &
\end{tikzcd},$$

By the above discussion, the induced maps f_*, C_*, J'_* and P'_* are isomorphisms. Therefore, $(B_{n',n})_*$ is an isomorphism and so is θ. $\square$

By construction of the map θ_m^{m+1}, the following diagram commutes.

$$
\begin{tikzcd}
G_*^{(a,b)}(\boldsymbol{\sigma}, m) \arrow[rr, "\theta_m^{m+1}", "\simeq"'] & & G_*^{(a,b)}(\boldsymbol{\sigma}, m+1) \\
HZ_*(\boldsymbol{\sigma}_{m,b}) \arrow[u] \arrow[d] \arrow[rr, "\cdot \circledast a_d^{(n_0-1)}"] & & HZ_*(\boldsymbol{\sigma}_{m+1,b}) \arrow[u] \arrow[dd] \\
H_{*+r}(\mathbb{C}\mathrm{P}^N) \arrow[d] & & \\
H_{*+r}(\mathbb{C}\mathrm{P}^{N+n_0(d+1)}) \arrow[rr, "\cdot * [\mathbb{C}\mathrm{P}^{(d+1)n_0/2}]"] & & H_{*+r+n_0(d+1)}(\mathbb{C}\mathrm{P}^{N+n_0(d+1)})
\end{tikzcd}
\tag{3.17}
$$
,

where the vertical arrows are induced by inclusions, $r := (n_1 + mn_0 - 1)(d+1)$, n_1 is the size of $\boldsymbol{\sigma}$, $N := (n_1 + mn_0)(d+1) - 1$.

Applying the commutativity of (3.14) to $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ and $\boldsymbol{\sigma'} = \boldsymbol{\varepsilon}^{n'}$, one has

$$a_k^{(n)} \circledast a_l^{(n')} = a_{k+l-d}^{(n+n'+1)},$$

for $-(d+1)(n-1)/2 \leq k \leq d$ and $-(d+1)(n'-1)/2 \leq l \leq d$. By associativity of $\circledast$ (Proposition 3.10), we deduce that the isomorphism $\theta_m^{m+k} := \theta_{m+k-1}^{m+k} \circ \cdots \circ \theta_m^{m+1}$ is $\theta_m^{m+k}(\alpha) = \alpha \circledast a_d^{(kn_0-1)}$.

By using the same construction as for θ_m^{m+k}, one can define an isomorphism

$$\eta_k : G_*^{(a,b)}(\boldsymbol{\sigma}, m) \xrightarrow{\simeq} G_*^{(a,b)}((\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma}), m),$$

sending α to $a_d^{(k-1)} \circledast \alpha$ for each even $k \in \mathbb{N}$. This isomorphism commutes with the direct system induced by the θ_m^{m+1}'s and the inclusion morphisms and makes a diagram similar to (3.17) commute where in particular $\boldsymbol{\sigma}_{m+1,b}$ is replaced by $(\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma})_{m,b}$. The commutation with the direct system induces a natural final isomorphism

$$\eta_k : G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_*^{(a,b)}((\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma})). \tag{3.18}$$

The definition of θ_m^{m+1} easily extends to the local homology of any capped fixed point $\bar{z}$ of action $t \in (-m, m)$ into an isomorphism

$$\theta_m^{m+1} : \mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}, m) \xrightarrow{\simeq} \mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}, m+1) \tag{3.19}$$

that is compatible with inclusion morphisms. This morphism is still defined by composition of the homology join with $a_d^{(n_0-1)}$ with the morphism induced by a composition map $B_{n',n}$. Proof of Proposition 3.11 still holds in this case. Therefore, local homology does not depend on m and one can define algebraically $\mathrm{C}_*(\boldsymbol{\sigma}; \bar{z})$ by a direct limit.

4.3. Interpolation isomorphisms. We start this section with a general statement that is easily deduced from Morse theory.

PROPOSITION 3.13. *Let X be a closed manifold and $m > 0$. Let $f_{s,t} : X \to \mathbb{R}$, $s \in [0,1]$, $t \in [-m, m]$, be a C^1-family of maps. We suppose that for all $s \in [0,1]$, $t \in (-m, m)$ and $x \in X$, $\frac{\mathrm{d}}{\mathrm{d}t} f_{s,t}(x) \leq 0$. If $a, b \in (-m, m)$, $a \leq b$, satisfy that 0 is a regular value of $f_{s,a}$ and $f_{s,b}$ for all $s \in [0,1]$, then the inclusion $X \hookrightarrow [0,1] \times X$, $x \mapsto (s, x)$, induces the following isomorphism in homology for all $s \in [0,1]$*

$$H_*\big(\{f_{s,b} \leq 0\}, \{f_{s,a} \leq 0\}\big) \xrightarrow{\simeq} H_*\big(\{(r,x) \mid f_{r,b}(x) \leq 0\}, \{(r,x) \mid f_{r,a}(x) \leq 0\}\big),$$

where (r, x) are describing $[0,1] \times X$ in the right hand side of the arrow. The analogous non-relative statement is also true: let $f_s : X \to \mathbb{R}$, $s \in [0,1]$, be a C^1-family of maps with 0 as a regular value. Then the inclusion $X \hookrightarrow [0,1] \times X$, $x \mapsto (s, x)$, induces the following isomorphism in homology for all $s \in [0,1]$

$$H_*\left(\{f_s \leq 0\}\right) \xrightarrow{\simeq} H_*\left(\{(r,x) \mid f_r(x) \leq 0\}\right).$$

Proof. For all interval $I \subset [0,1]$, let $f_{I,t} : I \times X \to \mathbb{R}$ be the map $f_{I,t}(r,x) := f_{r,t}(x)$. Let $a \leq b$ be real numbers as above. By compactness of $[0,1]$, there exists an $\varepsilon > 0$ such that $[-\varepsilon, 2\varepsilon]$ contains only regular values of $f_{s,a}$ and $f_{s,b}$ for all $s \in [0,1]$. By compactness, there also exists a $\delta > 0$ such that $\|f_{s,c} - f_{r,c}\|_\infty < \varepsilon$ for $|s-r| \leq \delta$ and $c \in \{a,b\}$.

We recall that if topological pairs $A \subset B \subset C \subset D$ satisfy that the inclusion morphisms $H_*(A) \to H_*(C)$ and $H_*(B) \to H_*(D)$ are isomorphisms, then the inclusion morphism $H_*(B) \to H_*(C)$ is also an isomorphism. We apply this result to $A := (\{f_{I,b} \leq -\varepsilon\}, \{f_{I,a} \leq -\varepsilon\})$, $B := (I \times \{f_{s,b} \leq 0\}, I \times \{f_{s,a} \leq 0\})$, $C := (\{f_{I,b} \leq \varepsilon\}, \{f_{I,a} \leq \varepsilon\})$ and $D := (I \times \{f_{s,b} \leq 2\varepsilon\}, I \times \{f_{s,a} \leq 2\varepsilon\})$ for $I \subset [0,1]$ an interval of length less than δ and $s \in I$. Indeed, these topological pairs are increasing for $\subset$ by definition of δ and the needed isomorphisms come from Morse deformation lemma which can be applied by definition of ε. We thus have the following commutative diagram:

$$\begin{tikzcd}
H_*(I \times \{f_{s,b} \leq 0\}, I \times \{f_{s,a} \leq 0\}) \arrow[r, "\simeq"] & H_*(\{f_{I,b} \leq \varepsilon\}, \{f_{I,a} \leq \varepsilon\}) \\
H_*(\{s\} \times \{f_{s,b} \leq 0\}, \{s\} \times \{f_{s,a} \leq 0\}) \arrow[r, "\simeq"] \arrow[u, "\simeq"] & H_*(\{f_{I,b} \leq 0\}, \{f_{I,a} \leq 0\}) \arrow[u, "\simeq"']
\end{tikzcd},$$

where every morphism is induced by inclusion. The top arrow is an isomorphism by the above general fact. The left hand side arrow is an isomorphism because I retracts on $\{s\}$. The right hand side arrow is an isomorphism by the Morse deformation lemma, since $[0,\varepsilon]$ contains only regular values of $f_{I,a}$ and $f_{I,b}$. Therefore, the bottom arrow is an isomorphism.

According to the Mayer-Vietoris long exact sequence, given topological pairs A and B, if the inclusion morphism $H_*(A \cap B) \to H_*(A)$ and $H_*(A \cap B) \to H_*(B)$ are isomorphisms, then the inclusion morphism $H_*(A \cap B) \to H_*(A \cup B)$ is an isomorphism. We apply this result to $A := (\{f_{I,b} \leq 0\}, \{f_{I,a} \leq 0\})$ and $B := (\{f_{J,b} \leq 0\}, \{f_{J,a} \leq 0\})$ for increasing length of I and $\mathrm{length}(J) \leq \delta$ to show inductively the result. $\square$

Another way to proceed is to remark that $(\{f_{I,b} \leq 0\} \setminus \{f_{I,a} < 0\}, \{f_{I,a} = 0\})$ retracts on $(\{f_{s,b} \leq 0\} \setminus \{f_{s,a} < 0\}, \{f_{s,a} = 0\})$ relative to boundaries through the gradient flow of the restriction of the projection $I \times X \to I$, which has no critical points under the hypothesis made on a and b.

Let $s \mapsto \boldsymbol{\sigma}^{(s)}$ be a C^1-family (or more generally C^1-piecewise) of tuples associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism Φ without $\mathbb{C}$-line of fixed points. We apply Proposition 3.13 to the following family of maps:

$$f_s := \widehat{F}_{\boldsymbol{\sigma}^{(s)}} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}.$$

Assumptions of Proposition 3.13 are fulfilled and we define Δ so that the following diagram commutes:

$$\begin{tikzcd}
HZ_*(\boldsymbol{\sigma}^{(0)}) \arrow[r, "\simeq"] \arrow[d, "\simeq"', "\Delta"] & H_{*+i_0}(A) \\
HZ_*(\boldsymbol{\sigma}^{(1)}) \arrow[ur, "\simeq"] &
\end{tikzcd},$$

where the non-vertical arrows are the inclusions maps and

$$A := \{(r,x) \mid f_r(x) \leq 0\}.$$

Since these isomorphisms are defined with inclusion maps the above way, it clearly commutes with inclusion and boundary morphisms. In the same way, let $(\eta_t^{(s)})$ be a C^1-family of tuples so that $s \mapsto \eta_t^{(s)}$ is associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism for a fixed t. If the associated map $f_{s,t} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ satisfies the assumption of Proposition 3.13, then we can define the associated isomorphism

$$\Delta : HZ_*(\boldsymbol{\eta}_b^{(0)}, \boldsymbol{\eta}_a^{(0)}) \xrightarrow{\simeq} HZ_*(\boldsymbol{\eta}_b^{(1)}, \boldsymbol{\eta}_a^{(1)}).$$

As an important example, Let $s \mapsto \boldsymbol{\sigma}^{(s)}$ be a C^1-family (or more generally C^1-piecewise) of tuples associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism Φ. If $a \leq b$ are not action values of $\boldsymbol{\sigma}$, then $\eta_t^{(s)} := \boldsymbol{\sigma}_{m,t}^{(s)}$ satisfies the above assumption and Δ is an isomorphism

$$\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}, m) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)}, m).$$

We will call Δ the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{(s)})$ and write in symbols $\Delta \longleftrightarrow s \mapsto \boldsymbol{\sigma}^{(s)}$.

Interpolation isomorphisms are compatible with the partition of small action windows into local homologies in the following sense. Let $(\boldsymbol{\eta}_t^{(s)})$, $t \in I$, be a C^1-family of tuples so that $s \mapsto \boldsymbol{\eta}_t^{(s)}$ is associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism $e^{-2i\pi t}\Phi$ for a fixed t and denote by φ the associated Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$. Let us assume that $t_0 \in I$ is an action value with only finitely many fixed points $z_1, \dots, z_n$. Let $\zeta_1^{(s)}, \dots, \zeta_n^{(s)}$ be the associated family of critical points of f_{s,t_0}.

Proposition 3.14. *Under the assumptions of the above paragraph, for every $\varepsilon > 0$ small enough, if the assumption of Proposition 3.13 are satisfied at $a = t_0 - \varepsilon$ and $b = t_0 + \varepsilon$, the following diagram commutes*

$$\begin{tikzcd}
\bigoplus_j \mathrm{C}_{*+i_0}\left(f_{0,t_0};\zeta_j^{(0)}\right) \arrow[r, "\simeq"] \arrow[d, "\oplus_j\delta_j", "\simeq"'] & HZ_*\left(\eta_{t_0+\varepsilon}^{(0)}, \eta_{t_0-\varepsilon}^{(0)}\right) \arrow[d, "\Delta", "\simeq"'] \\
\bigoplus_j \mathrm{C}_{*+i_0}\left(f_{1,t_0};\zeta_j^{(1)}\right) \arrow[r, "\simeq"] & HZ_*\left(\eta_{t_0+\varepsilon}^{(1)}, \eta_{t_0-\varepsilon}^{(1)}\right)
\end{tikzcd},$$

where the horizontal isomorphisms are defined in the same way as (3.11) and δ_j is an isomorphism from $\mathrm{C}_*(f_{0,t_0};\zeta_j^{(0)})$ to $\mathrm{C}_*(f_{1,t_0};\zeta_j^{(1)})$.

Proof. Assuming $f_{s,t} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$, with $s \in [0,1]$ and $t \in I$, let $M \subset [0,1] \times I \times \mathbb{C}\mathrm{P}^N$ be the submanifold

$$M := \left\{ f_{s,t}(\zeta) = 0 \mid (s,t,\zeta) \in [0,1] \times I \times \mathbb{C}\mathrm{P}^N \right\},$$

and $\mathcal{T} : M \to I$ be the projection onto the t coordinate. For $s \in [0,1]$, let $M_s := M \cap s \times I \times \mathbb{C}\mathrm{P}^N$ and $\mathcal{T}_s : \mathcal{T}|_{M_s}$. Fixing a Riemannian metric on M, let $(s \times \psi_s^\tau)_\tau$ be the reversed gradient flow of $\mathcal{T}_s$ for the induced metric. Let $U_1, \dots, U_n \subset M$ be respective neighborhoods of the paths $(s, t_0, \zeta_1^{(s)}), \dots, (s, t_0, \zeta_n^{(s)}) \in M$ that are disjoint restricted to the small action window $I \times [t_0 - \varepsilon, t_0 + \varepsilon] \times \mathbb{C}\mathrm{P}^N$ and that are invariant under the flow $((s,x) \mapsto (s, \psi_s^\tau(s,x))_\tau$. Taking back the notation of Section 2 of Chapter 1, one gets the following commutative diagram of inclusion morphisms

$$\begin{tikzcd}
\bigoplus_j H_*\left(\mathcal{T}_0^{\leq t_0+\varepsilon} \cap U_j, \mathcal{T}_0^{\leq t_0-\varepsilon} \cap U_j\right) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & H_*\left(\mathcal{T}_0^{\leq t_0+\varepsilon}, \mathcal{T}_0^{\leq t_0-\varepsilon}\right) \arrow[d, "\simeq"] \\
\bigoplus_j H_*\left(\mathcal{T}^{\leq t_0+\varepsilon} \cap U_j, \mathcal{T}^{\leq t_0-\varepsilon} \cap U_j\right) \arrow[r] & H_*\left(\mathcal{T}^{\leq t_0+\varepsilon}, \mathcal{T}^{\leq t_0-\varepsilon}\right) \\
\bigoplus_j H_*\left(\mathcal{T}_1^{\leq t_0+\varepsilon} \cap U_j, \mathcal{T}_1^{\leq t_0-\varepsilon} \cap U_j\right) \arrow[r, "\simeq"] \arrow[u, "\simeq"'] & H_*\left(\mathcal{T}_1^{\leq t_0+\varepsilon}, \mathcal{T}_1^{\leq t_0-\varepsilon}\right) \arrow[u, "\simeq"']
\end{tikzcd}.$$

The top and bottom arrows are the natural isomorphisms in a small window of regular values around t_0 (1.3). The vertical arrows come from Proposition 3.13 applied with the isomorphism of Lemma 3.5. We recall that inclusion morphisms define a natural isomorphism (1.2)

$$\mathrm{C}_*\left(\mathcal{T}_s; \left(s, t_0, \zeta_j^{(s)}\right)\right) \simeq H_*\left(\mathcal{T}_s^{\leq t_0+\varepsilon} \cap U_j, \mathcal{T}_s^{\leq t_0-\varepsilon} \cap U_j\right),$$

and that the left-hand side of this equation is naturally isomorphic to the local homology of f_{s,t_0} at $\zeta_j^{(s)}$ through (3.10). The conclusion follows from the definition of the interpolation isomorphisms and Lemma 3.5. $\square$

Proposition 3.15. *Let Δ, Δ' and Δ'' be the interpolation isomorphisms associated with $(\boldsymbol{\sigma}^{(s)})$, $(\boldsymbol{\eta}_t^{(s)})$ and $(\boldsymbol{\sigma}^{(s)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_t^{(s)})$ respectively. The following diagram commutes:*

$$\begin{tikzcd}
HZ_*\left(\boldsymbol{\sigma}^{(0)}\right) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(0)}, \boldsymbol{\eta}_0^{(0)}\right) \arrow[r, "\circledast"] \arrow[d, "\simeq"', "\Delta\otimes\Delta'"] & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}^{(0)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(0)}\right), \left(\boldsymbol{\sigma}^{(0)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(0)}\right)\right) \arrow[d, "\simeq"', "\Delta''"] \\
HZ_*\left(\boldsymbol{\sigma}^{(1)}\right) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(1)}, \boldsymbol{\eta}_0^{(1)}\right) \arrow[r, "\circledast"] & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(1)}\right), \left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(1)}\right)\right)
\end{tikzcd}.$$

Proof. One can assume that either $(\boldsymbol{\sigma}^{(s)})$ or $(\boldsymbol{\eta}_t^{(s)})$ is independent of s. This is a direct consequence of the naturality of $(B_{n,n'})_*$ and a slightly generalized version of pj_* to projective bundles. Let $I := [0,1]$ and let us extend pj_* to subsets of $I \times \mathbb{C}\mathrm{P}^N$ the following way. Let $A \subset \mathbb{C}\mathrm{P}^n$ and $B \subset I \times \mathbb{C}\mathrm{P}^m$, we set $B_s := B \cap s \times \mathbb{C}\mathrm{P}^m$ and define $A * B \subset I \times \mathbb{C}\mathrm{P}^{m+n+1}$ by $A * B := \bigcup_s s \times (A * B_s)$. Let $E_{A,B}$ be the set of those $(a, (s,b), (t,c))$'s with $a \in A$, $(s,b) \in B$ and $(t,c) \in A * B$ such that $s = t$ and $c \in (ab)$; let $p_1 : E_{A,B} \to A \times B$ and $p_2 : E_{A,B} \to A * B$ be associated projection maps. Now p_1 is a $\mathbb{C}\mathrm{P}^1$-fiber bundle and $\mathrm{pj}_* : H_*(A \times B) \to H_{*+2}(A * B)$ is defined by $(p_2)_* \circ (p_1)^*$. Since E_{A,B_s} is the restriction of the fiber bundle $E_{A,B}$ to $A \times B_s$, by naturality of the morphisms involved, the following diagram commutes for all $s \in I$:

$$\begin{tikzcd}
H_*(A \times B_s) \arrow[r, "\mathrm{pj}_*"] \arrow[d] & H_{*+2}(A * B_s) \arrow[d] \\
H_*(A \times B) \arrow[r, "\mathrm{pj}_*"] & H_{*+2}(A * B)
\end{tikzcd},$$

where the vertical arrows are inclusion morphisms induced by $s \hookrightarrow I$. By giving to $Z(\boldsymbol{\eta}_t^I)$ the meaning of $\bigcup_s s \times Z(\boldsymbol{\eta}_t^{(s)})$ and then extending the definition of HZ_* accordingly, we deduce that the following diagram commutes for all $s \in I$:

$$\begin{tikzcd}
HZ_*(\boldsymbol{\sigma}) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(s)}, \boldsymbol{\eta}_0^{(s)}\right) \arrow[r, "\circledast"] \arrow[d, "\simeq"] & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(s)}\right), \left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(s)}\right)\right) \arrow[d, "\simeq"] \\
HZ_*(\boldsymbol{\sigma}) \otimes HZ_*\left(\boldsymbol{\eta}_1^I, \boldsymbol{\eta}_0^I\right) \arrow[r, "\circledast"] & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^I\right), \left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^I\right)\right)
\end{tikzcd},$$

where the vertical arrows are inclusion morphisms. The conclusion follows. ☐

In particular, the interpolation isomorphisms $\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}, m) \to G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)}, m)$ commutes with the direct system (θ_m^{m+1}), so it ultimately defines the interpolation isomorphism

$$\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)})$$

that commutes with inclusion and boundary morphisms.

It can also be easily deduced from the definition that the interpolation morphisms are homotopy invariant.

Proposition 3.16. *Let $(\boldsymbol{\eta}_t^{(r,s)})$, $t \in I$, $(r,s) \in [0,1]^2$ be a C^1-family of tuples so that $(r,s) \mapsto \boldsymbol{\eta}_t^{(r,s)}$ is associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism $e^{-2i\pi t}\Phi$ for a fixed t and $r \mapsto \boldsymbol{\eta}_t^{(r,0)}$ and $r \mapsto \boldsymbol{\eta}_t^{(r,1)}$ are constant maps. In other words, $(\boldsymbol{\eta}_t^{(r,s)})$ is a homotopy with fixed extremities between $(\boldsymbol{\eta}_t^{(0,s)})$ and $(\boldsymbol{\eta}_t^{(1,s)})$. Then the interpolation isomorphisms associated with $(\boldsymbol{\eta}_t^{(0,s)})$ and $(\boldsymbol{\eta}_t^{(1,s)})$ are equal.*

We are now in position to prove that $G_*^{(a,b)}(\boldsymbol{\sigma})$ and its inclusion and boundary morphisms are independent, up to isomorphism, of the choice of continuous family of n-tuple of small Hamiltonian flows $(\boldsymbol{\sigma}^s)$ generating the $\mathbb{C}$-equivariant Hamiltonian flow (Φ_s) lifting (φ_s) with $\boldsymbol{\sigma}^0 = \boldsymbol{\varepsilon}^n$ and $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}$. Indeed, let $(\boldsymbol{\sigma}^s)$ and $((\boldsymbol{\sigma}')^s)$ be a n-tuple and a n'-tuple of small Hamiltonian flows generating (Φ_s) with $n \geq n'$. One can define an isomorphism $G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}')$ by composition of morphism η_k defined at (3.18) and interpolation maps Δ in the following way:

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "\simeq"] \arrow[d, "\eta_{2n}", "\simeq"'] & G_*^{(a,b)}(\boldsymbol{\sigma'}) \arrow[d, "\eta_{3n-n'}", "\simeq"'] \\ G_*^{(a,b)}((\boldsymbol{\varepsilon}^{2n}, \boldsymbol{\sigma})) \arrow[d, "\Delta", "\simeq"'] & G_*^{(a,b)}((\boldsymbol{\varepsilon}^{3n-n'}, \boldsymbol{\sigma'})) \\ G_*^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{2n})) \arrow[r, "\Delta'", "\simeq"'] & G_*^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}, \boldsymbol{\varepsilon}^{n-n'}, \boldsymbol{\sigma'})) \arrow[u, "\Delta''"', "\simeq"] \end{tikzcd},$$

where Δ, Δ' and Δ'' are interpolation isomorphisms associated with isotopies of $3n$-tuples generating the same Hamiltonian diffeomorphism Φ in the following way

$$\begin{aligned} \Delta \quad &\longleftrightarrow \quad s \mapsto \begin{cases} (\boldsymbol{\sigma}^{2s}, \boldsymbol{\sigma}^{-2s}, \boldsymbol{\sigma}), & 0 \leq s \leq 1/2, \\ (\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-2(1-s)}, \boldsymbol{\sigma}^{2(1-s)}), & 1/2 \leq s \leq 1, \end{cases} \\ \Delta' \quad &\longleftrightarrow \quad s \mapsto (\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-s}, \boldsymbol{\varepsilon}^{n-n'}, (\boldsymbol{\sigma'})^s), \\ \Delta'' \quad &\longleftrightarrow \quad s \mapsto (\boldsymbol{\sigma}^{1-s}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}^{n-n'}, \boldsymbol{\sigma'}). \end{aligned}$$

4.4. Composition of generating function homology groups. Let us fix 2 tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'}$ of odd respective sizes n and n', $a, b, c \in \mathbb{R}$ that are not action values of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma'}$ respectively. Let $m, m' \in \mathbb{N}$ such that $m > 2m' > 4n$. The composition map has the form

$$HZ_*(\boldsymbol{\sigma'}_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) \xrightarrow{\circledast} HZ_*(\boldsymbol{\eta}_b, \boldsymbol{\eta}_a),$$

where

$$\boldsymbol{\eta}_t := \left(\boldsymbol{\sigma'}, \boldsymbol{\delta}_c^{(m')}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right).$$

Let $(\boldsymbol{\eta}_t^s)_s$ be a homotopy of tuples of small Hamiltonians from $\boldsymbol{\eta}_t^0 = \boldsymbol{\eta}_t$ to $\boldsymbol{\eta}_t^1 = (\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{t+c, m+m'}$, for $2|t| < m$, generating the same diffeomorphism for a fixed value of t. The condition $m' > 2m > 4n'$ makes the construction of such a homotopy possible, we sketch the successive stages of it:

$$\begin{aligned} \boldsymbol{\eta}_t = \left(\boldsymbol{\sigma'}, \boldsymbol{\delta}_c^{(m')}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right) &\rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}^{m'n_0+1}, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}, \boldsymbol{\varepsilon}^k, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \\ &\rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{m'n_0}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m+m')}\right) = (\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{t+c, m+m'}. \end{aligned}$$

According to the previous section, this homotopy induces an interpolation isomorphism

$$\Delta : HZ_*(\boldsymbol{\eta}_b, \boldsymbol{\eta}_a) \xrightarrow{\simeq} G_*^{(a+c, b+c)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'}), m + m').$$

The composition of the above composition morphism with Δ gives this generating function homology version of the composition morphism:

$$HZ_*(\boldsymbol{\sigma'}_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) \to G_{*-2d}^{(a+c, b+c)}((\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m + m').$$

We define the same way composition morphism of absolute homology groups:

$$HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma'}_{m',t'}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})_{m+m', t+t'}).$$

We will denote these maps $\alpha \otimes \beta \mapsto \alpha \diamond \beta$ so that in symbols

$$\alpha \diamond \beta := \Delta(\alpha \circledast \beta).$$

Since interpolation isomorphisms commute with inclusion in the total space, the commutativity of (3.14) implies the commutativity of the analogous square

$$\begin{tikzcd} HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma'}_{m',t'}) \arrow[r, "\diamond"] \arrow[d] & HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})_{m+m', t+t'}) \arrow[d] \\ H_{*+r}(\mathbb{C}\mathrm{P}^N) \otimes H_{*+r'}(\mathbb{C}\mathrm{P}^{N'}) \arrow[r, "\mathrm{pj}_*"] & H_{*+r+r'+2}(\mathbb{C}\mathrm{P}^{N''}) \end{tikzcd} \tag{3.20}$$

This new form of the composition morphism is also associative.

COROLLARY 3.17. *The following diagram of composition morphisms commutes:*

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}_{m,c}) \otimes HZ_*(\boldsymbol{\sigma}'_{m',c'}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}'', m'') & \longrightarrow & HZ_*(\boldsymbol{\sigma}_{m,c}) \otimes G_{*-2d}^{(a+c',b+c')}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''), m'+m'') \\ \downarrow & & \downarrow \\ HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}'', m'') & \longrightarrow & G_{*-4d}^{(a+c+c',b+c+c')}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''), m+m'+m'') \end{array}.$$

In symbols, given $\alpha \in HZ_*(\boldsymbol{\sigma}_{m,c})$, $\beta \in HZ_*(\boldsymbol{\sigma}'_{m',c'})$ *and* $\gamma \in G_*^{(a,b)}(\boldsymbol{\sigma}'', m'')$,

$$(\alpha \diamond \beta) \diamond \gamma = \alpha \diamond (\beta \diamond \gamma).$$

PROOF. According to Proposition 3.15,

$$(\alpha \diamond \beta) \diamond \gamma = \Delta_2(\Delta_1(\alpha \circledast \beta) \circledast \gamma) = \Delta_2 \circ \widetilde{\Delta}_1((\alpha \circledast \beta) \circledast \gamma),$$

where the interpolation isomorphisms are associated with homotopies in the following way:

$$\begin{array}{lcl} \Delta_1 & \longleftrightarrow & (\boldsymbol{\sigma}_{m,c}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'_{m',c'}) \rightsquigarrow (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \\ \widetilde{\Delta}_1 & \longleftrightarrow & ((\boldsymbol{\sigma}_{m,c}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'_{m',c'}), \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}) \rightsquigarrow ((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}), \\ \Delta_2 & \longleftrightarrow & ((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}) \rightsquigarrow (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'')_{m+m'+m'',c+c'+t}. \end{array}$$

In the same way,

$$\alpha \diamond (\beta \diamond \gamma) = \Delta_2'(\alpha \circledast \Delta_1'(\beta \circledast \gamma)) = \Delta_2' \circ \widetilde{\Delta}_1'(\alpha \circledast (\beta \circledast \gamma)),$$

with convenient interpolation isomorphisms Δ_1', $\widetilde{\Delta}_1'$ and Δ_2'. According to the associativity of $\circledast$ (Proposition 3.10), it is enough to prove that $\Delta_2 \circ \widetilde{\Delta}_1 = \Delta_2' \circ \widetilde{\Delta}_1'$. These two interpolation isomorphisms are associated with homotopies that are themselves homotopic through homotopies preserving the associated family of diffeomorphisms. According to Proposition 3.16, they are thus equal. ☐

As a consequence, the following diagram commutes

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) & \xrightarrow{\diamond} & G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m+m') \\ \simeq \Big\downarrow \mathrm{id} \otimes \theta_m^{m+1} & & \simeq \Big\downarrow \theta_{m+m'}^{m+m'+1} \\ HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m+1) & \xrightarrow{\diamond} & G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m+m'+1) \end{array}.$$

It ultimately defines a morphism

$$HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\diamond} G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma})),$$

for almost all $a, b \in \overline{\mathbb{R}}$.

By naturality of the composition morphism, given $t \geq 0$ and $m \in \mathbb{N}^*$, the following diagram commutes

$$\begin{array}{ccc} G_*^{(a,b)}(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\varepsilon}_{m,0}) & \xrightarrow{\diamond} & G_{*-2d}^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^2)) \\ \Big\downarrow \mathrm{id} \otimes \mathrm{inc}_* & & \Big\downarrow \mathrm{inc}_* \\ G_*^{(a,b)}(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\varepsilon}_{m,t}) & \xrightarrow{\diamond} & G_{*-2d}^{(a+t,b+t)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^2)) \end{array}. \tag{3.21}$$

5. Properties of the generating function homology

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ be two different tuples of (h_s). We proved that the graded modules $G_*^{(a,b)}(\boldsymbol{\sigma})$ and $G_*^{(a,b)}(\boldsymbol{\sigma}')$ are isomorphic and that there exists a family of isomorphisms compatible with the inclusion morphism so that it makes sense to define $G_*^{(a,b)}(h_s)$ as the isomorphism class of these graded modules. We will keep track of the specific choice of $\boldsymbol{\sigma}$ in our statements for the sake of precision.

5.1. Generating function homology of the identity. Let us first focus on the special case $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}$. Let us denote by $T_{m,t}$ the family of generating functions associated with $(\boldsymbol{\varepsilon}_{m,t})_t$. Since the elementary generating function of δ_s is $u \mapsto -\tan(\pi s)\|u\|^2$, the map $T_{m,t}$ is a quadratic form. Since $T_{m,t}$ is a generating function, its kernel as a quadratic form has dimension $2(d+1)$. We had already remarked that $T_{m,0}$ is equivalent to $-T_{m,0}$ (they both generates the identity) which implies that $\operatorname{ind} T_{m,0} = mn_0(d+1)$ (Proposition 3.2). The variation of index is governed by the Maslov index of $(e^{-2i\pi t})$ so that

$$\operatorname{ind} T_{m,t} - \operatorname{ind} T_{m,0} = 2(d+1)\lfloor t \rfloor, \quad \forall t \in (-m-1, m+1) \setminus \mathbb{Z}$$

(See Proposition 1.1). Therefore, there exists an increasing sequence of complex projective subspaces $P_{m,-m} \subset P_{m,-m+1} \subset \cdots \subset P_{m,m}$ such that $P_{m,k} \simeq \mathbb{C}\mathrm{P}^{(d+1)(k+mn_0/2)}$ and $Z(\boldsymbol{\varepsilon}_{m,t})$ retracts on $P_{m,\lfloor t \rfloor}$ inducing an equivalence between the persistence modules $(H_*(P_{m,\lfloor t \rfloor}))$ and $(H_*(Z(\boldsymbol{\varepsilon}_{m,t})))$, $-m-1 < t < m+1$. Thus, as a graded R-module,

$$HZ_*(\boldsymbol{\varepsilon}_{m,t}) = \bigoplus_{k=-(d+1)mn_0/2}^{d+(d+1)\lfloor t \rfloor} R a_k^{(mn_0+1)}(t),$$

where $a_k^{(mn_0+1)}(t)$ is the generator of degree $2k$ identified with the class $[\mathbb{C}\mathrm{P}^l]$ of appropriate degree $2l = 2k+(d+1)mn_0$ under the isomorphism induced by $P_{m,\lfloor t \rfloor} \hookrightarrow Z(\boldsymbol{\varepsilon}_{m,t})$. The inclusion morphism $HZ_*(\boldsymbol{\varepsilon}_{m,t}) \to HZ_*(\boldsymbol{\varepsilon}_{m,s})$ maps each $a_k^{(mn_0+1)}(t)$ to $a_k^{(mn_0+1)}(s)$ (for $-m-1 < t \leq s < m+1$). Hence,

$$G_*^{(a,b)}(\boldsymbol{\varepsilon}, m) = \bigoplus_{k=d+(d+1)\lfloor a \rfloor}^{d+(d+1)\lfloor b \rfloor} R \alpha_k^{(m)}(a,b),$$

for $-m-1 < a \leq b < m+1$, where $\alpha_k^{(m)}(a,b)$ is the image of $a_k^{(mn_0+1)}(b)$ under the inclusion morphism $HZ_*(\boldsymbol{\varepsilon}_{m,b}) \to G_*^{(a,b)}(\boldsymbol{\varepsilon}, m)$. According to the commutativity of (3.17), one has $\theta_m^{m+1}\alpha_k^{(m)}(a,b) = \alpha_k^{(m+1)}(a,b)$. We set $\alpha_k(a,b) := \theta_m^{\infty}\alpha_k^{(m)}(a,b)$. For $a < b < c$, if $\alpha_k(b,c)$ is well-defined, then $\alpha_k(a,c)$ is also well-defined and sent to the former. We deduce that there exists a well-defined $\alpha_k(-\infty,c) \in G_{2k}^{(-\infty,c)}(\boldsymbol{\varepsilon})$ sent to $\alpha_k(a,c)$ for all $a \leq c$. Let α_k be the image of $\alpha_k(-\infty,c)$ under $G_{2k}^{(-\infty,c)}(\boldsymbol{\varepsilon}) \to G_{2k}^{(-\infty,+\infty)}(\boldsymbol{\varepsilon})$. Finally,

$$G_*^{(-\infty,+\infty)}(\boldsymbol{\varepsilon}) = \bigoplus_{k \in \mathbb{Z}} R\alpha_k,$$

we will show in Theorem 3.22 that this is also the case for any $\boldsymbol{\sigma}$.

5.2. "Periodicity" of the generating function homology. In order to show the "periodicity" of the persistence module of $\boldsymbol{\sigma}$, let us define a natural isomorphism

$$q : G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma}). \tag{3.22}$$

In order to simplify the exposition, let us set $a_d := a_d^{(mn_0+1)}(0) \in HZ_{2d}(\boldsymbol{\varepsilon}_{m,0})$ and $a_{2d+1} := a_{2d+1}^{(mn_0+1)}(1) \in HZ_{2(2d+1)}(\boldsymbol{\varepsilon}_{m,1})$. According to Proposition 3.11, the morphism $G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \to G_*^{(a+1,b+1)}((\boldsymbol{\varepsilon}^2, \boldsymbol{\sigma}))$, $\alpha \mapsto a_d \diamond \alpha$, is an isomorphism; let us write $\alpha \mapsto a_d^{-1} \diamond \alpha$ its inverse morphism. We define the morphism q by $\alpha \mapsto a_d^{-1} \diamond a_{2d+1} \diamond \alpha$.

PROPOSITION 3.18. *The morphism q is an isomorphism commuting with inclusion and boundary morphisms.*

PROOF. The naturality of this morphism comes from the naturality of $\alpha \mapsto a_d \diamond \alpha$ and $\alpha \mapsto a_{2d+1} \diamond \alpha$. It remains to prove that $\alpha \mapsto a_{2d+1} \diamond \alpha$ is an isomorphism. Let us set $a_{-1} := a_{-1}^{(mn_0+1)}(-1) \in HZ_{-2}(\boldsymbol{\varepsilon}_{m,-1})$. According to the commutativity of (3.20), $a_{2d+1} \diamond a_{-1} = a_{-1} \diamond$

$a_{2d+1} = a_d$ where a_d is identified with $a_d^{(2mn_0+3)}(0)$. Thus, the following diagram commutes

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "a_{2d+1}\diamond\cdot"] \arrow[dr, "\simeq", "a_d\diamond\cdot"'] & G_{*+2(d+1)}^{(a+1,b+1)}((\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma})) \arrow[d, "a_{-1}\diamond\cdot"] \arrow[dr, "\simeq", "a_d\diamond\cdot"'] & \\ & G_*^{(a,b)}((\boldsymbol{\varepsilon}^4,\boldsymbol{\sigma})) \arrow[r, "a_{2d+1}\diamond\cdot"'] & G_{*+2(d+1)}^{(a+1,b+1)}((\boldsymbol{\varepsilon}^6,\boldsymbol{\sigma})) \end{tikzcd}$$

so every arrow in it is an isomorphism. $\square$

We remark that this morphism commutes by definition with the isomorphism $G_*^{(a,b)}(\boldsymbol{\sigma},m) \to G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma},m)$ defined by the same formula, as long as $(a,b+1)\subset I_m$. We still denote it q. We recall that the powers of the oriented generator u of $H^2(\mathbb{C}\mathrm{P}^\infty)$ acts naturally on $G_*^{(a,b)}(\boldsymbol{\sigma},m)$ by cap-product since $Z(\boldsymbol{\sigma}_{m,b})\subset\mathbb{C}\mathrm{P}^\infty$. The action of u on the $G_*^{(a,b)}(\boldsymbol{\sigma},m)$'s by cap-product then extends easily on the $G_*^{(a,b)}(\boldsymbol{\sigma})$'s for $a,b\in\overline{\mathbb{R}}$ by taking direct (and projective if $b=+\infty$) limits.

Proposition 3.19. *Let $m\in\mathbb{N}$ and $a<b$ such that $(a,b+1)\subset I_m$ and let us assume that $\boldsymbol{\sigma}=(\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma}')$, the following diagram commutes:*

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma},m) \arrow[r, "q", "\simeq"'] \arrow[dr] & G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma},m) \arrow[d, "\cdot\frown u^{d+1}"] \\ & G_*^{(a+1,b+1)}(\boldsymbol{\sigma},m) \end{tikzcd} .$$

where the top arrow is the isomorphism $\alpha\mapsto a_d^{-1}\diamond a_{2d+1}\diamond\alpha$ and the diagonal arrow is the inclusion morphism. By taking the limit as m tends to ∞, we thus have for $a<b$ in $\overline{\mathbb{R}}$ the commutative diagram

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "q", "\simeq"'] \arrow[dr] & G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma}) \arrow[d, "\cdot\frown u^{d+1}"] \\ & G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \end{tikzcd} .$$

Proof. For all $\alpha\in G_*^{(a,b)}(\boldsymbol{\sigma},m)$, one has

$$a_d^{-1}\diamond a_{2d+1}\diamond\alpha = a_d^{-1}\circledast(\Delta'^{-1}\Delta(a_{2d+1}\circledast\alpha)),$$

where

$$\begin{aligned} \Delta &\quad\longleftrightarrow\quad (\boldsymbol{\varepsilon}_{m,1},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t}) \rightsquigarrow (\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma}')_{2m,t+1},\\ \Delta' &\quad\longleftrightarrow\quad (\boldsymbol{\varepsilon}_{m,0},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t+1}) \rightsquigarrow (\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma}')_{2m,t+1}. \end{aligned}$$

Let Δ'' be the interpolation isomorphism associated with $s\mapsto(\boldsymbol{\varepsilon}_{m,1-s},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t+s})$. By producing a homotopy between the interpolation associated with $\Delta'^{-1}\Delta$ and this last interpolation, we see that $\Delta''=\Delta'^{-1}\Delta$ (Proposition 3.16), so that

$$a_d^{-1}\diamond a_{2d+1}\diamond\alpha = a_d^{-1}\circledast(\Delta''(a_{2d+1}\circledast\alpha)),\quad \forall\alpha\in G_*^{(a,b)}(\boldsymbol{\sigma},m).$$

In order to simplify the expressions, we set with abuse of notation

$$\begin{aligned} HZ_*(\boldsymbol{\varepsilon}_{1-j},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{t+j}) &:= HZ_*((\boldsymbol{\varepsilon}_{m,1-j},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,b+j}),(\boldsymbol{\varepsilon}_{m,1-j},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,a+j}))\\ G_*^{(a+j,b+j)}(\boldsymbol{\sigma}) &:= G_*^{(a+j,b+j)}(\boldsymbol{\sigma},m), \end{aligned}$$

for $j\in\{0,1\}$. The simplified notation a_d is too abusive here and we will distinguish $a_d(0):=a_d^{(mn_0+1)}(0)\in HZ_{2d}(\boldsymbol{\varepsilon}_{m,0})$ from $a_d(1):=a_d^{(mn_0+1)}(1)\in HZ_{2d}(\boldsymbol{\varepsilon}_{m,1})$. According to Section 5.1, $a_d(1)$ is the image of $a_d(0)$ under the inclusion morphism.

In order to prove the result, we will show that the following diagram commutes:
(3.23)

$$\begin{tikzcd}
G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[rrr, "q", "\simeq"'] \arrow[dr, "a_{2d+1}", "\simeq"'] \arrow[ddr, "a_d(1)"'] \arrow[dddrrr, rounded corners] & & & G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma}) \arrow[dl, "a_d(0)"', "\simeq"] \arrow[ddd, "\cdot\frown u^{d+1}"] \\
& HZ_{*+2(d+1)}(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_t) \arrow[r, "\Delta''", "\simeq"'] \arrow[d, "\cdot\frown u^{d+1}"] & HZ_{*+2(d+1)}(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{t+1}) \arrow[d, "\cdot\frown u^{d+1}"] & \\
& HZ_*(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_t) \arrow[r, "\Delta''", "\simeq"'] & HZ_*(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{t+1}) & \\
& & & G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \arrow[ul, "a_d(0)", "\simeq"']
\end{tikzcd},$$

where arrows marked $a_d(0)$, $a_d(1)$ and a_{2d+1} correspond respectively to morphisms $\alpha \mapsto a_d(0) \circledast \alpha$ etc. Indeed, objects and arrows that bound this diagram represent the diagram we want to prove the commutativity of.

Our first computation proves that the top trapezoid commutes.

The commutativity of the left (and only) triangle is a direct consequence of the commutativity of (2.7), the definition of $\circledast$ and the fact that $B^* u = u$ for any composition map $B = B_{m,n}$ (defined in the beginning of Section 4.4). The relation $B^* u = u$ comes from the fact that B is a $\mathbb{C}$-projective map.

The commutativity of the middle square is a consequence of the compatibility of inclusion morphisms (that define Δ'') with respect to the cap-product.

The commutativity of the right trapezoid is a consequence of Corollary 2.10 and the definition of $\circledast$. In order to apply Corollary 2.10, one must show that the set

$$A := \left(Z(\boldsymbol{\varepsilon}_{m,1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{m,b}), Z(\boldsymbol{\varepsilon}_{m,1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{m,a})\right)$$

is homotopy equivalent to a projective stabilization $B * \mathbb{C}\mathrm{P}^d$ through projective maps. Since $\boldsymbol{\sigma} = (\boldsymbol{\varepsilon}^2, \boldsymbol{\sigma}')$ by hypothesis,

$$A \xleftarrow{f} \left\{\widehat{Q}_3(0,\cdot) \leq 0\right\} * B,$$

where f is the projective map induced by the automorphism (3.16) for $n' = 1$ up to a transposition of coordinates. According to the paragraph following (3.16), f induced an isomorphism in homology and $B * \{\widehat{Q}_3(0,\cdot) \leq 0\}$ retracts on $B * P$ where P is a $\mathbb{C}\mathrm{P}^d$. Therefore, the inclusion of $B * P$ composed with f gives a continuous map satisfying the hypothesis of Corollary 2.10.

Let us show that the bottom "square" commutes, that is composition of inclusion with $a_d(0)$ is the same as the composition of $a_d(1)$ with Δ''. Let us consider the following commutative diagram:

$$\begin{tikzcd}
HZ_*(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_t) \arrow[rr, "\Delta''", "\simeq"'] \arrow[dr, "\simeq"] & & HZ_*(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_{t+1}) \arrow[dl, "\simeq"'] \\
& H_{*+i_0}(E) & \\
& HZ_*(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}_t) \arrow[u] \arrow[uul] \arrow[uur] &
\end{tikzcd},$$

where

$$E := \bigcup_{s \in [0,1]} s \times \left(Z(\boldsymbol{\varepsilon}_{m,1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{m,b+s}), Z(\boldsymbol{\varepsilon}_{m,1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{m,a+s})\right),$$

and unlabelled arrow are inclusion morphisms. This diagram commutes by naturality of inclusion map and by definition of Δ'' (*i.e.* the commutativity of the top triangle). Let $\alpha \in G_*^{(a,b)}(\boldsymbol{\sigma})$. We must show that the element $a_d(1) \circledast \alpha$ in the top-left object is sent to $a_d(0) \circledast \alpha$ under Δ''. By naturality of the homology join (2.6) and by definition of $\circledast$, since $a_d(1)$ is the image of $a_d(0)$ under the inclusion morphism, $a_d(1) \circledast \alpha$ is the image of $a_d(0) \circledast \alpha$ under the inclusion map that is the arrow from the bottom to the top-left. It is then a consequence of the commutativity. $\square$

In the sequel, we will always assume without loss of generality that our $\boldsymbol{\sigma}$'s have the form $(\boldsymbol{\varepsilon}^2, \boldsymbol{\sigma}')$.

Corollary 3.20. *For $a < b$ in $\overline{\mathbb{R}}$ that are not action values, the following diagram commutes:*

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[d, "\cdot\frown u^{d+1}"] \arrow[dr] & \\ G_{*-2(d+1)}^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "q", "\simeq"'] & G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \end{tikzcd},$$

which also implies the commutativity of the following square:

$$\begin{tikzcd} G_*^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "q", "\simeq"'] \arrow[d, "\cdot\frown u^{d+1}"] & G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma}) \arrow[d, "\cdot\frown u^{d+1}"] \\ G_{*-2(d+1)}^{(a,b)}(\boldsymbol{\sigma}) \arrow[r, "q", "\simeq"'] & G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \end{tikzcd}.$$

These results are also true without taking limits as long as every map is well-defined.

Proof. In the same spirit as for the proof of Lemma 2.9, the commutativity of the triangle is due to Proposition 3.19 and the naturality of the isomorphism q with respect to inclusion morphisms. $\square$

Let us recall that $\alpha \mapsto a_d \circledast \alpha$ is compatible with the decomposition in local homologies (3.11) (see the argument surrounding (3.19)) and so is $\alpha \mapsto a_{2d+1} \circledast \alpha$ by the same argument. According to the compatibility of the interpolation isomorphism (Proposition 3.14), it follows that the isomorphism q is also compatible. That is: if there is finitely many capped fixed points $\bar{z}_1, \dots, \bar{z}_n$ with action value t, the following diagram commutes for all sufficiently small $\varepsilon > 0$:

$$\begin{tikzcd} \bigoplus_j \mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}_j, m) \arrow[r, "\simeq"] \arrow[d, "\oplus_j q_j", "\simeq"'] & G_*^{(t-\varepsilon,t+\varepsilon)}(\boldsymbol{\sigma}, m) \arrow[d, "q", "\simeq"'] \\ \bigoplus_j \mathrm{C}_{*+2(d+1)}(\boldsymbol{\sigma}; \bar{z}_j \# A_0, m) \arrow[r, "\simeq"] & G_{*+2(d+1)}^{(t+1-\varepsilon,t+1+\varepsilon)}(\boldsymbol{\sigma}, m) \end{tikzcd},$$

where the horizontal arrows are natural isomorphisms (3.11) and $\bar{z}_j \# A_0$ denote the recapping of $\bar{z}_j$ by the generator A_0 of $\pi_2(\mathbb{C}\mathrm{P}^d)$ such that $-\langle \omega, A_0 \rangle = \pi$, that is the capping of z_j with action $t+1$.

The above last propositions informally implies, that for an isolated capped fixed point $\bar{z}$,

$$\text{“}\mathrm{C}_*(\boldsymbol{\sigma}; \bar{z}\# A_0) \frown u^{d+1} = \mathrm{C}_{*-2(d+1)}(\boldsymbol{\sigma}; \bar{z})\text{”}.$$

This statement is rather informal since the two local homologies are not subgroups of the same group. However, if these groups “persist” in a common action window, a clear relation can be stated. Let us assume that $a(\bar{z}) = 0$ and let $\bar{z}_1 := \bar{z}\# A_0$ in order to simplify the notation. The decomposition in local homologies (3.11) gives us a natural injective morphism $\mathrm{C}_*(\bar{z}) \hookrightarrow G_*^{(-\varepsilon,\varepsilon)}(\boldsymbol{\sigma})$. We assume that the composition of this injection with the inclusion morphism gives an injection $\mathrm{C}_*(\bar{z}) \hookrightarrow G_*^{(-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma})$; we say that the local homology $\mathrm{C}_*(\bar{z})$ “persists in the action window $(-\varepsilon, 1+\varepsilon)$”. Moreover, we assume that there exists a graded subgroup Γ_* of $G_*^{(-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma})$ whose image under the inclusion morphism in the window $(1-\varepsilon, 1+\varepsilon)$ is $\mathrm{C}_*(\bar{z}_1)$; we also say that the local homology $\mathrm{C}_*(\bar{z}_1)$ “persists in the action window $(-\varepsilon, 1+\varepsilon)$ as Γ_*”. The last propositions together with the compatibility with the decomposition in local homologies imply the following proposition.

Proposition 3.21. *Let $i_* : G_*^{(-\varepsilon,\varepsilon)}(\boldsymbol{\sigma}) \to G_*^{(-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma})$ denote the inclusion morphism. Let us assume that both the isolated capped fixed point $\bar{z}$ with action 0 and $\bar{z}\# A_0$ persist in the action window $(-\varepsilon, 1+\varepsilon)$ as the subgroups $i_*\mathrm{C}_*(\bar{z})$ and Γ_* respectively. Then,*

$$\Gamma_* \frown u^{d+1} = i_* \mathrm{C}_{*-2(d+1)}(\boldsymbol{\sigma}; \bar{z}).$$

This is also true without taking limits (with $m \geq 1$).

We will use this fact in the proof of Ginzburg-Gürel theorems. Since this proof will be using the cohomology groups rather than the homology groups, let us explicitly write the dual equality:

$$u^{d+1} \smile \Gamma^* = i^* \mathrm{C}^{*+2(d+1)}(\boldsymbol{\sigma}; \bar{z}\# A_0),$$

where Γ^* is a subgroup of $G^*_{(-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma})$ whose image under the inclusion morphism in the window $(-\varepsilon,\varepsilon)$ is $\mathrm{C}^*(\bar{z})$ whereas i^* is the inclusion morphism $G^*_{(1-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma}) \to G^*_{(-\varepsilon,1+\varepsilon)}(\boldsymbol{\sigma})$. This dual analogue can be seen as a consequence of Proposition 3.21 and Poincaré duality stated below in Proposition 3.25.

5.3. Spectral invariants.

Theorem 3.22. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms associated with the Hamiltonian diffeomorphism φ of $\mathbb{C}\mathrm{P}^d$. As a graded R-module,*

$$G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma}) = \bigoplus_{k\in\mathbb{Z}} R\alpha_k$$

for some non-zero α_k's with $\deg \alpha_k = 2k$ and $\alpha_{k+1} \frown u = \alpha_k$ for all $k \in \mathbb{Z}$. For all $k\in\mathbb{Z}$, let

$$c_k(\boldsymbol{\sigma}) := \inf\left\{ t\in\mathbb{R} \mid \alpha_k \in \operatorname{im}\left(G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})\right)\right\}.$$

Then for all $k\in\mathbb{Z}$, $c_k(\boldsymbol{\sigma})\in\mathbb{R}$ is an action value of $\boldsymbol{\sigma}$ and $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma})+1$. Moreover

$$c_k(\boldsymbol{\sigma}) \leq c_{k+1}(\boldsymbol{\sigma})$$

for all $k\in\mathbb{Z}$, and if there exists $k\in\mathbb{Z}$ such that $c_k(\boldsymbol{\sigma}) = c_{k+1}(\boldsymbol{\sigma})$, then φ has infinitely many fixed points with action $c_k(\boldsymbol{\sigma})$. If $d+1$ consecutive $c_k(\boldsymbol{\sigma})$'s are equal then $\varphi = \mathrm{id}$.

We will call the classes α_k associated with $\boldsymbol{\sigma}$ the *spectral classes* of $\boldsymbol{\sigma}$ and the action values $c_k(\boldsymbol{\sigma})$, the *spectral values* of $\boldsymbol{\sigma}$.

Corollary 3.23 (Fortune-Weinstein theorem). *Every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has at least $d+1$ fixed points.*

Proof of Corollary 3.23. According to the end of Theorem 3.22, one sees that a Hamiltonian diffeomorphism φ of $\mathbb{C}\mathrm{P}^d$ with finitely many fixed points must have spectral values $c_k(\boldsymbol{\sigma})$ that are all distinct. Since $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma})+1$, it implies that φ has at least $d+1$ action values modulo $\mathbb{Z}$. □

The following proposition is contained in the proof of Theorem 3.22. It makes precise the fact that one could informally think of α_k as "the class $[\mathbb{C}\mathrm{P}^{k+\infty}] \in H_{2(k+\infty)}(\mathbb{C}\mathrm{P}^{2\infty})$".

Proposition 3.24. *Let $\boldsymbol{\sigma}$ be an n_1-tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms and let $k\in\mathbb{Z}$. Let $m\in\mathbb{N}$, $K\in\mathbb{R}$ and $t\in\mathbb{R}$ such that $-m-1<-K<c_k(\boldsymbol{\sigma})<t<m+1$. We set $r := (n_1+mn_0-1)(d+1)$ and $N := (n_1+mn_0)(d+1)-1$. Let $\alpha'_k \in G_{2k}^{(-\infty,t)}(\boldsymbol{\sigma})$ be a class sent to α_k under the inclusion morphism $G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$. Then the image $\alpha''_k \in G_{2k}^{(-K,t)}(\boldsymbol{\sigma})$ under the inclusion morphism is non-zero and there exists $b'_k \in G_{2k}^{(-K,t)}(\boldsymbol{\sigma},m)$ such that $\theta_m^\infty(b'_k) = \alpha''_k$. The class b'_k is the image of a class $b_k \in HZ_{2k}(\boldsymbol{\sigma}_{m,t})$ that is sent to $[\mathbb{C}\mathrm{P}^{k+r/2}] \in H_{2k+r}(\mathbb{C}\mathrm{P}^N)$ under the morphism induced by inclusion $Z(\boldsymbol{\sigma}_{m,t}) \hookrightarrow \mathbb{C}\mathrm{P}^N$.*

Proof of Theorem 3.22. In Section 5.1, we have proved the theorem for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}$; hence for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ for all odd $n\in\mathbb{N}$. Let us show that the persistence module of any n-tuple $\boldsymbol{\sigma}$ and the persistence module of $\boldsymbol{\varepsilon}^n$ are δ-interleaved for some $\delta>0$.

Let $\boldsymbol{\sigma}$ be an n-tuple and let us denote $f_1,\dots,f_n : \mathbb{C}^{d+1} \to \mathbb{R}$ the elementary generating functions of $\sigma_1,\dots,\sigma_n$ respectively. For all $\zeta \in (\mathbb{C}^{d+1})^{n+mn_0}$, one has

$$\left| \widehat{F}_{\boldsymbol{\sigma}_{m,t}}(\zeta) - \widehat{F}_{(\boldsymbol{\varepsilon}^n)_{m,t}}(\zeta) \right| \leq M(\boldsymbol{\sigma}) := \max_{z_1,\dots,z_n\in B} \left| f_1(z_1)+\cdots+f_n(z_n)\right|,$$

where $B\subset\mathbb{C}^{d+1}$ denotes the closed unit ball. It follows that the inclusion maps induce a natural (*i.e.* that commutes with inclusions and the direct system) $M(\boldsymbol{\sigma})$-interleaving between $G_*^{(-K,t)}(\boldsymbol{\sigma},m)$ and $G_*^{(-K,t)}(\boldsymbol{\varepsilon}^n,m)$ with $K\in(-m,1-m)$ fixed and almost every $t\in(-K,m)$. These natural interleavings induced a $M(\boldsymbol{\sigma})$-interleaving between $G_*^{(-\infty,t)}(\boldsymbol{\sigma})$ and $G_*^{(-\infty,t)}(\boldsymbol{\varepsilon}^n)$. The fact that $G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$ is isomorphic as a graded R-module to $G_*^{(-\infty,+\infty)}(\boldsymbol{\varepsilon}^n)$ is now a direct consequence of the existence of an interleaving between their associated persistence modules. The characterisation of $c_k(\boldsymbol{\sigma})$ and α_k given by Proposition 3.24 is true for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ by the above discussion. Since the $M(\boldsymbol{\sigma})$-interleaving is induced by inclusion, Proposition 3.24 is still true for $\boldsymbol{\sigma}$ that is "α_k corresponds to $[\mathbb{C}\mathrm{P}^{N(m)+k}]$" seen in $G_*^{(a,b)}(\boldsymbol{\sigma}, m)$ for m, $-a$ and b large enough. It is easy to see that $\alpha_{k+1} \frown u = \alpha_k$ in the case $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}$. Indeed,

$$\alpha_{k+1}^{(m+1)}(-m-\varepsilon, m+\varepsilon) \frown u = \alpha_k^{(m+1)}(-m-\varepsilon, m+\varepsilon)$$

for a small $\epsilon > 0$, $m \in \mathbb{N}^*$ and k in between $d - m(d+1) + 1$ and $d + m(d+1)$ since $[\mathbb{C}\mathrm{P}^{l+1}] \frown u = [\mathbb{C}\mathrm{P}^l]$. The conclusion for all $\boldsymbol{\sigma}$ then follows from the $M(\boldsymbol{\sigma})$-interleaving.

The fact that $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma}) + 1$ is a direct consequence of Proposition 3.18 applied to $a = -\infty$ and $b = c_k(\boldsymbol{\sigma}) + \varepsilon$ for $\varepsilon > 0$. The last statements of the Theorem 3.22 are consequences of the Lyusternik-Schnirelmann theory, as one can see that $G_*^{(a,b)}(\boldsymbol{\sigma}, m)$ is naturally isomorphic to the relative homology of sublevel sets of one single map according to Lemma 3.5 and $\alpha_{k+1} \frown u = \alpha_k$ for all $k \in \mathbb{Z}$. $\square$

The dual statement holds for the generating function cohomology groups with the additional structure of R-algebra induced by the cup-product: the R-algebra $G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma})$ is generated by a class e of degree 2 that is invertible, of infinite order and satisfies $eu = e^2$:

$$G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma}) = \bigoplus_{k\in\mathbb{Z}} Re^k.$$

Informally one could think of e^k as "the class $u^{k+\infty} \in H^{2(k+\infty)}(\mathbb{C}\mathrm{P}^{2\infty})$". Therefore, one can define alternatively the spectral values $c_k(\boldsymbol{\sigma})$ by

$$c_k(\boldsymbol{\sigma}) = \inf\left\{t \in \mathbb{R} \mid e^k \notin \ker\left(G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma}) \to G^*_{(-\infty,t)}(\boldsymbol{\sigma})\right)\right\}.$$

Given an n-tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, we denote by $\boldsymbol{\sigma}^{-1}$ the n-tuple $(\sigma_n^{-1}, \dots, \sigma_1^{-1})$. We recall that if $f : \mathbb{C}^{d+1} \to \mathbb{R}$ is an elementary generating function of σ then $-f$ is an elementary generating function of σ^{-1}. Therefore, one has

$$F_{\boldsymbol{\sigma}^{-1}}(v_1, v_2, \dots, v_n) = -F_{\boldsymbol{\sigma}}(v_1, v_n, v_{n-1}, \dots, v_2) \tag{3.24}$$

(this identity has already been stated in the special case $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ in Proposition 3.2).

Given an n-tuple $\boldsymbol{\sigma}$ and an m-tuple $\boldsymbol{\sigma}'$, one has

$$F_{(\boldsymbol{\sigma},\boldsymbol{\sigma}')}(\mathbf{v}, \mathbf{v}') = F_{(\boldsymbol{\sigma}',\boldsymbol{\sigma})}(\mathbf{v}', \mathbf{v}), \quad \forall \mathbf{v} \in (\mathbb{C}^{d+1})^n, \forall \mathbf{v}' \in (\mathbb{C}^{d+1})^m. \tag{3.25}$$

Proposition 3.25 (Poincaré duality). *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{n+1}$. There exists a duality isomorphism between generating function homology and cohomology*

$$PD : G^*_{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_{d-*}^{(-b,-a)}(\boldsymbol{\sigma}^{-1}),$$

with $-\infty \leq a \leq b \leq +\infty$ and a, b not action values. This isomorphism is natural: it commutes with inclusion, boundary maps and the action by u (that is $PD(v \smile u) = PD(v) \frown u$).

Proof. Let us recall a version of the classical Poincaré duality (see for instance [**50**, Theorem 3.43]). Let M be a compact R-orientable n-dimensional manifold whose boundary ∂M is the union of two disjoint manifolds A and B. Then the cap-product with the fundamental class $[M] \in H_n(M, \partial M)$ gives a natural isomorphism $H^*(M, A) \to H_{n-*}(M, B)$. This general statement can be applied to sublevel sets of a C^1 map of a closed R-orientable n-manifold $f : W \to \mathbb{R}$ in the following way. Let $a < b$ be regular values of f so that $A := \{f = a\}$ and $B := \{f = b\}$ are the disjoint pieces of the boundary of the compact R-orientable n-manifold $M := \{a \leq f \leq b\}$. Now, by excision (which can be used because the boundary of M admits a collar neighborhood)

$$H^*(M, A) \simeq H^*\big(\{f \leq b\}, \{f \leq a\}\big) \quad \text{and} \quad H_*(M, B) \simeq H_*\big(\{-f \leq -a\}, \{-f \leq -b\}\big).$$

Finally, one has the duality isomorphism

$$PD : H^*\big(\{f \leq b\}, \{f \leq a\}\big) \xrightarrow{\simeq} H_{n-*}\big(\{-f \leq -a\}, \{-f \leq -b\}\big).$$

Let us apply the same idea in $W := \mathbb{C}\mathrm{P}^N$ with $N := (n+mn_0)(d+1)-1$:

$$\begin{aligned} G^*_{(a,b)}(\boldsymbol{\sigma}, m) &= H^{(n+mn_0-1)(d+1)+*}\left(\left\{\widehat{F}_{\boldsymbol{\sigma}_{m,b}} \leq 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}_{m,a}} \leq 0\right\}\right) \\ &\simeq H_{2N-((n+mn_0-1)(d+1)+*)}\left(\left\{\widehat{F}_{\boldsymbol{\sigma}_{m,a}} \geq 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}_{m,b}} \geq 0\right\}\right), \end{aligned}$$

where we have used that 0 is not a critical value of either $\widehat{F}_{\boldsymbol{\sigma}_{m,a}}$ or $\widehat{F}_{\boldsymbol{\sigma}_{m,b}}$. This last homology group is isomorphic to

$$H_{(n+mn_0-1)(d+1)+(2d-*)}\left(\left\{\widehat{F}_{(\boldsymbol{\sigma}^{-1})_{m,-a}} \leq 0\right\}, \left\{\widehat{F}_{(\boldsymbol{\sigma}^{-1})_{m,-b}} \leq 0\right\}\right) = G^{(-b,-a)}_{2d-*}(\boldsymbol{\sigma}^{-1}, m).$$

Indeed, according to identity (3.24), this homology group is naturally isomorphic to the homology group of a pair of sublevel sets of functions $-\widehat{F}_{(\boldsymbol{\sigma}_{m,t})^{-1}}$. The conclusion follows by applying Proposition 3.13 to an interpolation from $(\boldsymbol{\sigma}_{m,t})^{-1}$ to $(\boldsymbol{\sigma}^{-1})_{m,-t}$.

Let us precise the interpolation argument. One has $(\boldsymbol{\sigma}_{m,t})^{-1} = ((\boldsymbol{\delta}_t^{(m)})^{-1}, \boldsymbol{\sigma}^{-1})$. According to (3.25), the sublevel set induced by this tuple is homeomorphic to the one induced by $(\boldsymbol{\sigma}^{-1}, (\boldsymbol{\delta}_t^{(m)})^{-1})$ so it is enough to find an interpolation from $(\boldsymbol{\delta}_t^{(m)})^{-1}$ to $\boldsymbol{\delta}_{-t}^{(m)}$. By definition,

$$(\boldsymbol{\delta}_t^{(m)})^{-1} = (\boldsymbol{\varepsilon}^{qn_0}, \delta_s^{-1}, \delta_1^{-1}, \dots, \delta_1^{-1})$$

with $s = t - \lfloor t \rfloor$. We remark that $\delta_t^{-1} = \delta_{-t}$. It is then easy to find the wanted homotopy among tuples of the form $(\delta_{t_1}, \delta_{t_2}, \dots)$ with $t_1 + t_2 + \dots = -t$. □

Using both equivalent definitions of the spectral values $c_k(\boldsymbol{\sigma})$, the Poincaré duality implies the following identity.

Corollary 3.26. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, then*

$$c_k(\boldsymbol{\sigma}^{-1}) = -c_{d-k}(\boldsymbol{\sigma}), \quad \forall k \in \mathbb{Z}.$$

Finally, let us apply the composition morphisms to spectral classes in order to prove the sub-additivity of the spectral values.

Proposition 3.27. *Let $m, m' \in \mathbb{N}$ and $k, l \in \mathbb{Z}$ be such that $c_k(\boldsymbol{\sigma}) \in I_m$ and $c_l(\boldsymbol{\sigma}') \in I_{m'}$ and let $t \in (c_k(\boldsymbol{\sigma}), m)$ and $t' \in (c_l(\boldsymbol{\sigma}'), m')$. Let $b_k \in HZ_{2k}(\boldsymbol{\sigma}_{m,t})$ and $b'_l \in HZ_{2l}(\boldsymbol{\sigma}'_{m',t'})$ be classes associated with the respective spectral classes α_k and α'_l in the way expressed in Proposition 3.24. Then the composition morphism*

$$HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma}'_{m',t'}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',t+t'})$$

maps the class $b_k \otimes b'_l$ to a class b''_{k+l-d} that is sent to the $[\mathbb{C}\mathrm{P}^r] \in H_*(\mathbb{C}\mathrm{P}^N)$ of appropriate degree under the inclusion morphism.

Proof. This is a direct consequence of the commutativity of (3.20). □

Corollary 3.28. *Given any tuples $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, one has*

$$c_{k+l-d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')) \leq c_k(\boldsymbol{\sigma}) + c_l(\boldsymbol{\sigma}'), \quad \forall k, l \in \mathbb{Z}.$$

5.4. Barcodes of the generating function homology groups. As explained in Section 1, we can now associate to every $\boldsymbol{\sigma}$ a persistence module $(G_*^{(-\infty,t)}(\boldsymbol{\sigma}))_t$ that satisfies the "periodicity" property $G_*^{(-\infty,t+1)}(\boldsymbol{\sigma}) \simeq G_{*+2(d+1)}^{(-\infty,t)}(\boldsymbol{\sigma})$, the isomorphism being an isomorphism of persistence modules according to the naturality of (3.22). While discussing barcodes properties, we assume that the persistence module is over a field. Since this periodicity property shifts the degree by a constant positive integer $2(d+1)$, it induces a permutation of the bars of the barcode sending a bar (a,b) on a bar $(a+1,b+1)$ that generates a free $\mathbb{Z}$-action on the bars (in order to simplify notation, we write (a,b) for the finite bar $(a,b]$). A family of representatives of the bars is given by the union of the barcodes of $(G_k^{(-\infty,t)}(\boldsymbol{\sigma}))_t$ for $0 \leq k \leq 2d+1$. For instance, Figure 2 represents a part of the barcode of some $\boldsymbol{\sigma}$ associated with a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^1$. This barcode has 2 $\mathbb{Z}$-orbits of finite bars and $d+1=2$ $\mathbb{Z}$-orbits of infinite bars corresponding to the spectral values $c_1 + \mathbb{Z}$ and $c_2 + \mathbb{Z}$ where $c_k := c_k(\boldsymbol{\sigma})$.

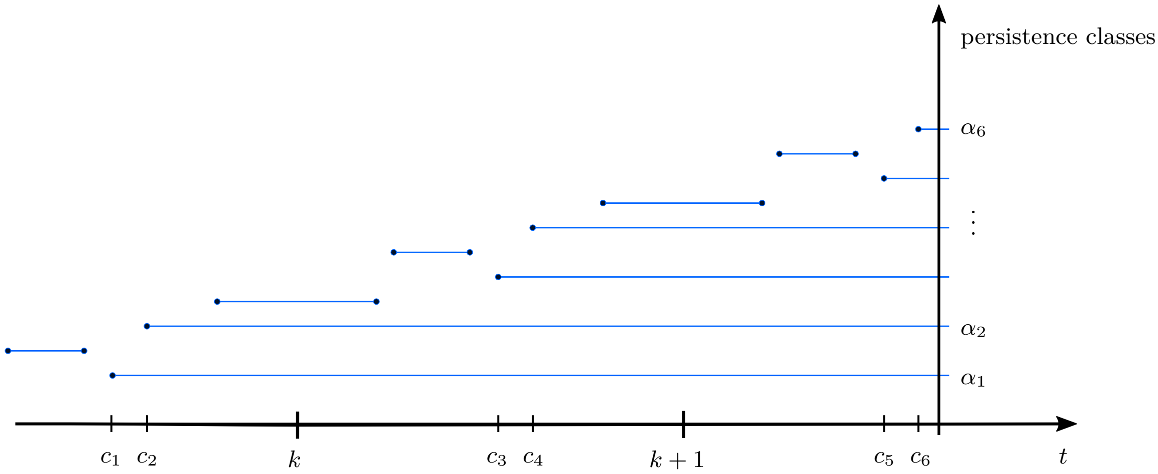

FIGURE 2. Barcode of a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^1$ in the neighborhood of $[k, k+1]$ for some $k \in \mathbb{Z}$ (bars of degree less than 2 are missing).

LEMMA 3.29. *Let $\boldsymbol{\sigma}$ be a tuple of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{n+1}$ with a finite number of fixed $\mathbb{C}$-lines. For every coefficient field $\mathbb{F}$ and $a < b$ that are not action values of $\boldsymbol{\sigma}$, the number of bars of the barcode of $\boldsymbol{\sigma}$ over the field $\mathbb{F}$ that intersect $t = a$ or $t = b$ but not both is equal to* $\dim G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F})$.

PROOF. Let us consider the long exact sequence

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F}) \xrightarrow{\partial_*} G_{*-1}^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_{*-1}^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to \cdots . \tag{3.26}$$

Applying the normal form theorem of persistence modules, one can find bases $((\alpha_i), (\delta_j^-))$ and $((\beta_k), (\delta_j^+))$ of the $\mathbb{F}$-vector spaces $G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F})$ and $G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F})$ that are in a canonical bijection with bars of the barcode intersecting $t = a$ and $t = b$ respectively, δ_j^- and δ_j^+ being associated with the same bar for each j while the bars associated with the α_i's do not intersect $t = b$ and the bars associated with the β_k's do not intersect $t = a$ (see Figure 3). In other words, the following diagram commutes

$$\begin{array}{ccc} G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) = & \bigoplus_i \mathbb{F}\alpha_i \oplus \bigoplus_j \mathbb{F}\delta_j^- & \\ \big\downarrow & \big\downarrow \simeq & , \\ G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) = & \bigoplus_j \mathbb{F}\delta_j^+ \oplus \bigoplus_k \mathbb{F}\beta_k & \end{array}$$

where the left arrow is the inclusion morphism and the right arrow sends δ_j^- to δ_j^+ for all j. Let us recall that, according to the finiteness assumption on the number of fixed points, the number of α_i's and β_k's is finite (here a and b are finite). With the above diagram, one can extract a short exact sequence of finite dimensional vector spaces from the long exact sequence (3.26)

$$0 \to \bigoplus_k \mathbb{F}\beta_k \to G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to \bigoplus_i \mathbb{F}\alpha_i \to 0.$$

Hence the result. $\square$

PROPOSITION 3.30. *Given a tuple $\boldsymbol{\sigma}$ of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms with a finite number of fixed $\mathbb{C}$-lines, for every field $\mathbb{F}$,*

$$N(\boldsymbol{\sigma}; \mathbb{F}) = d + 1 + 2K(\boldsymbol{\sigma}; \mathbb{F}),$$

where $K(\boldsymbol{\sigma}; \mathbb{F})$ is the number of $\mathbb{Z}$-orbits of finite bars of the persistence module of $\boldsymbol{\sigma}$ over the field $\mathbb{F}$. In other words, $N(\boldsymbol{\sigma}; \mathbb{F})$ is the number of (finite) extremities of a set of representative bars.

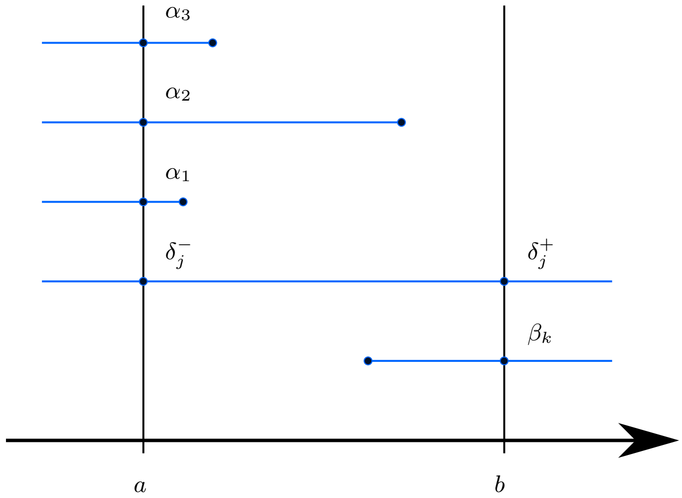

FIGURE 3. Relationship between the barcode of $\boldsymbol{\sigma}$ in the interval (a,b) and its persistence module (there are infinitely many bars that does not appear in the figure). The value $\dim G_*^{(a,b)}(\boldsymbol{\sigma};\mathbb{F})$ gives the number of α_i's and β_k's.

PROOF. According to the $\mathbb{Z}$-symmetry of the barcode, it boils down to proving that the numbers of extremities of the barcode lying inside $[0,1)$ is equal to $N(\boldsymbol{\sigma};\mathbb{F})$. Let $0 \leq t_1 < t_2 < \cdots < t_n < 1$ be the action values of $\boldsymbol{\sigma}$ in $[0,1)$, that is the points where extremities of the barcode could appear. Let $t_j^{\pm} := t_j \pm \varepsilon$ where $\varepsilon > 0$ is strictly less than the minimum distance between two action values so that t_j is the only action value inside $[t_j^-, t_j^+]$. According to Lemma 3.29, $\dim G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma};\mathbb{F})$ equals the number of extremities at $t = t_j$. Therefore, we just have to prove that

$$N(\boldsymbol{\sigma};\mathbb{F}) = \sum_{j=1}^{n} \dim G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma};\mathbb{F}).$$

Let us denote by φ the Hamiltonian diffeomorphism associated with $\boldsymbol{\sigma}$. Since t_j is the only action value in $[t_j^-, t_j^+]$, by (3.11)

$$G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma}) \simeq \bigoplus_z \mathrm{C}_*(\boldsymbol{\sigma}; z, t_j),$$

where the direct sum is over the fixed points $z \in \mathbb{C}\mathrm{P}^d$ of φ with action value t_j. By taking these isomorphisms over the field $\mathbb{F}$ for all j, the conclusion follows. $\square$

CHAPTER 4

Two theorems of Ginzburg and Gürel

1. Statements of the theorems

A compact invariant set $K \subset \mathbb{C}\mathrm{P}^d$ of a homeomorphism φ is said to be isolated if there exists a neighborhood U of K such that, for all $p \in U \setminus K$, $\varphi^k(p) \notin U$ for some $k \in \mathbb{Z}$. A fixed point of a Hamiltonian diffeomorphism is said to be homologically visible if its local Floer homology is non-trivial. The purpose of this chapter is to provide an elementary proof of the following theorem of Ginzburg-Gürel [**42**]:

Theorem 4.1. *Every Hamiltonian diffeomorphism φ of $\mathbb{C}\mathrm{P}^d$ has infinitely many periodic points provided it has a fixed point that is homologically visible for all iterates φ^k, $k \in \mathbb{N}^*$, and isolated as an invariant set.*

As Ginzburg-Gürel already pointed out, Theorem 4.1 has two important corollaries. If x is a hyperbolic point then it is always isolated as an invariant set and the local cohomology of every iterate has rank 1.

Corollary 4.2. *Every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ with a hyperbolic fixed point has infinitely many periodic points.*

In fact, this theorem of Ginzburg-Gürel was originally proven in [**41**] in a more general setting, including some complex Grassmannians, $\mathbb{C}\mathrm{P}^d \times P^{2k}$ where P is symplectically aspherical and $k \leq d$, monotone products $\mathbb{C}\mathrm{P}^d \times \mathbb{C}\mathrm{P}^d$. We mention that the case of $\mathbb{C}\mathrm{P}^d \times \mathbb{T}^{2k}$, when $k \leq d$, can be deduced as well from our techniques.

In the special case of pseudo-rotations, every fixed point arises from a spectral invariant and thus has a non-trivial local cohomology.

Corollary 4.3. *No fixed point of a pseudo-rotation of $\mathbb{C}\mathrm{P}^d$ is isolated as an invariant set.*

The original proof of Theorem 4.1 involves a non-trivial estimate on the energy of Floer trajectories leaving a periodic orbit called crossing energy theorem by Ginzburg-Gürel [**41**, Theorem 3.1] [**42**, Theorem 6.1] and proved with a Gromov compactness like theorem on J-holomorphic curves due to Fish [**34**]. The second ingredient of the original proof is quantum homology, which is defined by means of Gromov-Witten invariants. Although we closely follow the original argument, our proof employs only Morse theory and classical algebraic topology.

2. Proof of Theorem 4.1 and its corollaries

In this section, we prove Theorem 4.1, postponing the proof of the crossing energy theorem to Section 3. We then provide the proofs of Corollaries 4.2 and 4.3 sketched in the introduction.

2.1. Preliminaries. Let $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ be the time-one map of a Hamiltonian flow (φ_s) with a fixed point $x \in \mathbb{C}\mathrm{P}^d$ which is isolated as an invariant set. Moreover, let (Φ_s) be a $\mathbb{C}$-equivariant Hamiltonian flow of $\mathbb{C}^{d+1}$ lifting (φ_s) and let us suppose that the action value $a(x) = 0$ (by adding a constant to the Hamiltonian map (h_s), see (3.1) for the definition of a) and that the local cohomology groups of x associated with the iterations of φ are all non-zero.

Let us fix a C^1-family of n_1-tuples $(\boldsymbol{\sigma}_s)$ associated with (Φ_s) with n_1 even of the form $\boldsymbol{\sigma}_s = (\boldsymbol{\varepsilon}, \boldsymbol{\sigma}'_s)$ (in order to apply Proposition 3.21). We will exclusively work inside the small action window $I := (-\varepsilon, 1+\varepsilon)$ where $\varepsilon > 0$ is small enough. In order to study iterations of φ, let us set for all $m \in \mathbb{N}^*$ and $t \in I$

$$F_t^m := F_{(\varepsilon, \boldsymbol{\sigma}^m)_{1,t}} : \mathbb{C}^{N(m)+1} \to \mathbb{R},$$

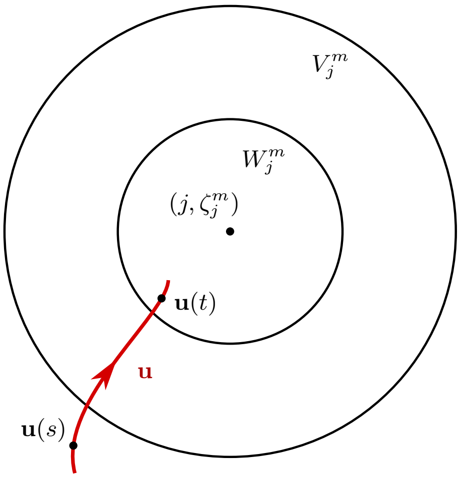

FIGURE 4. Crossing energy theorem

with $N(m) = (d+1)(n_1 m + n_0 + 1) - 1$. Let M^m be the submanifold of $I \times \mathbb{C}\mathrm{P}^{N(m)}$ defined by

$$M^m := \Big\{(t,[z]) \in I \times \mathbb{C}\mathrm{P}^{N(m)} \mid F_t^m(z) = 0\Big\}$$

and $\mathcal{T}^m : M^m \to I$ be the projection onto the first coordinate. Therefore, according to Lemma 3.5, for all $a \le b$ in I, one has a natural isomorphism

$$H^*\big(\{\mathcal{T}^m \le b\}, \{\mathcal{T}^m \le a\}\big) \simeq G^{*-i(m)}_{(a,b)}((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^m), 1),$$

with $i(m) := (n_1 m + n_0)(d+1)$.

We will prove Theorem 4.1 by contradiction: let us assume that φ has only finitely many periodic points so that $\mathcal{T}^m$ has only isolated critical points in a finite number for all $m \in \mathbb{N}^*$. In our construction of $\mathcal{T}^m$, we take $\varepsilon < 1/2$ so that any fixed point of φ have at most 2 associated critical points. Taking an iteration, we might suppose that any periodic point is a fixed point of φ. For all $m \in \mathbb{N}^*$, let $(j, \zeta_j^m) \in M^m$, $j \in \{0,1\}$, be the critical points of $\mathcal{T}^m$ associated with x. In Section 3, we prove the crossing energy theorem which applies to our point x, isolated as an invariant set, in the following way:

THEOREM 4.4. *There exist $c_\infty > 0$, families of open neighborhoods $V_j^m, W_j^m \subset M^m$ of (j,ζ_j^m) with $\overline{W_j^m} \subset V_j^m$ which do not intersect* $\mathrm{Crit}(\mathcal{T}^m) \setminus (j,\zeta_j^m)$ *and an adapted pseudo-gradient X_m of $\mathcal{T}^m$, such that any flow or reversed-flow line $\mathbf{u} : \mathbb{R} \to M^m$, $\dot{\mathbf{u}} = \pm X_m(\mathbf{u})$, with $\mathbf{u}(s) \notin V_j^m$ and $\mathbf{u}(t) \in W_j^m$ for some $m \in \mathbb{N}^*$ and $j \in \{0,1\}$ satisfies*

$$|\mathcal{T}^m(\mathbf{u}(s)) - \mathcal{T}^m(\mathbf{u}(t))| > c_\infty.$$

Let $c_\infty > 0$ be given by the above result. Without loss of generality, we suppose that

$$0 < c_\infty < \frac{1}{2(d+1)}. \tag{4.1}$$

2.2. Augmented action. We will follow an arithmetic trick due to Ginzburg-Gürel in order to find a good power $m \in \mathbb{N}^*$ so that the actions of the fixed points of φ^m are well distributed with respect to their indices.

For a fix point $y \in \mathbb{C}\mathrm{P}^d$, let us denote by y^m the unique associated capped fixed point of φ^m that has action $a(y^m) \in [0,1)$. With a slight abuse of notation, $a(y) \in [0,1)$ will mean $a(y^1)$. Hence,

$$a(y^m) = ma(y) - \lfloor ma(y) \rfloor.$$

According to (3.12),

$$\operatorname{supp} \mathrm{C}^*((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^m); y^m) \subset [\mathrm{mas}(y^m), \mathrm{mas}(y^m) + \nu(y^m)]\,,$$

so Bott's iteration inequalities (1.10) imply

$$\operatorname{supp} \mathrm{C}^*((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^m); y^m) \subset [\overline{\mathrm{mas}}(y^m) - d, \overline{\mathrm{mas}}(y^m) + d]\,. \tag{4.2}$$

LEMMA 4.5. *Let $y \in \mathbb{CP}^d$ be a fixed point of φ, then*

$$\overline{\mathrm{mas}}(y^m) = m\,\overline{\mathrm{mas}}(y) - 2(d+1)\lfloor ma(y)\rfloor, \quad \forall m \in \mathbb{N}^*.$$

PROOF. Let $y \in \mathbb{CP}^d$ be fixed by φ, $\tilde{y} \in \mathbb{C}^{d+1} \setminus 0$ a lift, $m \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$. In $\mathrm{Sp}(2(d+1))$, the path $s \mapsto \mathrm{d}\left(e^{-2i\pi kma(y)s}\Phi_{kms}\right)(\tilde{y})$ is homotopic relative to endpoints to the concatenation of the path $s \mapsto \mathrm{d}\left(e^{-2i\pi ka(y^m)s}\Phi_{kms}\right)(\tilde{y})$ and the loop $\Gamma : s \mapsto e^{-2i\pi k\lfloor ma(y)\rfloor s}$, thus Proposition 1.1 (1) and (5) imply

$$\mathrm{mas}\left(\tilde{y}, \left(e^{-2i\pi kma(y)s}\Phi_{kms}\right)\right) = \mathrm{mas}\left(\tilde{y}, \left(e^{-2i\pi ka(y^m)s}\Phi_{kms}\right)\right) + \mathrm{mas}(\Gamma),$$

According to Proposition 1.1, $\mathrm{mas}(\Gamma) = 2(d+1)k\lfloor ma(y)\rfloor$, thus, dividing by k and letting $k \to \infty$, we get

$$\overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi ma(y)s}\Phi_{ms}\right)\right) = \overline{\mathrm{mas}}\left(\tilde{y}, \left(e^{-2i\pi a(y^m)s}\Phi_{ms}\right)\right) + 2(d+1)\lfloor ma(y)\rfloor.$$

It implies the result according to Proposition 1.2. ☐

By analogy with Ginzburg-Gürel augmented action, for any fixed point $y \in \mathbb{CP}^d$ we define the real number

$$\tilde{a}(y^m) := m\left(a(y) - \frac{1}{2(d+1)}\,\overline{\mathrm{mas}}(y)\right) = m\tilde{a}(y).$$

According to Lemma 4.5,

$$\begin{aligned}\overline{\mathrm{mas}}(y^m) - \overline{\mathrm{mas}}(x^m) &= m\big(\overline{\mathrm{mas}}(y) - \overline{\mathrm{mas}}(x)\big) - 2(d+1)\lfloor ma(y)\rfloor \\ &= 2m(d+1)\big(\tilde{a}(x) - \tilde{a}(y)\big) + 2(d+1)a(y^m).\end{aligned} \tag{4.3}$$

By Dirichlet's lemma, one can find $m \in \mathbb{N}^*$ such that, for all fixed point y, the fraction part of each $ma(y)$, which is $a(y^m)$, satisfies

$$a(y^m) \in [0, c_\infty) \cup (1 - c_\infty, 1)$$

with m taken sufficiently large so that

$$|\tilde{a}(y) - \tilde{a}(x)| = 0 \quad \text{or} \quad m|\tilde{a}(y) - \tilde{a}(x)| > 3.$$

Thus Equation (4.3) together with assumption (4.1) implies the following lemma.

LEMMA 4.6. *With this specific choice of $m \in \mathbb{N}^*$, given any fixed point $y \in \mathbb{CP}^d$, we have:*
- $|\overline{\mathrm{mas}}(y^m) - \overline{\mathrm{mas}}(x^m)| \leq 2d+1$ *implies* $a(y^m) < c_\infty$,
- $|\overline{\mathrm{mas}}(y^m) - (\overline{\mathrm{mas}}(x^m) + 2(d+1))| \leq 2d+1$ *implies* $a(y^m) > 1 - c_\infty$.

Given two subsets $A, B \subset \mathbb{R}$, we denote the smallest distance among their points by

$$\mathrm{dist}(A, B) := \inf\{|a - b| \mid a \in A, b \in B\} \in [0, +\infty].$$

Let (x^m, j), $j \in \{0, 1\}$, denote the capped fixed point of φ^m associated with x with action j. According to the inclusion of support (4.2), Lemma 4.6 implies

COROLLARY 4.7. *With this specific choice of $m \in \mathbb{N}^*$, given any capped fixed point $\bar{y}$ of φ^m of action $t \in (-\varepsilon, 1+\varepsilon)$, if*

$$\mathrm{dist}\left(\mathrm{supp}\,\mathrm{C}^*\big((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^m); \bar{y}\big), \mathrm{supp}\,\mathrm{C}^*\big((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^m); (x^m, j)\big)\right) \leq 1,$$

then $|t - j| < c_\infty$, for $j \in \{0, 1\}$.

2.3. Subordinated min-max. By taking the m-th iteration of $\varphi \in \mathbb{CP}^d$, we can suppose that φ satisfies Corollary 4.7 for $m = 1$ and write $\mathcal{T} := \mathcal{T}^m$ and $M := M_m$. Let $G_t : M \to M$ be the gradient flow associated with the pseudo-gradient of Theorem 4.4 at time $t \in \mathbb{R}$. In order to simplify notation in this section, given any subset $U \subset M$ and any $b \in I$, we set $U^{\leq b} := U \cap \{\mathcal{T} \leq b\}$ and $U^{<b} := U \cap \{\mathcal{T} < b\}$, whereas we denote by $\mathrm{C}^*(y)$ the local cohomology associated with a critical points $y \in M$. For all critical points $y \in M$, let us define a specific flow-out $U(y)$, that is an open neighborhood of y which is invariant under G_t for all $t \geq 0$. We take a small neighborhood B of y, then we set

$$U'(y) := \bigcup_{t \geq 0} G_t(B),$$

let $\{z_j\}$ be the family of critical points in the closure of $U'(y)$, we then define

$$U(y) := U'(y) \cup \bigcup_j U'(z_j).$$

Let $z_0 := (0, \zeta_0)$ and $z_1 := (1, \zeta_1)$ be the two critical points associated with the fixed point $x \in \mathbb{C}\mathrm{P}^d$. Applying Theorem 4.4 together with Corollary 4.7, we choose B small enough such that $z_j \in U(y)$ implies that

$$\operatorname{dist}\left(\operatorname{supp} \mathrm{C}^*(y), \operatorname{supp} \mathrm{C}^*(z_j)\right) > 1, \tag{4.4}$$

and in the case where $y = z_j$, we do the same so that this last equation holds also for critical points $y \in U(z_j)$ (we recall that the local cohomologies with respect to $\mathcal{T}$ are isomorphic to the local cohomologies with respect to $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ up to a shift in degree that is independent on the critical points, according to Section 3.3). We first prove that the local cohomology $\mathrm{C}^*(z_0)$ "persists in the action window $[0, 1+\varepsilon)$". Let $v_0 \in \mathrm{C}^*(z_0)$ be a non-zero class, which exists by hypothesis.

Lemma 4.8. *For all $b \in [0, 1+\varepsilon)$, there exists a class $v \in H^*(M^{\leq b}, M^{<0})$ such that its image under the morphism induced by the inclusion $H^*(M^{\leq b}, M^{<0}) \to H^*(M^{\leq 0}, M^{<0})$ is v_0. Moreover, given one of the above flow-outs $U = U(y)$,*

$$v \notin \ker\left(H^*(M^{\leq b}, M^{<0}) \to H^*(U^{\leq b}, U^{<0})\right)$$

if and only if $z_0 \in U$, where the morphism is induced by inclusion.

Proof. According to Morse deformation lemma, if the lemma is true for b and $(b, c] \subset I$ does not contain any critical value, then the lemma is also true for c. Since there is a finite number of critical values, we can thus prove this lemma inductively on the critical value $b \geq 0$. We start with the case $b = 0$. As we have seen, by excision $\mathrm{C}^*(z_0) \subset H^*(M^{\leq 0}, M^{<0})$ and taking $v = v_0$ under this injection is enough.

Let us assume that $b > 0$ is a critical value and that the lemma is true on $[0, b)$. Let (y_k) be the family of critical points of value b and $U_k := U(y_k)$ be their associated flows-out. We will work with the following commutative diagram:

$$\begin{tikzcd} H^*(M^{\leq b}, M^{<b}) \arrow[r, "j^*"] \arrow[d] & H^*(M^{\leq b}, M^{<0}) \arrow[r, "i^*"] \arrow[d] & H^*(M^{<b}, M^{<0}) \arrow[d] \\ H^*(U_k^{\leq b}, U_k^{<b}) \arrow[r, "j_k^*"] & H^*(U_k^{\leq b}, U_k^{<0}) \arrow[r, "i_k^*"] & H^*(U_k^{<b}, U_k^{<0}) \end{tikzcd} \tag{4.5}$$

where every arrow is induced by inclusion. By Morse deformation lemma the $M^{<b}$ and $U^{<b}$ in the right hand side of the diagram can be replaced by $M^{\leq c}$ and $U^{\leq c}$ for some $c < b$ close enough. By induction, there thus exists $v' \in H^*(M^{<b}, M^{<0})$ satisfying the lemma (with symbol $\leq b$ replaced by $< b$). Let us first show that v' is in the image of i^*. According to the long exact sequence of the triple $(M^{\leq b}, M^{<b}, M^{<0})$, it boils down to showing that $\partial^* v' = 0$ where ∂^* is the coboundary morphism. By contradiction let us assume that $\partial^* v' \neq 0$. By excision, we recall that

$$H^*(M^{\leq b}, M^{<b}) \simeq \bigoplus_k \mathrm{C}^*(y_k),$$

thus if $\partial^* v' \neq 0$ then $\partial_k^* v' \neq 0$ for some k, where ∂_k^* is the composition of the coboundary morphism with the projection on $\mathrm{C}^*(y_k)$ with respect to the above direct sum. Identifying $H^*(U_k^{\leq b}, U_k^{<b})$ with $\mathrm{C}^*(y_k)$ by excision, one has the following commutative diagram:

$$\begin{tikzcd} H^*(M^{<b}, M^{<0}) \arrow[r, "\partial^*"] \arrow[d] \arrow[dr, "\partial_k^*"] & H^{*+1}(M^{\leq b}, M^{<b}) \arrow[d] \\ H^*(U_k^{<b}, U_k^{<0}) \arrow[r] & \mathrm{C}^{*+1}(y_k) \end{tikzcd}$$

where the vertical arrows are induced by inclusions and the horizontal are coboundary morphisms. Thus we see that v' is not in the kernel of the left hand side arrow, so that by induction hypothesis $z_0 \in U_k$. But according to Equation (4.4), if ℓ is the degree of v' (which maps to $v_0 \in \mathrm{C}^\ell(z_0)$), then $\mathrm{C}^{\ell+1}(y_k) = 0$, a contradiction.

Hence $\partial^* v' = 0$ and there exists $v'' \in H^*(M^{\leq b}, M^{<0})$ such that $i^* v'' = v'$. This v'' maps to v_0 as required but does not satisfy the second conclusion of the lemma *a priori*. We now explain how to build v in the inverse image of v'. For a fixed k, let $v''_k \in H^*(U_k^{\leq b}, U_k^{<0})$ be the image of v'' under the vertical arrow of (4.5). For v'' to satisfy the conclusions of the lemma, we need v''_k to be zero if and only if the image of $v' = i^* v''$ under its vertical arrow is zero. If $i_k^* v''_k = 0$, then there exists $w'_k \in H^*(U_k^{\leq b}, U_k^{<b})$ such that $j_k^* w'_k = v''_k$. We recall that the left hand side arrow is equivalent to the projection

$$\bigoplus_{\ell} \mathrm{C}^*(y_\ell) \to \mathrm{C}^*(y_k),$$

let $w_k \in H^*(M^{\leq b}, M^{<b})$ be then the image of w'_k under the inclusion $\mathrm{C}^*(y_k) \subset H^*(M^{\leq b}, M^{<b})$. We finally set

$$v := v'' - \sum_k j^*(w_k) \in H^*(M^{\leq b}, M^{<0})$$

to be the wanted solution.

The conclusion is true for the $U = U_k$ with this choice of v, by construction. Let U be the flow-out of some critical point. If U does not contain any of the y_k, by the Morse deformation lemma, $U^{\leq b}$ retracts on $U^{<b}$ so that the conclusion follows by induction. Otherwise, let (y_{k_q}) be the sub-family of (y_k) included in U, so that $U_{k_q} \subset U$ by construction of our flows-out. If $z_0 \in U$, then by hypothesis, v' is not in the kernel of $H^*(M^{<b}, M^{<0}) \to H^*(U^{<b}, U^{<0})$ and, as $i^* v = v'$, v is neither in the kernel of $H^*(M^{\leq b}, M^{<0}) \to H^*(U^{\leq b}, U^{<0})$. Conversely, if v is not in the above kernel, either v' is not in the kernel of its restriction to U, in which case $z_0 \in U$ by induction, or its image $v_U \in H^*(U^{\leq b}, U^{<0})$ under the above map has the form $j_U^* w_U$ where $j_U^* : H^*(U^{\leq b}, U^{<b}) \to H^*(U^{\leq b}, U^{<0})$. Now by excision $H^*(U^{\leq b}, U^{<b})$ is isomorphic to the direct sum of the $H^*(U_{k_q}^{\leq b}, U_{k_q}^{<b})$'s. Thus, there is a k_q such that w_U projects on $w_{k_q} \neq 0$. The commutativity of the left hand square of (4.5) for $k = k_q$ together with the construction of v yields a contradiction. ☐

Corollary 4.9. *There exists a subgroup $G \subset H^*(M^{\leq 1}, M^{<0})$ which image under the morphism $H^*(M^{\leq 1}, M^{<0}) \to H^*(M^{\leq 0}, M^{<0})$ is $\mathrm{C}^*(z_0)$ and such that its image under $H^*(M^{\leq 1}, M^{<0}) \to H^*(U^{\leq 1}, U^{<0})$ is non-zero if and only if $z_0 \in U$, where $U := U(y)$ is a flow-out.*

Proof. It is enough to take G to be the subgroup of $H^*(M^{\leq 1}, M^{<0})$ generated by every v given by Lemma 4.8 for $b = 1$ and each $v_0 \in \mathrm{C}^*(z_0) \setminus 0$. ☐

By using (4.4) for z_1, one can prove dually the following lemma.

Lemma 4.10. *The subgroup $\mathrm{C}^*(z_1) \subset H^*(M^{\leq 1}, M^{<1})$ trivially intersects the kernel of the morphism*

$$H^*(M^{\leq 1}, M^{<1}) \to H^*(U^{\leq 1}, U^{<0})$$

induced by inclusion, where $U := U(z_1)$ is the flow-out of z_1.

Proof of Theorem 4.1. Let $v \in H^*(M^{\leq 1}, M^{<0})$ be the class given by Lemma 4.8 for $b = 1$. Thus, applying Proposition 3.21 (see also the paragraph just after) to Corollary 4.9, there exists a class $w \in \mathrm{C}^*(z_1)$ that maps to $u^{d+1} v \in H^*(M^{\leq 1}, M^{<0})$ (here we have implicitly used the isomorphism of Lemma 3.5 that naturally commutes with cup-product). Considering the restriction map $H^*(M^{\leq 1}, M^{<0}) \to H^*(U^{\leq 1}, U^{<0})$, where $U := U(z_1)$ is the flow-out of z_1, the image of $u^{d+1} v$ is non-zero by Lemma 4.10, thus the image v' of v is non-zero. Therefore, Lemma 4.8 implies that $z_0 \in U(z_1)$. For $t \in [0,1]$, let i_t^* be the morphism induced by inclusion

$$i_t^* : H^*(U^{\leq 1}, U^{<0}) \to H^*(U^{\leq t}, U^{<0}),$$

and, for $0 \leq k \leq d+1$, let $\tau_k \in [0,1]$ be the family of min-max values

$$\tau_k := \inf \left\{ t \geq 0 \mid u^k v' \notin \ker i_t^* \right\}. \tag{4.6}$$

As we have just seen, $u^{d+1} v' \neq 0$ so that $u^k v' \neq 0$ for $k \leq d+1$. Lemma 4.8 implies that $\tau_0 = 0$. By the long exact sequence of the triple $(U^{\leq 1}, U^{<1}, U^{<0})$, the image of $u^{d+1} v'$ under $H^*(U^{\leq 1}, U^{<0}) \to H^*(U^{<1}, U^{<0})$ is zero, thus $\tau_{d+1} = 1$. By Lyusternik-Schnirelmann theory, τ_1 is a critical value of $\mathcal{T}|_U$ and $\tau_0 < \tau_1 < \tau_{d+1}$ since $\mathcal{T}$ has a finite number of critical values (we recall that $d \geq 1$). Let (y_j) be the family of critical points of value τ_1 in U. According to Theorem 4.4, if the flow-out U has been taken small enough, $\tau_1 \leq 1 - c_\infty$, thus $\tau_1 < c_\infty$ since there are no critical points with value in $[c_\infty, 1-c_\infty]$. Since $H^*(U^{\leq \tau_1}, U^{<\tau_1})$ decomposes in the direct sum of the local cohomologies of the y_j's, we find by similar arguments as before that there exists a j such that the image of uv' on $H^*(U_j^{\leq \tau_1}, U_j^{<0})$ is non-zero. For this j, the image of v' under the same map is thus also non-zero, hence $z_0 \in U_j$ by Lemma 4.8. But according to Theorem 4.4, for U_j taken small enough in our proofs, one must have $\tau_1 \geq c_\infty$, a contradiction. □

2.4. Corollaries.

Proof of Corollary 4.2. Let $x \in \mathbb{C}\mathrm{P}^d$ be a hyperbolic fixed point of $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$ and $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$ be a lift of φ. According to Theorem 4.1, it is enough to prove that the local cohomology group $\mathrm{C}^*((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^k); x)$ is non zero for all iteration $k \in \mathbb{N}^*$. In Section 3.3, we have seen that $\mathrm{C}^*((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^k); x)$ and $\mathrm{C}^*(\widehat{F}_{t(x^k)}; \zeta_k)$ are isomorphic up to a shift in degree, where $\zeta_k \in \mathbb{C}\mathrm{P}^{N(k)}$ is the critical point of the map $\widehat{F}_{t(x^k)} : \mathbb{C}\mathrm{P}^{N(k)} \to \mathbb{R}$ induced by the generating function $F_{t(x^k)}$ of $e^{-2i\pi t(x^k)}\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$. Since x is hyperbolic, $\dim\ker(\mathrm{d}\varphi(x)^k - \mathrm{id}) = 0$ for all $k \in \mathbb{N}^*$, thus $\mathrm{d}^2\widehat{F}_{t(x^k)}$ is non-degenerate according to (3.7) and $\mathrm{C}^*((\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^k); x)$ has rank 1. □

Proof of Corollary 4.3. According to Theorem 4.1, it is enough to prove that the local cohomology groups of each the $d+1$ fixed points of any pseudo-rotation of $\mathbb{C}\mathrm{P}^d$ are non-zero. This is a direct consequence of Theorem 3.22. Indeed, since a pseudo-rotation has finitely many fixed points, its spectral values c_k are all distinct. Since $c_{k+d+1} = c_k + 1$, there are exactly $d+1$ distinct spectral values in the action window $[0,1)$. By definition of the action value c_k as min-max, there are at least one homologically visible capped fixed point with action c_k for all $k \in \mathbb{Z}$. □

3. Ginzburg-Gürel crossing Theorem for generating functions

In this section, we prove the analogue of Ginzburg-Gürel crossing theorem for generating functions. Since the proof in $\mathbb{C}\mathrm{P}^d$ is essentially the same as the one in $\mathbb{C}^d$ with some technical changes which could make it less transparent to the reader, we first provide the argument for $\mathbb{C}^d$, even though the $\mathbb{C}^d$ setting will not be employed here.

3.1. Crossing energy theorem in $\mathbb{C}^d$. If $\boldsymbol{\sigma} := (\sigma, \ldots, \sigma)$ is a tuple of even size associated with Φ, then $\boldsymbol{\sigma}^m$ is a tuple of even size of the iterated diffeomorphism Φ^m. Given any $x \in \mathbb{C}^d$, let $B_r^{2d}(x) := \{z \in \mathbb{C}^d \mid |z - x| < r\}$ or simply $B_r(x)$. We will denote by A_m the linear isomorphism of $(\mathbb{C}^d)^{mn+1}$ defined by $A_m(\mathbf{v}) := \mathbf{w}$ where $w_k = \frac{v_k + v_{k+1}}{2}$. Throughout this section, we will study the generating functions $F_{(\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon})}$ of Φ^m with a linear change of coordinates: let $F^m(w) := F_{(\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon})} \circ A_m^{-1}(w)$. Given a tuple $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$, $x \in \mathbb{C}^d$ and a radius $r > 0$, we denote by $B_r(x, \boldsymbol{\eta}) \subset (\mathbb{C}^d)^m$ the open set

$$B_r\left(\frac{x + \eta_1(x)}{2}\right) \times B_r\left(\frac{\eta_1 + \eta_2 \circ \eta_1(x)}{2}\right) \times \cdots \times B_r\left(\frac{\eta_{m-2} \circ \cdots \circ \eta_1(x) + \eta_{m-1} \circ \cdots \circ \eta_1(x)}{2}\right),$$

that is $B_r(x, \boldsymbol{\eta}) = \prod_j B_r(w_j)$ where the m-tuple $\mathbf{w}$ is associated with the discrete trajectory

$$(x, \eta_1(x), \ldots, \eta_{m-1} \circ \cdots \circ \eta_1(x))$$

of the discrete dynamics of $\boldsymbol{\eta}$.

Lemma 4.11. *Let $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_n)$ be such an n-tuple and $x \in \mathbb{C}^d$ be a fixed point of $\sigma_n \circ \cdots \circ \sigma_1$. Suppose there exists a sequence $(m_j)_{j \geq 0}$ such that there exists a sequence $(\mathbf{w}^j)_{j \geq 0}$ with $\mathbf{w}^j \in B_r(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id})) \setminus B_{r/2}(x, (\boldsymbol{\sigma}^{m_j}, \mathrm{id}))$ satisfying,*

$$\left|\nabla F^{m_j}\left(\mathbf{w}^j\right)\right|^2 = \sum_{k=1}^{m_j n+1} \left|\partial_{w_k} F^{m_j}\left(\mathbf{w}^j\right)\right|^2 \xrightarrow{j \to \infty} 0. \tag{4.7}$$

Let $(a_1, \ldots, a_n) \in (\mathbb{C}^d)^n$ be such that $B_r(x, \boldsymbol{\sigma}) = B_r(a_1) \times \cdots \times B_r(a_n)$. Then, there exist a sequence $(z_j)_{j \in \mathbb{Z}} \in (\mathbb{C}^d)^{\mathbb{Z}}$ and some integer $1 \leq q \leq n$ such that $z_{j+1} = \sigma_j(z_j)$ with

$$\begin{cases} \frac{z_j + \sigma_j(z_j)}{2} \in \overline{B_r^{2d}(a_{j \bmod n})} & \textit{for all } j \in \mathbb{Z}, \\ \frac{z_q + \sigma_q(z_q)}{2} \notin B_{r/2}^{2d}(a_q) \textit{ or } z_q = z_1 \notin B_{r/2}^{2d}(x). & \end{cases} \tag{4.8}$$

Let us remark that Proposition 3.1 and (4.7) imply

$$|z_k^j - \sigma_{k-1}(z_{k-1}^j)| \xrightarrow{j\to\infty} 0 \text{ for } 1 < k \leq m_j n + 1 \quad \text{and} \quad |z_1^j - z_{m_j n+1}^j| \xrightarrow{j\to\infty} 0,$$

where $\mathbf{z}^j$ is the discrete trajectory associated with $\mathbf{w}^j$ *via* relations (3.2). Indeed, $\partial_{v_k} F^{m_j} = \frac{1}{2}(\partial_{w_k} F^{m_j} + \partial_{w_{k-1}} F^{m_j})$. Thus, the proof essentially consists in an elementary application of the Cantor's diagonal argument to ultimately get a discrete trajectory of the dynamics $\sigma_n \circ \cdots \circ \sigma_1$ whose special property (4.8) comes from the domain of the $\mathbf{w}^j$'s.

Proof. We first prove the case where $(m_j)_{j\geq 0}$ admits a bounded subsequence for a better understanding of the general case. Taking an extracted subsequence, we might suppose that $m_j \equiv m \in \mathbb{N}^*$. Then by relative compactness, we might suppose that $\mathbf{w}^j \to \mathbf{w} \in \overline{B_r(x, (\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon}))} \setminus B_{r/2}(x, (\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon}))$. Let us define $(z'_j)_{1\leq j\leq mn+1} \in (\mathbb{C}^d)^{mn+1}$ by the relations (3.2) for $\mathbf{w}$. According to Proposition 3.1, since $\nabla F^m(\mathbf{w}) = 0$, one has $z'_{j+1} = \sigma_{j \bmod n}(z'_j)$ for $1 \leq j \leq mn$ and $z'_1 = z'_{mn+1}$. Since $w \notin B_{r/2}(x, (\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon}))$, there is some integer $1 \leq q' \leq mn+1$ such that $w_{q'} \notin B^{2d}_{r/2}(a_{q' \bmod n})$ if $q' \neq mn+1$ or $w_{q'} \notin B^{2d}_{r/2}(x)$ otherwise. If $q' \neq mn+1$, let $k \in \mathbb{N}$ be such that $kn+1 \leq q' < (k+1)n+1$ and let $1 \leq q \leq n$ be the integer $q = q' - kn$. The wanted sequence $(z_j)_{j\in\mathbb{Z}}$ is then the mn-periodic sequence such that

$$z_j = z'_{j+kn}, \quad -kn+1 \leq j \leq (m-k)n.$$

In this case,

$$\frac{z_q + \sigma_q(z_q)}{2} = w_{q'} \notin B^{2d}_{r/2}(a_q).$$

If $q' = mn+1$, then the wanted sequence is the mn-periodic sequence such that $z_j = z'_j$ for $1 \leq j \leq mn$ with $q = 1$ and in this case

$$z_q = z_1 = w_{mn+1} \notin B^{2d}_{r/2}(x).$$

Now suppose that $(m_j)_{j\geq 0}$ admits no bounded infinite subsequence. Taking an extracted subsequence, we might suppose that (m_j) is increasing. For all $j \in \mathbb{N}$, let $1 \leq q'_j < m_j n + 1$ be such that $w^j_{q'_j} \notin B_{r/2}(a_{q'_j \bmod n})$ or $q'_j = m_j n + 1$ if such an integer does not exist. Similarly to the bounded case, we first suppose that we can take an extracted subsequence $q'_j \neq m_j n + 1$ for all $j \geq 0$. Let $k_j \in \mathbb{N}$ be such that $k_j n + 1 \leq q'_j < (k_j+1)n + 1$ and let $1 \leq q_j \leq n$ be the integer $q_j = q'_j - k_j n$. Taking an extracting subsequence, we might suppose $q_j \equiv q$. For all $j \geq 0$, let $(\mathbf{w}'^j) \in (\mathbb{C}^d)^{\mathbb{Z}}$ be the $m_j n$-periodic sequence such that $w'^j_k = w^j_{k+k_j n}$ for $-k_j n+1 \leq j \leq (m_j - k_j)n$.

Let $M \in \mathbb{N}^*$ and let us consider the sequence $(\mathbf{w}^{M,j})_{j\geq 0}$ in $(\mathbb{C}^d)^{2M+1}$ defined by restriction: for all $j \geq 0$, $(w^{M,j}_k)_{-M-1\leq k\leq M} := (w'^j_k)_{-M-1\leq k\leq M}$. Now, we can extract a subsequence $j^M_p \to \infty$, such that $(\mathbf{w}^{M,j^M_p})_p$ converges. Since $\partial_{w_{k+k_j n}} F^{m_j}(\mathbf{w}^j) \to 0$ for all $-M-1 \leq k \leq M$, the associated $(2M+1)$-tuple $(z_j)_{-M\leq j\leq M}$, now satisfies $z_{j+1} = \sigma_j(z_j)$ for $-M \leq j \leq M-1$. By a diagonal extraction associated with subsequences $(j^M_p)_p$ as M goes to infinity, we extend our $(2M+1)$-tuples (z_j) to a sequence in $\mathbb{Z}$ with the wanted properties. In particular $\frac{z_q + \sigma_q(z_q)}{2} \notin B^{2d}_{r/2}(a_q)$.

If one cannot extract a subsequence such that $q'_j \neq m_j n + 1$, we can extract a subsequence such that $q'_j \equiv m_j n + 1$. Then take $q = 1$ and define $(\mathbf{w}'^j)$ to be the $m_j n$-periodic sequence such that $w'^j_k = w^j_k$ for $1 \leq k \leq m_j n$. By the same way as above, one gets the wanted (z_j) by a diagonal extraction and in this case $z_q = z_1 \notin B^{2d}_{r/2}(x)$. □

Theorem 4.12. *Let $\Phi \in \mathrm{Ham}(\mathbb{C}^d)$ admitting C^1-small n-tuples $\boldsymbol{\sigma}$. Suppose that $x \in \mathbb{C}^d$ is a fixed point of Φ which is isolated as an invariant set. Then for every sufficiently small $r > 0$, there exists $c_\infty > 0$ and a n-tuple $\boldsymbol{\sigma}$ associated with Φ, with n even, such that for all $m \geq 1$, any gradient or reverse-gradient flow line $\mathbf{u} : \mathbb{R} \to (\mathbb{C}^d)^{mn+1}$, $\dot{\mathbf{u}} = \pm\nabla F^m(\mathbf{u})$, with $\mathbf{u}(0) \in \partial B_r(x, (\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon}))$ and $\mathbf{u}(\tau) \in B_{r/2}(x, (\boldsymbol{\sigma}^m, \boldsymbol{\varepsilon}))$ for some $\tau \in \mathbb{R}$ satisfies*

$$|F^m(\mathbf{u}(0)) - F^m(\mathbf{u}(\tau))| > c_\infty.$$

Proof. Since x is isolated as an invariant set, there exists some $R > 0$ such that for all $z \in B^{2d}_R(x) \setminus \{x\}$, there exists $k \in \mathbb{Z}$ such that $\Phi^k(z) \notin B^{2d}_R(x)$. Fix such an $R > 0$ and choose an even tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ such that $|z - \sigma_j(z)| < R/8$ for all $z \in B^{2d}_R(x)$. Let $m \geq 1$ and

$\mathbf{u} : \mathbb{R}_+ \to (\mathbb{C}^d)^{mn+1}$ be as the statement of the theorem, we may suppose that $\mathbf{u}$ takes its values in $B_{R/2}(x,(\boldsymbol{\sigma}^m,\boldsymbol{\varepsilon}))$. Let $\tau > 0$ be such that $\mathbf{u}(\tau) \in \partial B_{R/4}(x,(\boldsymbol{\sigma}^m,\boldsymbol{\varepsilon}))$. In order to prove the theorem, it is enough to show that there exists $c_\infty > 0$ independent of $m \geq 1$ and $\mathbf{u}$ satisfying

$$|F^m(\mathbf{u}(0)) - F^m(\mathbf{u}(\tau))| > c_\infty.$$

By contradiction, suppose there exist a sequence $(m_j)_{j\geq 0}$ and a sequence of gradient or reverse-gradient flow lines $\mathbf{u}^j : [0,\tau_j] \to B_{R/2}(x,(\boldsymbol{\sigma}^{m_j},\boldsymbol{\varepsilon}))$, $\dot{\mathbf{u}}^j = \pm\nabla F^{m_j}(\mathbf{u}^j)$, with $\mathbf{u}^j(0) \in \partial B_{R/2}(x,(\boldsymbol{\sigma}^{m_j},\boldsymbol{\varepsilon}))$ and $\mathbf{u}^j(\tau_j) \in \partial B_{R/4}(x,(\boldsymbol{\sigma}^{m_j},\boldsymbol{\varepsilon}))$ such that

$$\left|F^{m_j}(\mathbf{u}^j(0)) - F^{m_j}(\mathbf{u}^j(\tau_j))\right| \to 0.$$

For some $1 \leq k_j \leq m_j n + 1$, one has $|u^j_{k_j}(0) - u^j_{k_j}(\tau_j)| \geq R/4$ so

$$R/4 \leq \int_0^{\tau_j} |\dot{u}^j_{k_j}(s)|\mathrm{d}s \leq \int_0^{\tau_j} |\dot{\mathbf{u}}^j(s)|\mathrm{d}s = \int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|\mathrm{d}s,$$

but

$$\left(\int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|\mathrm{d}s\right)^2 \leq \tau_j \int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}^j(s))|^2\mathrm{d}s = \tau_j\left(F^{m_j}(\mathbf{u}^j(0)) - F^{m_j}(\mathbf{u}^j(\tau_j))\right),$$

thus $\tau_j \to +\infty$. Combined with $\int_0^{\tau_j} |\nabla F^{m_j}(\mathbf{u}(s))|^2\mathrm{d}s \to 0$, it implies that there exists a sequence $(s_j)_{j\geq 0}$ with $s_j \in [0,\tau_j]$, such that the sequence $(\mathbf{u}^j(s_j)))_{j\geq 0}$ satisfies the hypothesis of Lemma 4.11 with $r = R/2$.

Therefore, according to Lemma 4.11, there exist a sequence $(z_j)_{j\in\mathbb{Z}} \in (\mathbb{C}^d)^{\mathbb{Z}}$ and some integer $1 \leq q \leq n$, such that $|z_j + \sigma_j(z_j) - 2x| \leq R$ which implies that $|z_j - x| \leq R/2 + R/16 < R$ by the specific choice of σ_j, $|z_q - x| > R/4$ and $z_{j+1} = \sigma_j(z_j)$. Thus, for all $k \in \mathbb{Z}$, $\Phi^k(z_1) = z_{kn+1} \in B^{2d}_R(x)$ with $z_1 \neq x$ since

$$\sigma_{q-1} \circ \cdots \circ \sigma_1(z_1) = z_q \neq x = \sigma_{q-1} \circ \cdots \circ \sigma_1(x),$$

a contradiction. ☐

3.2. Crossing energy theorem in $\mathbb{CP}^d$. We employ the notation of Section 2. We recall that $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_{n_1})$ is a specific n_1-tuple, with n_1 even, associated with Φ, $\boldsymbol{\delta}_t := \boldsymbol{\delta}_t^{(1)}$ is a n_0-tuple, with n_0 even, associated with $e^{-2i\pi t}$, $F^m_t = F_{(\boldsymbol{\varepsilon},\boldsymbol{\sigma}^m,\boldsymbol{\delta}_t)}$ is a conical generating function of the conical Hamiltonian diffeomorphism $e^{-2i\pi t}\Phi^m$, $M^m := \{(t,[z]) \in I \times \mathbb{CP}^{N(m)} \mid F^m_t(z) = 0\}$ is the domain of the projection map $\mathcal{T}^m : M^m \to I$ with $N(m) = (d+1)(n_1 m + n_0 + 1) - 1$. Similarly to the $\mathbb{C}^d$-case, we apply a linear change of coordinates. First, we apply a cyclic permutation so that F^m_t is replaced by the generating function of $(\boldsymbol{\sigma}^m,\boldsymbol{\delta}_t,\boldsymbol{\varepsilon})$ and then one take w-coordinates, so that F^m_t is replaced by $F^m_t \circ A_m^{-1}$ and by a slight abuse of notation we will still denote by M^m and $\mathcal{T}^m$ domains and functions seen in the induced projective chart.

The proof of the crossing energy theorem in $\mathbb{CP}^d$ follows the same lines as the $\mathbb{C}^d$ case. First, we need an analogue to Lemma 4.11. We have to define a neighborhood of $\mathbb{CP}^{N(m)}$ similar to $B(x,(\boldsymbol{\sigma}^m,\boldsymbol{\varepsilon}))$ in the $\mathbb{C}^d$ case. Let $B_1 \subset \mathbb{C}^{d+1}$ be the unit Euclidean ball centered at the origin, so that, for $k \in \mathbb{N}^*$, $\partial(B_1^k) \subset (\mathbb{C}^{d+1})^k$ denotes the sphere

$$\partial(B_1^k) = \bigcup_{1\leq j\leq k} B_1^{j-1} \times \mathbb{S}^{2d+1} \times B_1^{k-j}.$$

Let $\pi_m : \partial(B_1^{mn_1+n_0+1}) \to \mathbb{CP}^{N(m)}$ be the quotient map by the diagonal action of S^1. We define now a S^1-equivariant neighborhood in the sphere $\partial(B_1^{mn_1+n_0+1})$ of the normalized w-coordinates of some point $x \in \mathbb{C}^{d+1}\setminus 0$ relative to F^m_t. Let $\mathbf{a} = (a_1,\ldots,a_{mn_1+n_0+1}) \in (\mathbb{C}^{d+1})^{mn_1+n_0+1}$ be the w-coordinates of x, that is $B_r(x,(\boldsymbol{\sigma}^m,\boldsymbol{\delta}_t,\boldsymbol{\varepsilon})) = \prod_j B_r(a_j)$. Let $\lambda > 0$ such that $\lambda\mathbf{a} \in \partial(B_1^{mn_1+n_0+1})$. For $r > 1$ we define

$$U_r(x,m,t) := S^1 \cdot B_r(\lambda x,(\boldsymbol{\sigma}^m,\boldsymbol{\delta}_t,\boldsymbol{\varepsilon})) \cap \partial(B_1^{mn_1+n_0+1}),$$

where $S^1 \cdot E := \{\mu z \mid z \in E,\ \mu \in S^1\}$ for any subset $E \subset (\mathbb{C}^d)^{mn_1+n_0+1}$. Let $V_r(x,m,t) \subset \mathbb{CP}^{N(m)}$ be the projection of this neighborhood on $\mathbb{CP}^{N(m)}$.

LEMMA 4.13. *Let $x \in \mathbb{C}^{d+1} \setminus 0$ be a fixed point of Φ. Suppose there exists an increasing sequence of positive integers $(m_j)_{j\geq 0}$ such that there exist a sequence $(t_j)_{j\geq 0}$ in I satisfying $(t_j) \to t \in \{0,1\}$ and a sequence $(\mathbf{w}^j)_{j\geq 0}$ with $\mathbf{w}^j \in U_r(x, m_j, t) \setminus U_{r/2}(x, m_j, t)$ satisfying,*

$$\left|\nabla F_{t_j}^{m_j}\left(\mathbf{w}^j\right)\right|^2 = \sum_{k=1}^{m_j n_1 + n_0 + 1} \left|\partial_{w_k} F_{t_j}^{m_j}\left(\mathbf{w}^j\right)\right|^2 \xrightarrow{j\to\infty} 0.$$

Let $\mathbf{a} := (a_1, \dots, a_{n_1}) \in (\mathbb{C}^{d+1})^{n_1}$ be such that $B_r(x, \boldsymbol{\sigma}) = \prod_j B_r(a_j)$ and $\tilde{\mathbf{a}} := (\tilde{a}_1, \dots, \tilde{a}_{n_0+1}) \in (\mathbb{C}^{d+1})^{n_0+1}$ be the (n_0+1)-tuple

$$\tilde{a}_k := \frac{\delta_{(k-1)t/n_0}(x) + \delta_{kt/n_0}(x)}{2} \quad \text{for} \quad 1 \leq k \leq n_0 \quad \text{and} \quad \tilde{a}_{n_0+1} = x.$$

Then, there exists a possibly infinite integer $\kappa \in \mathbb{Z} \cup \{+\infty\}$ such that there exist a sequence $(b_j) \in (\mathbb{C}^{d+1})^{\mathbb{Z}}$ defined by

$$b_j := \begin{cases} a_{j \bmod n_1} & \text{if } j \leq \kappa n_1, \\ \tilde{a}_{j-\kappa n_1} & \text{if } \kappa n_1 + 1 \leq j \leq \kappa n_1 + n_0 + 1, \\ a_{j-(n_0+1) \bmod n_1} & \text{if } j \geq \kappa n_1 + n_0 + 2, \end{cases}$$

a sequence $(z_j)_{j\in\mathbb{Z}} \in (\mathbb{C}^{d+1} \setminus 0)^{\mathbb{Z}}$ satisfying

$$z_{j+1} = \begin{cases} \sigma_{j \bmod n_1}(z_j) & \text{if } j \leq \kappa n_1, \\ g_{t/n_0}(z_j) & \text{if } \kappa n_1 + 1 \leq j \leq \kappa n_1 + n_0 + 1, \\ \sigma_{j-n_0-1 \bmod n_1}(z_j) & \text{if } j \geq \kappa n_1 + n_0 + 2, \end{cases}$$

and some integer $1 \leq q \leq n_1 + n_0 + 1$ such that

$$\begin{cases} \frac{z_j + z_{j+1}}{2} \in \mathbb{C} \cdot \overline{B_r^{2d}(b_j)} & \text{for all } j \in \mathbb{Z}, \\ \frac{z_q + z_{q+1}}{2} \notin \mathbb{C} \cdot B_{r/2}^{2d}(b_q). \end{cases}$$

PROOF. The proof follows along the same lines as Lemma 4.11 with just additional calligraphic difficulties, we will only underline the key changes.

Let $x \in \mathbb{C}^{d+1} \setminus 0$, $\mathbf{a} \in (\mathbb{C}^{d+1})^{n_1}$ and $\tilde{\mathbf{a}} \in (\mathbb{C}^{d+1})^{n_0+1}$ satisfying the assumptions of the lemma. Let $\lambda > 0$ be such that $(\lambda \mathbf{a}, \lambda \tilde{\mathbf{a}}) \in \partial(B_1^{n_1+n_0+1})$ (it exists since $x \neq 0$), then

$$U_r(x, m, t) = S^1 \cdot \left[\left(\prod_{k=1}^{n_1} B_r^{2(d+1)}(\lambda a_k) \right)^m \times \prod_{k=1}^{n_0+1} B_r^{2(d+1)}(\lambda \tilde{a}_k) \right] \cap \partial(B_1^{mn_1+n_0+1}).$$

Let $(\mathbf{w}^j)$ be satisfying the assumptions of the lemma. By S^1-invariance of the function $|\partial F_{t_j}^{m_j}|$ and the neighborhood $U_r(x, m_j, t)$, we can suppose that

$$\mathbf{w}^j \in \left(\prod_{k=1}^{n_1} B_r^{2(d+1)}(\lambda a_k) \right)^{m_j} \times \prod_{k=1}^{n_0+1} B_r^{2(d+1)}(\lambda \tilde{a}_k).$$

The result follows from Cantor's diagonal argument applied to the sequence $(\mathbf{w}^j/\lambda)$ in the same way as in the proof of Lemma 4.11. □

In order to state the crossing energy theorem in $\mathbb{C}\mathrm{P}^d$, we will need to define a "good" pseudo-gradient X_m for the function $\mathcal{T}^m$. For technical reasons, the projection $\pi_m : \partial(B_1^{mn_1+n_0+1}) \to \mathbb{C}\mathrm{P}^{N(m)}$ is the most natural for our problem. However the sphere $\partial(B_1^{mn_1+n_0+1})$ is not smooth, we thus introduce a smooth S^1-invariant sphere $\Sigma_m \subset (\mathbb{C}^{d+1})^{mn_1+n_0+1}$:

$$\Sigma_m := \left\{ \mathbf{z} \in (\mathbb{C}^{d+1})^{mn_1+n_0+1} \mid \sum_{k=1}^{mn_1+n_0+1} |z_k|^{p_m} = 1 \right\},$$

where $p_m \geq 2$ is chosen such that,

$$\forall \mathbf{z} \in \Sigma_m, \exists \lambda \in [1,2], \quad \lambda \mathbf{z} \in \partial(B_1^{mn_1+n_0+1}),$$

(necessarily $(p_m) \to \infty$). We endow $\mathbb{C}\mathrm{P}^{N(m)}$ with the Riemannian metric induced by the S^1-invariant projection $\pi'_m : \Sigma_m \to \mathbb{C}\mathrm{P}^{N(m)}$. Since

$$\mathrm{dist}(\partial U_r(x, m, t), U_{r/2}(x, m, t)) \geq r/2,$$

the condition on p_m implies that

$$\operatorname{dist}(\partial V_r(x,m,t), V_{r/2}(x,m,t)) \geq r/4. \tag{4.9}$$

Let $f^m : I \times \mathbb{C}\mathrm{P}^{N(m)} \to \mathbb{R}$ be the C^1 function satisfying $f^m(t, \pi'_m(\mathbf{z})) = F_t^m(\mathbf{z})$ for all $\mathbf{z} \in \Sigma_m$, so that $M^m = \{(t,\zeta) \in I \times \mathbb{C}\mathrm{P}^{N(m)} \mid f^m(t,\zeta) = 0\}$. The pseudo-gradient X_m of $\mathcal{T}^m$ is defined by

$$X_m(t,\zeta) := \partial_t f^m(t,\zeta)\nabla f^m(t,\zeta) - |\nabla f^m(t,\zeta)|^2 \frac{\partial}{\partial t}$$

We have $\left\langle X_m, -\frac{\partial}{\partial t}\right\rangle \geq 0$ with equality if and only if $\nabla f^m = 0$, that is to say $\mathrm{d}\mathcal{T}^m = 0$.

Theorem 4.14. *Let $\Phi \in \mathrm{Ham}_{\mathbb{C}}(\mathbb{C}^{d+1})$ be a lift of $\varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^d)$. Suppose that $x \in \mathbb{C}^{d+1} \setminus 0$ is a fixed point of Φ such that $[x] \in \mathbb{C}\mathrm{P}^d$ is isolated as an invariant set of φ. Then for every sufficiently small $r > 0$, there exist $c_\infty > 0$ and a tuple $\boldsymbol{\sigma}$ associated with Φ such that for all $m \geq 1$, if $(t, \zeta_t^m) \in M^m$ denotes the critical point of $\mathcal{T}^m$ with critical value $t \in \{0,1\}$ associated with x, any gradient flow line $\mathbf{u} : \mathbb{R} \to M^m$, $\dot{\mathbf{u}} = \pm X_m(\mathbf{u})$, with $\mathbf{u}(0) \in I \times \partial V_r(x,m,t)$ and $\mathbf{u}(\tau) \in I \times V_{r/2}(x,m,t)$ for some $\tau \in \mathbb{R}$ satisfies*

$$|\mathcal{T}^m(\mathbf{u}(0)) - \mathcal{T}^m(\mathbf{u}(\tau))| > c_\infty.$$

Proof. We follow the steps of the proof of Theorem 4.12. By contradiction, suppose there exist a sequence $(m_j)_{j\geq 0}$ and a sequence of pseudo-gradient flow line $\mathbf{u}^j : [0,\tau_j] \to U_r(x,m_j,t)$, $\dot{\mathbf{u}}^j = \pm X_{m_j}(\mathbf{u}^j)$ with $\mathbf{u}^j(0) \in \partial V_r(x,m_j,t)$ and $\mathbf{u}^j(\tau_j) \in V_{r/2}(x,m_j,t)$ such that

$$\left|\mathcal{T}^{m_j}(\mathbf{u}^j(0)) - \mathcal{T}^{m_j}(\mathbf{u}^j(\tau_j))\right| \xrightarrow{j\to+\infty} 0 \quad \text{and} \quad \mathcal{T}^{m_j}(\mathbf{u}^j(0)) \xrightarrow{j\to+\infty} 0.$$

First we must show that $\tau_j \not\to 0$. Let $p_2 : I \times \mathbb{C}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ be the projection on the second factor, then (4.9) implies that

$$\frac{r}{4} \leq \int_0^{\tau_j} |\mathrm{d}p_2 \cdot \dot{\mathbf{u}}^j| \mathrm{d}s,$$

so

$$\left(\frac{r}{4}\right)^2 \leq \tau_j \int_0^{\tau_j} |\mathrm{d}p_2 \cdot X_{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s = \tau_j \int_0^{\tau_j} (\partial_t f^{m_j}(\mathbf{u}^j))^2 |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

Remark that there exists some $C > 0$ independent of m (it only depends on $(\boldsymbol{\delta}_t)$) such that $0 \leq -\partial_t f^m < C$, thus

$$\int_0^{\tau_j} |\mathrm{d}p_2 \cdot X_{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s \leq C^2 \int_0^{\tau_j} |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

This last term goes to 0 since

$$\left|\mathcal{T}^{m_j}(\mathbf{u}^j(0)) - \mathcal{T}^{m_j}(\mathbf{u}^j(\tau_j))\right| = \int_0^{\tau_j} \left\langle -\frac{\partial}{\partial t}, X_{m_j}(\mathbf{u})\right\rangle \mathrm{d}s = \int_0^{\tau_j} |\nabla_\zeta f^{m_j}(\mathbf{u}^j)|^2 \mathrm{d}s.$$

Therefore, $\tau_j \to +\infty$ and thus there exists a sequence $(s_j)_{j\geq 0}$ in $I \times V_r(x,m,t) \setminus I \times V_{r/2}(x,m,t)$ such that $|\nabla_\zeta f^{m_j}(s_j)| \to 0$.

Let $(t^j; \lambda_j \mathbf{w}^j) \in I \times U_r(x,m_j,t)$ be lifted from s_j with $\mathbf{w}^j \in \Sigma_{m_j}$ and $\lambda_j \in [1,2]$ such that $\lambda_j \mathbf{w}^j \in \partial(B_1^{m_j n_1 + n_0 + 1})$ (which exists by definition of Σ_{m_j}). Since $t^j = \mathcal{T}^{m_j}(s_j)$, one has $t^j \to t$. Since $|\nabla_\zeta f^{m_j}(s_j)| \to 0$, the norm of the orthogonal projection of $\nabla F_{t^j}(\mathbf{w}^j) \in \mathbb{C}^{N(m_j)+1}$ on the sphere Σ_{m_j} goes to zero as $j \to \infty$. The radial component is $\left\langle \mathbf{w}^j, \nabla F_{t^j}(\mathbf{w}^j)\right\rangle = 2F_{t^j}(\mathbf{w}^j) = 0$, hence $|\nabla F_{t^j}^{m_j}(\mathbf{w}^j)| \to 0$. Since $\lambda_j \in [1,2]$, the homogeneity of $F_{t^j}^{m_j}$ implies that

$$\left|\nabla F_{t^j}^{m_j}(\lambda_j \mathbf{w}^j)\right| \xrightarrow{j\to\infty} 0.$$

We can thus apply Lemma 4.13 to the sequences (m_j), $(\lambda_j \mathbf{w}^j)$ and the fixed point $x \in \mathbb{C}^{d+1}$. We then find a sequence $(z_j)_{j\in\mathbb{Z}}$ in $\mathbb{C}^{d+1}$ such that $\varphi^k([z_0])$ stays close to $[x]$ for all $k \in \mathbb{Z}$ with $[z_0] \neq [x]$. $\square$

CHAPTER 5

On the Hofer-Zehnder conjecture

1. Statement of the theorem

In this chapter, we give an alternative proof of the theorem of Shelukhin on the Hofer-Zehnder conjecture "that could have been given in the 90s". Let φ be a Hamiltonian diffeomorphism of $\mathbb{CP}^d$ with finitely many fixed points and let (h_t) be the associated Hamiltonian map. Shelukhin introduced a homology count over a field $\mathbb{F}$ of the number of fixed points of φ. In our setting, it has been defined as $N((h_s);\mathbb{F})$ at (3.13). In fact, one can prove that $\mathrm{C}_*((h_s);x;\mathbb{F})$ is isomorphic to the local Floer homology of x over the field $\mathbb{F}$, so that $N((h_s);\mathbb{F})$ equals the homological count defined by Shelukhin. One always has $N((h_s);\mathbb{F}) \geq d+1$, according to Théret's proof of Fortune-Weinstein theorem (*i.e.* the existence of spectral invariants stated in Theorem 3.22). The theorem of Shelukhin that we prove is the following.

THEOREM 5.1 ([**69**, Theorem A for $M = \mathbb{CP}^d$]). *Every Hamiltonian diffeomorphism φ of $\mathbb{CP}^d$ such that $N((h_s);\mathbb{F}) > d+1$ for some field $\mathbb{F}$ has infinitely many periodic points. Moreover, when φ has finitely many fixed points, if $\mathbb{F}$ has characteristic 0 in the former assumption, there exists $A \in \mathbb{N}$ such that, for all prime $p \geq A$, φ has a p-periodic point that is not a fixed point; if $\mathbb{F}$ has characteristic $p \neq 0$, φ has infinitely many periodic points with period in $\{p^k \mid k \in \mathbb{N}\}$.*

In the special case where every fixed point of φ is non-degenerate, one has $\dim \mathrm{C}_*((h_s);x;\mathbb{F}) = 1$ for every fixed point; hence, $N((h_s);\mathbb{F})$ equals the number of fixed points of φ. As a special case, every Hamiltonian diffeomorphism of $\mathbb{CP}^d$ that has at least $d+2$ non-degenerate periodic points has infinitely many periodic points and the number grows at least like the sum of prime numbers (*i.e.* the number of periodic points of period less than k is $\gtrsim \frac{k^2}{\log k}$).

Our proof follows the same main steps as the original one and takes advantage of the new proof given by Shelukhin in [**3**, Appendix B] of inequality (5.1) below. Let us give a short outline of it. Our analogue of the Floer homology of the Hamiltonian diffeomorphism associated with the Hamiltonian map (h_s) defines with its inclusion morphisms a persistence module $(G_*^{(-\infty,t)}(h_s;\mathbb{F}))_t$. Such a persistence module can be represented in a graphical way by a barcode (see Figure 2). Infinite bars of the barcodes always exist in the same cardinality: they are associated with the spectral invariants of (h_s). On the other hand, finite bars exist if and only if $N((h_s);\mathbb{F}) > d+1$. The barcodes always have infinitely many bars but there is a natural free $\mathbb{Z}$-action on their collection, preserving the length of bars in such a way that, if φ has finitely many fixed points, there are only finitely many $\mathbb{Z}$-orbits of finite bars. Moreover, if φ has finitely many periodic points, the number of finite bars associated with its iterations is uniformly bounded. The proof consists in showing that the existence of a finite bar implies an unbounded growth of the number of $\mathbb{Z}$-orbits of finite bars.

Since $\mathbb{Z}$ acts on the collection of finite bars by preserving their length, one can define the sequence of bar length of each $\mathbb{Z}$-orbit $0 < \beta_1((h_s);\mathbb{F}) \leq \beta_2((h_s);\mathbb{F}) \leq \cdots \leq \beta_n((h_s);\mathbb{F})$. We denote by $\beta_{\mathrm{tot}}((h_s);\mathbb{F})$ the sum of these lengths. The proof relies on two results: the length of a finite bar is bounded by 1 (with our normalization, see (3.1) below), the sum of the lengths satisfies the Smith-type inequality

$$\beta_{\mathrm{tot}}((h_{ps});\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}((h_s);\mathbb{F}_p), \tag{5.1}$$

for all prime p. These two steps easily imply Theorem 5.1.

2. Outline of the proof

Here, we introduce the main tools of the proof. We postpone the proof of technical statements and the definition of technical objects to the remaining sections in order to give the proof of Theorem 5.1 at the end of this section.

Theorem 5.1 is proved by studying the length of the finite bars of the barcode of (h_s). Let us denote by $I_1, \dots, I_n \subset \mathbb{R}$ representatives of the $\mathbb{Z}$-orbit of finite bars over the field $\mathbb{F}$. Up to a permutation, one can assume that $(\operatorname{length} I_k)_k$ is non-decreasing. Let $\beta_k((h_s);\mathbb{F})$ be the length of I_k, $\beta((h_s);\mathbb{F}) := \beta_n((h_s);\mathbb{F})$ be the length of the longest bar and

$$\beta_{\mathrm{tot}}((h_s);\mathbb{F}) := \sum_k \beta_k((h_s);\mathbb{F}).$$

The number $\beta((h_s);\mathbb{F})$ was first introduced by Usher and called the boundary depth of (h_s) [**78**]. The first important property is that every finite bar has a length less than 1 (see Theorem 5.2). This is the analogue of [**69**, Theorem B] in the special case $M = \mathbb{C}\mathrm{P}^d$ and the proof follows the same key ideas: we define a product between GF-homologies and use it to find an interleaving between the GF-homology of (h_s) and the one of $(h'_s) \equiv 0$ that does not have any finite bar. The second important property is the Smith inequality (5.1) stated in Corollary 5.6 which is the analogue of [**69**, Theorem D] in the special case $M = \mathbb{C}\mathrm{P}^d$. The general strategy follows a new proof of [**69**, Theorem D] given by Shelukhin in [**3**, Appendix B]. In the realm of generating functions, the proof is rather short and very elementary: it essentially relies on the classical Smith inequality (5.4); it might seem surprising regarding the extraordinary machinery necessary to prove its Floer theoretical analogue (although the Floer theoretical proof is available for every closed monotone symplectic manifold).

Proof of Theorem 5.1. Let $\boldsymbol{\sigma}$ be any tuple of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms associated with φ, so that $N(\boldsymbol{\sigma};\mathbb{F}) = N((h_s);\mathbb{F})$. Let us denote by $K(\boldsymbol{\sigma};\mathbb{F})$ the number of $\mathbb{Z}$-orbits of finite bars of the barcode associated with $\boldsymbol{\sigma}$ over the field $\mathbb{F}$. According to the universal coefficient theorem, one can assume that $\mathbb{F} = \mathbb{Q}$ if $\mathbb{F}$ has characteristic 0 and $\mathbb{F} = \mathbb{F}_p$ if it has characteristic $p \neq 0$.

Let us assume that $\mathbb{F} = \mathbb{Q}$. According to Proposition 3.30, $N(\boldsymbol{\sigma};\mathbb{Q}) > d+1$ implies that $K(\boldsymbol{\sigma};\mathbb{Q}) > 0$ so $\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{Q}) > 0$. According to Corollary 5.6, for all prime number $p \geq 3$,

$$K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq \beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p).$$

Thus, by Proposition 5.7, for all sufficiently large prime p,

$$K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{Q}),$$

that is to say that $K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ grows at least linearly with prime numbers p. According to Theorem 5.2, $\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p) \leq 1$ so $K(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ must diverge to $+\infty$ with prime numbers p and so must $N(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ by Proposition 3.30. Let $z_1, \dots, z_n \in \mathbb{C}\mathrm{P}^d$ be the fixed points of φ. According to Corollary 3.7, there exists $B > 0$ such that $\dim \mathrm{C}_*(\boldsymbol{\sigma}^p; z_k;\mathbb{F}_p) < B$ for all k and all prime p. Let $A \in \mathbb{N}$ be such that for all prime $p \geq A$, $N(\boldsymbol{\sigma}^p;\mathbb{F}_p) > nB$. Then, for all prime $p \geq A$, there must be at least one fixed point of φ^p that is not one of the z_k's, that is, there must be at least one p-periodic point that is not a fixed point. Hence, the conclusion for the case $\mathbb{F}$ of characteristic 0.

Let us assume that $\mathbb{F} = \mathbb{F}_p$ for some prime number p. By contradiction, let us assume that φ has only finitely many periodic points with period in $\mathbb{P} := \{p^k \mid k \in \mathbb{N}\}$. According to Corollary 5.6,

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) \geq p^k\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p), \quad \forall k \in \mathbb{N},$$

in particular, $N(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) > d+1$ for all $k \in \mathbb{N}$. Thus, by replacing φ with φ^{p^k} for a sufficiently large k, one can assume that every periodic point of φ with period belonging to $\mathbb{P}$ has only admissible periods (see the paragraph just above Proposition 3.6 for the definition of an admissible period). According to Proposition 3.6, it implies that $N(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) = N(\boldsymbol{\sigma};\mathbb{F}_p)$ for all $k \in \mathbb{N}$. But Corollary 5.6 together with Proposition 3.30 imply that the left-hand side of this equation must diverge to $+\infty$ as k grows, a contradiction. $\square$

3. Uniform bound on β

THEOREM 5.2. *For all tuples of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms $\boldsymbol{\sigma}$ generating a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ with finitely many fixed points, the longest finite bar of its barcode is less than 1:*

$$\beta(\boldsymbol{\sigma}) \leq 1.$$

As a matter of fact, the proof allows us to give the more precise bound:

$$\beta(\boldsymbol{\sigma}) \leq c_{d+k}(\boldsymbol{\sigma}) - c_k(\boldsymbol{\sigma}),$$

for all $k \in \mathbb{Z}$ (in particular, one can always replace ≤ 1 by < 1). Moreover, the finiteness of the set of fixed points is irrelevant if one take the more general definition of β given by (5.2) below (which is closer to the original definition of the boundary depth introduced by Usher in [**78**]).

We will essentially prove that the persistence modules of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are $(c_d(\boldsymbol{\sigma}), -c_0(\boldsymbol{\sigma}))$-interleaved. The isometry theorem between the interleaving distance and the barcode distance states that the distance between the two associated barcodes is not more than $c_d(\boldsymbol{\sigma}) - c_0(\boldsymbol{\sigma})$. Since the barcode of $\boldsymbol{\varepsilon}$ does not have any finite bar, the conclusion follows.

In order to simplify the proof, we will use a slightly weaker result than the isometry theorem. We recall that the maximal length of a finite bar $\beta(V^t) \geq 0$ in the persistence module (V^t) can alternatively be defined by

$$\beta(V^t) = \sup\left\{\beta \geq 0 \mid \exists t \in \mathbb{R},\ \ker(V^t \to V^{t+\beta}) \neq \ker(V^t \to V^{+\infty})\right\}. \tag{5.2}$$

LEMMA 5.3. *Let $((V^t), \pi)$ be a persistence module and $((W^t), \kappa)$ be a persistence module without any finite bar. If there exist $\delta, \delta' \in \mathbb{R}$ with $\delta + \delta' \geq 0$ and $f : (V^t) \to (W^{t+\delta})$ and $g : (W^t) \to (V^{t+\delta'})$ that are morphisms of persistence modules such that $g_{t+\delta} f_t = \pi_t^{t+\delta+\delta'}$ for all $t \in \mathbb{R}$, then $\beta(V^t) \leq \delta + \delta'$.*

PROOF OF LEMMA 5.3. By contradiction, let us assume that there exist $t \in \mathbb{R}$ and $v \in V^t$ such that $\pi_t^{t+\delta+\delta'} v \neq 0$ and $\pi_t^{+\infty} v = 0$. Since $\pi_t^{+\infty} v = 0$, there exists $s \geq t + \delta + \delta'$ such that $\pi_t^s v = 0$. By hypothesis, $\pi_t^{t+\delta+\delta'} = g_{t+\delta} f_t$ so $w := f_t v \neq 0$. Since (W^t) does not have any finite bars, $\kappa_{t+\delta}^s w \neq 0$. Since f is a morphism of persistence modules, $f_s \pi_t^s = \kappa_{t+\delta}^{s+\delta} f_t$ so $f_s \pi_t^s v = \kappa_{t+\delta}^{s+\delta} w \neq 0$. A contradiction with $\pi_t^s v = 0$. □

PROOF OF THEOREM 5.2. Let $\varepsilon > 0$ and let $\eta := c_d(\boldsymbol{\sigma}) + \varepsilon/2$ and $\eta' := c_d(\boldsymbol{\sigma}^{-1}) + \varepsilon/2$. One can assume that the size n of $\boldsymbol{\sigma}$ satisfies $2n = mn_0$ for some integer $m > \max(|\eta|, |\eta'|, |c_d(\boldsymbol{\sigma})|, |c_d(\boldsymbol{\sigma}^{-1})|)$, by concatenation of $\boldsymbol{\sigma}$ with some $\boldsymbol{\varepsilon}^k$. Let $b_d \in HZ_{2d}(\boldsymbol{\sigma}_{m,\eta})$ and $b'_d \in HZ_{2d}(\boldsymbol{\sigma}_{m,\eta'}^{-1})$ be classes associated with the spectral class α_d in the sense of Proposition 3.24 (well-defined by definition of η, η' and m). In order to simplify the exposition, let us set $a_d := a_d^{(2n+1)}(0) \in HZ_{2d}(\boldsymbol{\varepsilon}^{2n+1})$ and $a'_d := a_d^{(2n+1)}(\eta + \eta') \in HZ_{2d}(\boldsymbol{\varepsilon}_{m,\eta+\eta'})$ (we recall that $\boldsymbol{\varepsilon}^{2n+1} = \boldsymbol{\varepsilon}^{mn_0+1} = \boldsymbol{\varepsilon}_{m,0}$ by our assumption on n). Let us denote by Δ_1 the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{1-s})_{s\in[0,1]}$ and by Δ_2 the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1})_{s\in[0,1]}$. Let us denote by $\widetilde{\Delta}_1$ the interpolation isomorphism associated with $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{1-s})_{s\in[0,1]}$ and by $\widetilde{\Delta}_2$ the one associated with $(\boldsymbol{\sigma}^{1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{s\in[0,1]}$.

We define the following morphisms of persistence modules:

$$f_t : \begin{cases} G_*^{(-\infty,t)}(\boldsymbol{\sigma}) & \to & G_*^{(-\infty,t+\eta')}(\boldsymbol{\varepsilon}^{2n+1}), \\ \alpha & \mapsto & \Delta_1(b'_d \diamond \alpha), \end{cases}$$

$$g_t : \begin{cases} G_*^{(-\infty,t)}(\boldsymbol{\varepsilon}^{2n+1}) & \to & G_*^{(-\infty,t+\eta)}(\boldsymbol{\sigma}), \\ \alpha & \mapsto & a_d^{-1} \diamond \widetilde{\Delta}_2 \circ \widetilde{\Delta}_1^{-1}(b_d \diamond \alpha). \end{cases}$$

Indeed, these morphisms commute with inclusion morphisms by naturality of the morphisms involved in their definitions. Let us denote by $\pi_t^s : G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,s)}(\boldsymbol{\sigma})$ the inclusion morphism for $t \leq s$.

In order to apply Lemma 5.3, one just needs to show that $g_{t+\eta'} \circ f_t = \pi_t^{t+\eta+\eta'}$ for all $t \in \mathbb{R}$. For all $\alpha \in G_*^{(-\infty,t)}(\boldsymbol{\sigma})$, one has

$$\begin{aligned} g_{t+\eta'} \circ f_t(\alpha) &= a_d^{-1} \diamond \widetilde{\Delta}(b_d \diamond \Delta_1(b'_d \diamond \alpha)) \\ &= a_d^{-1} \diamond \widetilde{\Delta}_2(b_d \diamond b'_d \diamond \alpha) \\ &= a_d^{-1} \diamond \Delta_2(b_d \diamond b'_d) \diamond \alpha \\ &= a_d^{-1} \diamond a'_d \diamond \alpha \\ &= \pi_t^{\eta+\eta'} \alpha, \end{aligned}$$

where the second and third equalities come from Proposition 3.15 and the identity $a_d^{-1} \diamond a'_d = \pi_t^{\eta+\eta'}$ is a direct consequence of the commutativity of (3.21). The conclusion follows from Lemma 5.3 since the persistence module of $\boldsymbol{\varepsilon}^{2n+1}$ does not have any finite bar (see Section 5.1) and

$$\eta + \eta' = c_d(\boldsymbol{\sigma}) + c_d(\boldsymbol{\sigma}^{-1}) + \varepsilon = c_d(\boldsymbol{\sigma}) - c_0(\boldsymbol{\sigma}) + \varepsilon \le 1 + \varepsilon,$$

where the second equality comes from Corollary 3.26. □

4. Smith inequality

In this section, we show how the classical Smith inequality (5.4) can be applied to the sublevel sets of generating functions to prove inequality (5.1). Çineli-Ginzburg used the same kind of argument to prove a Smith inequality between the dimension of the local homology of a Hamiltonian orbit and its p-iterate for p prime [**29**].

4.1. $\mathbb{Z}/p\mathbb{Z}$-symmetry of a p-iterated generating function. Let us fix a prime number $p \ge 3$. Let us fix $t \in \mathbb{R}$ and study generating function of $(e^{-2i\pi t}\Phi_1)$. In order to fix notation, we recall that

$$F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v}) := \sum_{k=1}^{n} f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\langle v_k, iv_{k+1}\rangle,$$

where $\mathbf{v} := (v_1, \dots, v_n) \in (\mathbb{C}^{d+1})^n$ and the $f_k : \mathbb{C}^{d+1} \to \mathbb{R}$ are S^1-invariant and 2-homogeneous. Thus $F_{\boldsymbol{\sigma}^p_{m,t}} : (\mathbb{C}^{n(d+1)})^p \to \mathbb{R}$ is invariant under the action of $\mathbb{Z}/p\mathbb{Z}$ by cyclic permutation of coordinates generated by

$$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p) \mapsto (\mathbf{v}_p, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}),$$

(here $\boldsymbol{\sigma}^p_{m,t}$ means $(\boldsymbol{\sigma}_{m,t})^p$). The induced $\widehat{F}_{\boldsymbol{\sigma}^p_{m,t}} : \mathbb{C}\mathrm{P}^{pn(d+1)-1} \to \mathbb{R}$ is then invariant under the $\mathbb{Z}/p\mathbb{Z}$-action by permutation of homogeneous coordinates induced by

$$[\mathbf{v}_1 : \mathbf{v}_2 : \dots : \mathbf{v}_p] \mapsto [\mathbf{v}_p : \mathbf{v}_1 : \dots : \mathbf{v}_{p-1}].$$

Fixed points $(\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}/p\mathbb{Z}}$ of this action are the disjoint union $\bigsqcup_q P_q$ of the p following $(n(d+1)-1)$-complex projective subspaces:

$$P_q := \left\{ \left[\mathbf{v} : \zeta^q \mathbf{v} : \zeta^{2q}\mathbf{v} : \dots : \zeta^{(p-1)q}\mathbf{v}\right] \mid [\mathbf{v}] \in \mathbb{C}\mathrm{P}^{n(d+1)-1} \right\}, \quad \zeta := e^{\frac{2i\pi}{p}},$$

where q is an integer in $\left\{\frac{1-p}{2}, \dots, \frac{p-1}{2}\right\}$.

Using the fact that the f_k's are S^1-invariant and 2-homogeneous,

$$\begin{aligned} \frac{1}{p} F_{\boldsymbol{\sigma}^p_{m,t}}(\mathbf{v}, \zeta^q \mathbf{v}, \dots, \zeta^{q(p-1)}\mathbf{v}) = \sum_{k=1}^{n-1} \left[f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\langle v_k, iv_{k+1}\rangle \right] \\ + f_n\left(\frac{v_n + \zeta^q v_1}{2}\right) + \frac{1}{2}\langle v_n, i\zeta^q v_1\rangle. \end{aligned}$$

We apply the linear change of variables $\mathbf{v} \mapsto \mathbf{u}$ given by $u_k := v_k + (-1)^k \frac{1-\zeta^q}{2} v_1$ so that

$$\begin{cases} u_1 + u_2 &= v_1 + v_2, \\ u_2 + u_3 &= v_2 + v_3, \\ &\vdots \\ u_{n-1} + u_n &= v_{n-1} + v_n, \\ u_n + u_1 &= v_n + \zeta^q v_1. \end{cases}$$

A direct computation gives

$$\sum_{k=1}^{n-1} \langle v_k, iv_{k+1}\rangle + \langle v_n, i\zeta^q v_1\rangle = \sum_{k=1}^{n} \langle u_k, iu_{k+1}\rangle - 2\tan\left(\frac{q\pi}{p}\right)\|u_1\|^2,$$

for all integer $q \in \left\{\frac{1-p}{2}, \ldots, \frac{p-1}{2}\right\}$, so that

$$F_{\boldsymbol{\sigma}^p_{m,t}}(\mathbf{v}, \zeta^q\mathbf{v}, \ldots, \zeta^{q(p-1)}\mathbf{v}) = p\left[F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{u}) - \tan\left(\frac{q\pi}{p}\right)\|u_1\|^2\right] =: pG_{t,q}(\mathbf{u}),$$

for $q \in \left\{\frac{1-p}{2}, \ldots, \frac{p-1}{2}\right\}$. This last function $G_{t,q}$ is the fiberwise sum of a generating function of $e^{-2i\pi t}\Phi_1$ and a generating function of $e^{-2i\pi q/p}$. We recall that in this case, a $\mathbb{C}$-line of critical points of this function are in one-to-one correspondence with a $\mathbb{C}$-line of fixed points of the composed diffeomorphism $e^{-2i\pi(t+q/p)}\Phi_1$ (see the paragraph surrounding Equation (1.8)). Let $(f_{s,t})$ be the family of functions

$$f_{s,t}(\mathbf{u}) := F_{\boldsymbol{\sigma}_{m,t+(1-s)q/p}}(\mathbf{u}) - \tan\left(s\frac{q\pi}{p}\right)\|u_1\|^2, \quad s \in [0,1].$$

The function $f_{s,t}$ is the fiberwise sum of a generating function of $e^{-2i\pi(t+(1-s)q/p)}\Phi_1$ and a generating function of $e^{-2i\pi sq/p}$, so 0 is a regular value of $f_{s,t}$ if and only if $e^{-2i\pi(t+q/p)}\Phi_1$ does not have any $\mathbb{C}$-line of fixed points, that is if and only if $t + q/p$ is not an action value of $\boldsymbol{\sigma}$. According to Proposition 3.13,

$$H_*\left(\left\{\widehat{G}_{b,q} \le 0\right\}, \left\{\widehat{G}_{a,q} \le 0\right\}\right) \simeq G_{*-i_0}^{(a+q/p,b+q/p)}(\boldsymbol{\sigma}, m) \simeq G_{*-i_0}^{(a+q/p,b+q/p)}(\boldsymbol{\sigma}), \tag{5.3}$$

where $-m \le a + q/p \le b + q/p \le m$, i_0 is some integer and $a + q/p$ and $b + q/p$ are not action values of $\boldsymbol{\sigma}$.

4.2. Application of Smith inequality. According to Smith inequality,

$$\dim H_*(X; \mathbb{F}_p) \ge \dim H_*(X^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{F}_p), \tag{5.4}$$

where X is a locally compact space or pair such that $H_*(X; \mathbb{F}_p)$ is finitely generated, a space on which the group $\mathbb{Z}/p\mathbb{Z}$ acts (see for instance [**20**, Chapter IV, §4.1]). Here $\dim H_*$ means the total dimension $\sum_k \dim H_k$.

Proposition 5.4. *Given any tuple $\boldsymbol{\sigma}$ of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, for every prime number p and every $a \le b$ such that $a + q/p$ and $b + q/p$ are not action values of $\boldsymbol{\sigma}$ and pa and pb are not action values of $\boldsymbol{\sigma}^p$,*

$$\dim G_*^{(pa,pb)}(\boldsymbol{\sigma}^p; \mathbb{F}_p) \ge \sum_{(1-p)/2 \le q \le (p-1)/2} \dim G_*^{(a+q/p,b+q/p)}(\boldsymbol{\sigma}; \mathbb{F}_p).$$

Proof. Let us assume that $p \ge 3$ and refer the reader to Section 4.4 for the modifications specific to $p = 2$. By concatenating $\boldsymbol{\sigma}$ with some $\boldsymbol{\varepsilon}^k$ if needed, it is not difficult to find a homotopy interpolating $\boldsymbol{\sigma}^p_{m,t} := (\boldsymbol{\sigma}_{m,t})^p$ and $(\boldsymbol{\sigma}^p)_{pm,pt}$. Therefore, the associated interpolation isomorphism gives us

$$HZ_*(\boldsymbol{\sigma}^p_{m,b}, \boldsymbol{\sigma}^p_{m,a}) \simeq G_*^{(pa,pb)}(\boldsymbol{\sigma}^p, pm) \simeq G_*^{(pa,pb)}(\boldsymbol{\sigma}^p). \tag{5.5}$$

Now, we apply the Smith inequality (5.4) to the couple

$$X := \left(\left\{\widehat{F}_{\boldsymbol{\sigma}^p_b} \le 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}^p_a} \le 0\right\}\right).$$

According to the last section,

$$X^{\mathbb{Z}/p\mathbb{Z}} \simeq \bigsqcup_{(1-p)/2 \le q \le (p-1)/2} \left(\left\{\widehat{G}_{b,q} \le 0\right\}, \left\{\widehat{G}_{a,q} \le 0\right\}\right).$$

Therefore, Smith inequality (5.4), (5.5) and (5.3) yield the conclusion. □

4.3. Computation of β_{tot}. The arguments of this section follow the proof of [**3**, Theorem B.1] given by Shelukhin in the realm of Floer theory that applies to every closed monotone symplectic manifold.

PROPOSITION 5.5. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1}$ with a finite number of associated fixed points in CP^d. For all $a \in \mathbb{R}$, all integer $n \in \mathbb{N}^*$ and all field $\mathbb{F}$,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}) = \frac{1}{2}\left(\int_0^1 \dim G_*^{(a+t,a+t+n)}(\boldsymbol{\sigma};\mathbb{F})\,\mathrm{d}t - n(d+1)\right).$$

PROOF. Let $I_1,\ldots,I_n \subset \mathbb{R}$ be representatives of each $\mathbb{Z}$-orbit of finite bars of the persistence module $(G_*^{(-\infty,t)}(\boldsymbol{\sigma};\mathbb{F}))_t$ and let $J_1,\ldots,J_{d+1} \subset \mathbb{R}$ be representatives of each $\mathbb{Z}$-orbit of infinite bars of this persistence module. Given an interval $I \subset \mathbb{R}$, let $\chi_I : \mathbb{R} \to \{0,1\}$ denote its characteristic map. According to Lemma 3.29, for $a < b$ which are neither infinite nor action values of $\boldsymbol{\sigma}$, one has

$$\dim G_*^{(a,b)}(\boldsymbol{\sigma};\mathbb{F}) = \sum_{k\in\mathbb{Z}}\left[\sum_{r=1}^{n}\left|\chi_{I_r}(b+k) - \chi_{I_r}(a+k)\right| + \sum_{s=1}^{d+1}\left|\chi_{J_s}(b+k) - \chi_{J_s}(a+k)\right|\right].$$

Therefore, in order to prove the statement, it is enough to prove that for all $I \in \{I_1,\ldots,I_n\}$,

$$\sum_{k\in\mathbb{Z}}\int_0^1 \left|\chi_I(a+t+k+n) - \chi_I(a+t+k)\right|\mathrm{d}t = 2\,\mathrm{length}\,I, \tag{5.6}$$

while for all $J \in \{J_1,\ldots,J_{d+1}\}$,

$$\sum_{k\in\mathbb{Z}}\int_0^1 \left|\chi_J(a+t+k+n) - \chi_J(a+t+k)\right|\mathrm{d}t = n. \tag{5.7}$$

In order to prove (5.6), let us write $I = (u,v)$. According to Theorem 5.2, $\mathrm{length}\,I = v-u \leq 1$. If the integer parts $\lfloor u-a\rfloor$ and $\lfloor v-a\rfloor$ are equal, then only the terms $k = \lfloor u-a\rfloor$ and $k = \lfloor u-a\rfloor+n$ are non-zero, both equal to $\mathrm{length}\,I$. Otherwise, one must compute the four non-zero terms and gets

$$(1-\{u-a\}) + \{v-a\} + (1-\{u-a\}) + \{v-a\} = 2(v-u) = 2\,\mathrm{length}\,I,$$

where $\{x\}$ denotes the fractional part $x - \lfloor x\rfloor$ for $x \in \mathbb{R}$ (at the first equality, we have used $\lfloor v-a\rfloor = \lfloor u-a\rfloor + 1$).

Identity (5.7) is proven by a similar straightforward computation. □

COROLLARY 5.6 ([**69**, Theorem D for $M = \mathrm{CP}^d$]). *For all tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1}$ with a finite number of associated fixed points in CP^d, for all prime number p,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p).$$

PROOF. Let us take the integral over almost all $t \in [0,1]$ of the inequality stated in Proposition 5.4 for $a = t$ and $b = 1+t$. On the left hand side,

$$\begin{aligned}
\int_0^1 \dim G_*^{(pt,p+pt)}(\boldsymbol{\sigma}^p;\mathbb{F}_p)\mathrm{d}t &= \frac{1}{p}\int_0^p \dim G_*^{(s,p+s)}(\boldsymbol{\sigma}^p;\mathbb{F}_p)\mathrm{d}s \\
&= \frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \dim G_*^{(k+s,k+s+p)}(\boldsymbol{\sigma}^p;\mathbb{F}_p)\mathrm{d}s \\
&= \frac{1}{p}\sum_{k=0}^{p-1}\left[2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p) + p(d+1)\right] \\
&= 2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p) + p(d+1),
\end{aligned}$$

where we have applied Proposition 5.5 at the third line. On the right hand side, by applying Proposition 5.5 once again,

$$\sum_{(1-p)/2\leq q\leq (p-1)/2} \int_0^1 \dim G_*^{(q/p,q/p+t)}(\boldsymbol{\sigma};\mathbb{F}_p)\mathrm{d}t = \sum_{(1-p)/2\leq q\leq (p-1)/2} \left[2\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p)+d+1\right]$$
$$= 2p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p)+p(d+1).$$

Therefore,

$$2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p)+p(d+1)\geq 2p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p)+p(d+1)$$

and the conclusion follows. □

Proposition 5.7. *For all tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms $\boldsymbol{\sigma}$ of $\mathbb{C}^{d+1}$ with a finite number of associated fixed points in $\mathbb{C}\mathrm{P}^d$, there exists an integer $N\in\mathbb{N}$ such that for all prime number $p\geq N$,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p)=\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{Q}).$$

Proof. According to Proposition 5.5, it is enough to prove that for some $N\in\mathbb{N}$ every prime number such that $p\geq N$ satisfies

$$\dim G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{F}_p)=\dim G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{Q}),\quad \forall t\in[0,1]. \tag{5.8}$$

If there is no action value of $\boldsymbol{\sigma}$ in $[a,b]$ then $\dim G_*^{(a,a+1)}(\boldsymbol{\sigma};\mathbb{F})=\dim G_*^{(b,b+1)}(\boldsymbol{\sigma};\mathbb{F})$ for all field $\mathbb{F}$. Since there is a finite number with action values in $[0,1]$, it is enough to prove (5.8) for a finite number of value t (one in between each critical value of $[0,1]$). For each t (that is not an action value), $G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{F})\simeq H_{*+i_0}(A_t,B_t;\mathbb{F})$ for a topological pair (A_t,B_t) of some complex projective space with a finitely generated homology group with integral coefficients. According to the universal coefficient theorem, there exists $N_t\in\mathbb{N}$ such that $\dim H_*(A_t,B_t;\mathbb{Q})=\dim H_*(A_t,B_t;\mathbb{F}_p)$ for all prime number $p\geq N_t$. The conclusion follows by taking the maximum among the N_t's for our finite set of t's. □

4.4. The special case $p=2$. Here, we briefly explain how to modify the above arguments in the special case $p=2$ — that is only useful to prove Theorem 5.1 when $\mathbb{F}$ has characteristic 2.

In order to study the $\mathbb{Z}/2\mathbb{Z}$-symmetry of a generating function associated with (Φ_t^2), one cannot take the generating function of $\boldsymbol{\sigma}_{m,t}^2$ since it is an even tuple. Instead, we take the generating function of $(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})$ which is invariant under the following action of $\mathbb{Z}/2\mathbb{Z}$ written in w-variables:

$$(\mathbf{w}^1,w^2,\mathbf{w}^3)\mapsto\left(\mathbf{w}^3,-w^2+\sum_{k=1}^n(-1)^k(w_k^1+w_k^3),\mathbf{w}^1\right).$$

Indeed, Q_{2n+1} is invariant under this action and, in w-variables,

$$F_{(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})}(\mathbf{w}^1,w^2,\mathbf{w}^3)=F'(\mathbf{w}^1)+F'(\mathbf{w}^3)+Q_{2n+1}(\mathbf{w}^1,w^2,\mathbf{w}^3),$$

where F' is the direct sum of the elementary generating functions of $\boldsymbol{\sigma}_{m,t}$. The set of fixed points of the induced action on $\mathbb{C}\mathrm{P}^{(2n+1)(d+1)-1}$ is the disjoint union of the complex projective spaces P_0 and P_1 defined by

$$P_0:=\left\{\left[\mathbf{w}:\sum_{k=1}^n(-1)^k w_k:\mathbf{w}\right]\ \middle|\ \mathbf{w}\in(\mathbb{C}^{d+1})^n\right\},$$
$$P_1:=\left\{[\mathbf{w}:w':-\mathbf{w}]\ |\ (\mathbf{w},w')\in(\mathbb{C}^{d+1})^n\times\mathbb{C}^{d+1}\right\}.$$

The restriction of the generating function to P_0 gives us back the generating function $\widehat{F}_{\boldsymbol{\sigma}_{m,t}}$ whereas, still in w-variables,

$$\frac{1}{2}F_{(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})}(\mathbf{w},w',-\mathbf{w})=F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{w})+2\left\langle\sum_{k=1}^n(-1)^{k+1}w_k,iw'\right\rangle,$$

By the change of variables $A_n\mathbf{v}=\mathbf{w}$ and $\xi=2w'$, one gets, in v-variables, the function

$$(\mathbf{v},\xi)\mapsto F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v})+\langle v_1,i\xi\rangle$$

which is the fiberwise sum of a generating function of $e^{-i\pi t}\Phi_1$ with the generating function $(x;\xi) \mapsto \langle x, i\xi\rangle$ that generates $-\mathrm{id}$. This time, we can take $(f_{s,t})$ to be the family of functions

$$f_{s,t}(\mathbf{v}, v_{n+1}) := F_{\boldsymbol{\sigma}_{m,t+(1-s)/2}}(\mathbf{v}) + \sin\left(s\frac{\pi}{2}\right)\left\langle v_1 - i\cos\left(s\frac{\pi}{2}\right)\frac{\xi}{2}, i\xi\right\rangle,\ s \in [0,1],$$

that interpolates the latter with $(\mathbf{v}, \xi) \mapsto F_{\boldsymbol{\sigma}_{m,t+1/2}}(\mathbf{v})$. That being said, it is not difficult to conclude.

Part 2

Geodesic loops and geodesic chords

CHAPTER 6

Homologically visible closed geodesics on complete surfaces

This chapter is a joint work with Tobias Soethe.

1. Introduction

Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $M \simeq S^1 \times \mathbb{R}$ be a complete Riemannian cylinder. Let ΛM be its loop space. Two loops $\alpha, \beta \in \Lambda M$ are said to be geometrically distinct if their images are distinct: $\alpha(S^1) \neq \beta(S^1)$. Throughout this chapter, by writing that two closed geodesics are distinct we will always mean that they are geometrically distinct. Given a ring R, a closed geodesic $\gamma \in \Lambda M$ is said to be homologically visible over R if the local homology of the critical circle $S^1 \cdot \gamma \subset \Lambda M$ of the energy functional is non-zero over the coefficients ring R (see Section 2 for precise definitions). With the exception of the Möbius band, every result is true over any coefficients ring R (once fixed), so the ring R will not be mentioned explicitly.

THEOREM 6.1. *Let M be a complete Riemannian cylinder where all closed geodesics are isolated and assume one of the following hypothesis:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics.*

Then M contains infinitely many homologically visible closed geodesics intersecting some common compact set $K \subset M$ and at least one without self-intersection.

Notice that according to Bott's iteration theory, a closed geodesic c has a non-zero average index if and only if some iterate c^m has a non-zero index. The fact that hypothesis 5 implies that there exists infinitely many homologically visible closed geodesics proves a conjecture of Abbondandolo:

COROLLARY 6.2. *Any complete Riemannian cylinder where all closed geodesics are isolated has zero, one or infinitely many homologically visible closed geodesics.*

By essentially taking the double cover (see Section 7 for details), one can thus deduce the following counterpart of Theorem 6.1 when M is a complete Möbius band.

COROLLARY 6.3. *Let M be a complete Riemannian Möbius band where all closed geodesics are isolated and assume one of the following hypothesis:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics over $\mathbb{F}_2$.*

Then M contains infinitely many closed geodesics intersecting some common compact set $K \subset M$ that are homologically visible over $\mathbb{F}_2$.

According to Thorbergsson [**76**, Theorem 3.2], any complete Möbius band has at least one homologically visible closed geodesic without self-intersection (it is homologically visible as a local minimum of the energy, see Section 2 below).

COROLLARY 6.4. *Any complete Riemannian Möbius band where all closed geodesics are isolated has one or infinitely many homologically visible closed geodesics over $\mathbb{F}_2$.*

Similar results can also be obtained when $M \simeq \mathbb{R}^2$ is a complete plane:

THEOREM 6.5. *Let M be a complete Riemannian plane where all closed geodesics are isolated and assume one of the following hypothesis:*

1. there exists a self-intersecting closed geodesic,
2. there exist two distinct closed geodesics that intersect,
3. there exists a closed geodesic of non-zero average index,
4. there exists a homologically visible closed geodesic.

Then M contains infinitely many homologically visible closed geodesics intersecting some common compact set $K \subset M$ and at least one without self-intersection.

COROLLARY 6.6. *Any complete Riemannian plane where all closed geodesics are isolated has zero or infinitely many homologically visible closed geodesics.*

It is easy to give counter-examples to Theorem 6.1 when none of the assumptions 1-5 hold by considering embedded cylinders of revolution

$$(\theta, z) \mapsto (r(z)\cos\theta, r(z)\sin\theta, z), \tag{6.1}$$

for well-chosen smooth maps $r : \mathbb{R} \to (0, +\infty)$. A complete cylinder may have no closed geodesic at all: take $r' > 0$. It can have an arbitrary large finite number $k \in \mathbb{N}$ of homologically invisible closed geodesics: take $r'(z) > 0$ for all $z \in \mathbb{R} \setminus \{z_1, \ldots, z_k\}$ and $r'(z_i) = 0$. It can also have a unique visible closed geodesic: take $r' < 0$ on $(-\infty, 0)$, $r'(0) = 0$ and $r' > 0$ on $(0, +\infty)$ (one can as well add to this cylinder an arbitrary large finite number of homologically invisible closed geodesic the same way as before). By taking such an r even and taking the quotient under the involution $(\theta, z) \mapsto (\theta + \pi, -z)$, one gets Möbius bands with only one homologically visible closed geodesic and as many homologically invisible closed geodesic as wanted. Remark that in our examples closed geodesics are without self-intersections and not contractible as implied by the theorem. Counter-examples where the theorem fails by lack of completeness can be found as well by choosing embedded cylinders of revolution restricting the domain of the embedding (6.1) to $(\theta, z) \in \mathbb{R}/2\pi\mathbb{Z} \times (a, b)$ for $a, b \in \mathbb{R}$. We could proceed as follows: take an even $r : [-1, 1] \to (0, +\infty)$ with $r' > 0$ on $[-1, 0)$ such that $z = 0$ is the only closed geodesic of the associated compact embedded cylinder. One can find such an r by slightly modifying a Tannery surface: a sufficient condition is that the metric g on the interior of the cylinder can be written as

$$g = [\alpha + h(\cos\rho)]^2 \,\mathrm{d}\rho^2 + \sin^2\rho \,\mathrm{d}\theta^2,$$

for a good choice of coordinates $(\rho, \theta) \in (0, \pi) \times S^1$, where α is irrational and $h : (-1, 1) \to (-\alpha, \alpha)$ is a smooth odd function (see for instance [**18**, Theorem 4.13]). Then extend r to a smooth map $(-3, 1] \to (0, +\infty)$ with $r|_{(-3,-1)} < r(-1)$, $r' < 0$ on $(-3, -2)$ and $r' > 0$ on $(-2, -1)$. Then $z = -2$ and $z = 0$ are the only closed geodesic of the cylinder embedded by $r|_{(-3,1)}$ and are both visible.

In a similar way, we can give examples of complete planes with only an arbitrary finite number of homologically invisible closed geodesics by considering surfaces of revolution (6.1) parametrized by $\mathbb{R}/2\pi\mathbb{Z} \times [0, +\infty)$ with $r : [0, +\infty) \to [0, +\infty)$ being increasing and smooth on $(0, +\infty)$ with $r(0) = 0$ and $r'(z) \to +\infty$ when $z \to 0$ in a suitable way (*i.e.* so that the surface is smooth at the origin). Then, as above, we get homologically invisible closed geodesics on the inflexion points of r, and nowhere else.

We say that $C_- \subset M$ (resp. C_+) is a neighborhood of $-\infty$ (resp. of $+\infty$) if C_- contains $S^1 \times (-\infty, a)$ for some $a \in \mathbb{R}$ (resp. $S^1 \times (b, +\infty)$ for some $b \in \mathbb{R}$) for an arbitrarily fixed identification of M with $S^1 \times \mathbb{R}$. In order to prove Theorem 6.1, we will extensively use the following theorem due to Bangert (where for the notion of local convexity, we refer to Section 2.2):

THEOREM 6.7 ([**9**, Theorem 3, Remark 2]). *Let M be a complete Riemannian cylinder where all closed geodesics are isolated and suppose there exist disjoint locally convex open neighborhoods C_- and C_+ of $-\infty$ and $+\infty$ respectively such that the boundaries $\partial C_\pm$ are not totally geodesic. Then M contains infinitely many homologically visible closed geodesics intersecting $M \setminus (C_- \cup C_+)$ and at least one without self-intersections.*

Since Bangert did not give the precise proof of that statement, for the sake of completeness we give a comprehensive proof in this chapter. The proof of Theorem 6.5 is quite similar and relies extensively on the analogous theorem of Bangert when M is a plane where all closed geodesics are isolated: if there exists a locally convex open neighborhood of infinity $C \neq M$ with a boundary ∂C which is not totally geodesic, M contains infinitely many homologically visible closed geodesics [**9**, Theorem 3]. These two theorems were originally used by Bangert to prove that any complete Riemannian plane of finite area has infinitely many closed geodesics.

In fact, our results extend *verbatim* to the case where M is a complete reversible Finsler manifold as we will essentially use variational properties of geodesics in our proof with no concern for geometric notion specific to Riemannian manifold. However, nothing can be said concerning the more general case of a complete (asymmetrical) Finsler manifold. The major issue is that, in the asymmetrical case, a closed subset of M which is bounded by a geodesic is not locally convex. In this direction, we point out that the related question of whether or not infinitely many closed geodesics exist on every irreversible Finsler cylinder of finite area is still open [**22**, Question 2.3.2].

In Section 2, we fix notation and recall results of the variational theory of geodesics that we will need. In Section 3, we give a comprehensive proof of Theorem 6.7 of Bangert. In Section 4, we prove Theorem 6.1 when hypothesis 1, 2 or 3 is assumed. In Section 5, we prove Theorem 6.1 when hypothesis 4 is assumed. In Section 6, we prove the last case of Theorem 6.1. In Section 7, we prove Corollary 6.3. In Section 8, we prove Theorem 6.5.

2. Preliminaries

In this section, we recall some results of Riemannian geometry that we will use in our proofs and fix some notation. For the extension of these notions to the Finsler case, the reader may for instance look at [**23**, Section 2].

2.1. The energy functional. Given a complete Riemannian manifold with boundary W, we denote by ΛW the space of H^1-maps $S^1 \to W$. In fact, if one wants to avoid analytic questions, we can always reduce our space to a finite-dimensional manifold of broken geodesics. For $\gamma \in \Lambda W$ and $m \in \mathbb{N}^*$, the iterated loop $\gamma^m \in \Lambda W$ is defined by $t \mapsto \gamma(mt)$. A geodesic is an immersed path $\gamma : \mathbb{R} \to W$ such that
$$\nabla_{\dot\gamma}\dot\gamma = 0,$$
where ∇ denotes the Levi-Civita connection of the metric and $\dot\gamma$ stands for the derivative of γ. Therefore, in our convention, geodesics have constant speed. A closed geodesic is a geodesic γ which is periodic: $\gamma(t+1) = \gamma(t)$ so that $\gamma \in \Lambda W$. Closed geodesics of W are the critical points with non-zero critical value of the energy functional $E : \Lambda W \to [0, +\infty)$,
$$E(\gamma) := \int_{S^1} g_\gamma(\dot\gamma, \dot\gamma)\mathrm{d}t, \quad \forall \gamma \in \Lambda W.$$
The energy functional E is C^2. If W is a locally convex compact manifold (possibly with boundary), E also satisfies the Palais-Smale condition and the $(-\nabla E)$-flow is defined for all time $t \geq 0$ (for a real-valued map f on a Riemann-Hilbert manifold, ∇f denotes its gradient). We notice that every closed geodesic lies on a critical circle $S^1 \cdot \gamma$, where S^1 acts on ΛW by $t \cdot \gamma := \gamma(t + \cdot)$. In our study we assume that E has only isolated critical circles (except for the constant loops which have zero value). Two closed geodesics c_1 and c_2 are said to be geometrically distinct if they do not have the same image in W.

2.2. Finite-dimensional approximation of the loop space. Morse's finite-dimensional approximation of the curve space over W, as presented by Bangert in [**9**] consists of the following data: an open set $\mathcal{O} \subset W$, an energy bound $\kappa > 0$ and a parameter $j \in \mathbb{N}$ satisfying $\frac{1}{j} < \frac{\varepsilon^2}{\kappa}$ where $\varepsilon > 0$ is smaller than the injectivity radius on $\mathcal{O}$. The positivity of ε will be fulfilled if for instance $\mathcal{O}$ has compact closure, as will be the case in our considerations. The finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$ is constructed as follows: it is the set of all curves $\gamma \in \Lambda W$ such that $E(\gamma) < \kappa$, $\gamma(i/j) \in \mathcal{O}$ and such that $\gamma|_{[i/j,(i+1)/j]}$ is a geodesic of length less than ε for $0 \leq i \leq j-1$.

Let $d(p,q) \in \mathbb{R}$ denote the Riemannian distance between two points $p, q \in W$ and $B_r(p) := \{q \in W \mid d(p,q) < r\}$ denote the Riemannian ball of radius $r \geq 0$ centered at p. In this chapter, given any open subset $C \subset W$, we will say that C is locally convex if there exists an $\varepsilon > 0$ so that for every $p \in C$ and every pair of points $q, q' \in C$ lying in the connected component of $B_{2\varepsilon}(p) \cap C$ containing p, there exists a unique geodesic of length $d(q,q')$ joining q and q' and it is entirely contained in C. Let Ω be a finite-dimensional approximation of ΛW and $C \subset W$ a locally convex set with compact boundary such that $C \subset \mathcal{O}$ and let us fix an $\varepsilon > 0$ given by the definition of local convexity. The negative gradient of the restriction of the energy functional to Ω is given by

$$-\nabla E|_{\Omega}(\gamma) = -2\big(\dot\gamma_1(1/j) - \dot\gamma_2(1/j), \ldots, \dot\gamma_{j-1}((j-1)/j) - \dot\gamma_j((j-1)/j)\big)$$

for $\gamma \in \Omega$, where $\gamma_i = \gamma|_{[(i-1)/j, i/j]}$ for $1 \le i \le j$ (see [**45**, p. 252]). Now from our choice of j and Cauchy-Schwarz inequality, we get

$$d(\gamma((i-1)/j), \gamma(i/j))^2 \le \frac{1}{j} E(\gamma|_{((i-1)/j, i/j)}) \le \frac{\varepsilon^2}{\kappa}\kappa = \varepsilon^2$$

and consequently by local convexity of C, the negative gradient flow of the finite-dimensional approximation of the energy functional respects C.

2.3. Index of a closed geodesic. The index of a closed geodesic γ is the Morse index of E:

$$\mathrm{ind}(\gamma) := \mathrm{ind}(E, \gamma).$$

It is always finite. The behavior of this index under iteration $k \mapsto \mathrm{ind}(\gamma^k)$ was extensively studied by Bott in [**21**]. We simply recall that

$$\mathrm{ind}(\gamma^k) \ge k\,\overline{\mathrm{ind}}(\gamma) - \dim(W) + 1, \quad k \in \mathbb{N}, \tag{6.2}$$

where $\overline{\mathrm{ind}}(\gamma) \ge 0$ is the average index of γ defined by

$$\overline{\mathrm{ind}}(\gamma) := \lim_{k\to\infty} \frac{\mathrm{ind}(\gamma^k)}{k}.$$

Let $p \in W$ and $\Omega_p W \subset \Lambda W$ be the set of loops based at p, that is H^1-paths $\gamma : [0,1] \to W$ such that $\gamma(0) = \gamma(1) = p$. Given a closed geodesic $\gamma \in \Lambda W$, we denote by $\mathrm{ind}_\Omega(\gamma) \in \mathbb{N}$ the Morse index

$$\mathrm{ind}_\Omega(\gamma) := \mathrm{ind}\left(E|_{\Omega_{\gamma(0)}W}, \gamma\right).$$

By inclusion, $\mathrm{ind}_\Omega(\gamma) \le \mathrm{ind}(\gamma)$. In fact, we have the concavity inequality [**7**, Eq. (1.5)]:

$$\mathrm{ind}(\gamma) - \dim(W) + 1 \le \mathrm{ind}_\Omega(\gamma) \le \mathrm{ind}(\gamma). \tag{6.3}$$

A Jacobi field of the geodesic path γ is a smooth map $J : \mathbb{R} \to \gamma^* TW$, satisfying

$$J(t) \in T_{\gamma(t)}W, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \nabla^2_{\dot\gamma} J = \mathrm{R}(\dot\gamma, J)\dot\gamma,$$

where R denotes the Riemann tensor. Let $\mu(t)$ be the number of linearly independent Jacobi fields of γ such that $J(0) = J(t) = 0$; the Morse index theorem states that

$$\mathrm{ind}_\Omega(\gamma) = \sum_{0<t<1} \mu(t). \tag{6.4}$$

The local homology of an isolated critical circle $S^1 \cdot \gamma$ over the ring R is by definition

$$\mathrm{C}_*(S^1\cdot\gamma; R) := H_*(\{E < E(\gamma)\} \cup S^1 \cdot \gamma, \{E < E(\gamma)\}; R),$$

where the set $\{E < E(\gamma)\}$ is $\{\delta \in \Lambda W \mid E(\delta) < E(\gamma)\}$, and H_* denotes the singular homology. When the choice of fixed ring R is irrelevant, the symbol R will not be written. According to the Gromoll-Meyer theory, local homology groups are finitely generated (see [**46**, remark following Lemma 1] for the case of an isolated critical point and [**6**, Proposition 3.1] for the reduction to this case). A closed geodesic is said to be homologically visible if $\mathrm{C}_*(S^1\cdot\gamma) \ne 0$ and is said to be homologically invisible otherwise. Although this notion depends on the choice of coefficients ring R, the universal coefficients theorem implies that a closed geodesic is homologically invisible over every ring R if and only if it is homologically invisible over $\mathbb{Z}$. By excision, for all neighborhood $U \subset \Lambda W$ of $S^1 \cdot \gamma$,

$$\mathrm{C}_*(S^1\cdot\gamma) \simeq H_*\left(U \cap \left(\{E < E(\gamma)\} \cup S^1\cdot\gamma\right), U \cap \{E < E(\gamma)\}\right). \tag{6.5}$$

Therefore, every local minimum of E is homologically visible. We will be interested in the properties of the local homology $\mathrm{C}_*(S^1\cdot\gamma)$ especially in the case where γ is a closed geodesic of average index $\overline{\mathrm{ind}}(\gamma) = 0$ and whose image $\gamma(S^1)$ lies in the interior of W ($\overline{\mathrm{ind}}(\gamma) = 0$ is equivalent to the fact that $\mathrm{ind}(\gamma^m)$ vanishes for all $m \ge 1$). Let $\gamma \in \Lambda W$ be such a closed geodesic. Given $m \in \mathbb{N}$, we denote by $\psi_m : \Lambda W \to \Lambda W$ the iteration map $\psi_m(\delta) := \delta^m$. According to a theorem of Gromoll-Meyer

[**47**, Theorem 3], the local homology $\mathrm{C}_d(S^1 \cdot \gamma)$ is zero in degrees $d \geq 2 \dim W$ and there exist infinitely many positive integers m such that the induced map in homology

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot \gamma) \to \mathrm{C}_*(S^1 \cdot \gamma^m) \tag{6.6}$$

is an isomorphism. On the other hand, a theorem of Bangert-Klingenberg [**8**, Corollary 1] states that there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$, there exists $e_m > m^2 E(\gamma)$ such that the composition

$$\mathrm{C}_*(S^1 \cdot \gamma) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1 \cdot \gamma^m) \xrightarrow{\mathrm{inc}_*} H_* \left(\{E < e_m\}, \{E < m^2 E(\gamma)\}\right) \tag{6.7}$$

is zero.

3. Proof of the Bangert theorem

A closed geodesic γ is a mountain pass if, for all neighborhoods $U \subset \Lambda M$ of $S^1 \cdot \gamma$, $U \cap E^{-1}([0, E(\gamma))$ is not connected. For the proof of Theorem 6.7, we need the following statement, which tells us that isolated closed geodesics cannot remain mountain pass critical points of the energy functional when sufficiently iterated. A geometric proof is given by Bangert [**9**].

Theorem 6.8 ([**9**, Theorem 2]). *Let γ be an isolated closed geodesic on M, where $\dim M = 2$. Then there exists an integer $m_\gamma \in \mathbb{N}$ such that the following holds: For all integer $m \in \mathbb{N}$ with $m \geq m_\gamma$, there is a neighborhood U of $S^1 \cdot \gamma$ in ΛM such that $U \cap E^{-1}([0, E(\gamma^m)))$ is connected.*

According to Gromoll-Meyer [**47**], given an isolated closed geodesic γ, there exists a connected neighborhood $U \subset \Lambda M$ of the critical circle $S^1 \cdot \gamma$ such that

$$\mathrm{C}_*(S^1 \cdot \gamma) \simeq H_* \left(U, U \cap E^{-1}\left([0, E(\gamma))\right)\right).$$

If γ and all its iterates are homologically invisible, Theorem 6.8 is thus true for $m_\gamma = 1$.

Proof of Theorem 6.7. We begin by taking care of a technical problem: we will need to have that ∂C_- and ∂C_+ do not contain any closed geodesic. By assumption, if ∂C_- contains a closed geodesic, then ∂C_- is not connected and the complement of C_- has a bounded connected component. We then find a simple closed curve which is contractible in M, but non-contractible in $M \setminus C$. We can apply the negative gradient flow in a finite dimensional approximation (see Section 2.2) to obtain a closed geodesic which is contractible in M. Then we can apply the argument presented in Section 4 (which works independently of Bangert's theorem) to conclude that there already have to exist infinitely many homologically visible closed geodesics. We now assume that both ∂C_- and ∂C_+ do not contain any closed geodesic.

Assume there are only finitely many prime closed geodesics $\gamma_1, \ldots, \gamma_k$ which have homologically visible iterates and which intersect $W := M \setminus (C_- \cup C_+)$. We will now derive a contradiction from this assumption. We will define a suitable finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$. Now as the statement of Theorem 6.8 remains true in a finite-dimensional approximation, we get that there exists $m_0 \in \mathbb{N}$ such that for all integers $m \geq m_0$ and for all $i \in \{1, \ldots, k\}$ the following holds:

i) There exists a neighborhood U of $S^1 \cdot \gamma_i^m$ in Ω such that $U \cap E^{-1}([0, E(\gamma_i^m)))$ is connected.

Set $A := \max\{E(\gamma_i^{m_0}) \,|\, i \in \{1, \ldots, k\}\}$, and notice that A is larger than the energy of a closed geodesic of mountain pass type. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We define the following sets of curves:

$$P_n^\pm := \{\gamma \in \Omega \,|\, \gamma(S^1) \subset C_\pm,\ [\gamma] = n\}.$$

In the following for each $U, V \subset M$, we will denote

$$\mathrm{dist}(U, V) := \inf_{x \in U,\ y \in V} d(x, y).$$

Choose $\delta > 0$. Then there exists an $n \in \mathbb{N}$ such that for any curve $\gamma \in P_n^\pm$ such that $\mathrm{dist}(\gamma(S^1), W)$ is less than δ it holds that $E(\gamma) \geq A$. We can now say how exactly the finite-dimensional approximation has to be chosen:

- Choose a $\kappa > 0$ large enough such that there exists a homotopy $h : [0, 1] \to E^{-1}([0, \kappa))$ in Ω from $h_0 \in P_n^-$ to $h_1 \in P_n^+$ with
$$\mathrm{dist}\left(h_t(S^1), W\right) < \delta, \quad \forall t \in [0, 1].$$
- Set $\mathcal{O} := \{p \in M \mid \mathrm{dist}(p, W) < R\}$, where $R > 2\kappa^{\frac{1}{2}} + \delta$ such that $\mathcal{O}$ contains $\gamma_1, \ldots, \gamma_k$.

- Choose j such that the $(-\nabla E)$-flow of the finite-dimensional approximation respects $C_\pm$, as described above.

A technical issue is given by the fact that the gradient flow of $-\nabla E$ may not be defined for all times as the sublevel sets of $E|_\Omega$ are not compact. Ultimately we are only going to be interested in curves intersecting the compact set W, *i.e.* the subset

$$K := \{\gamma \in \Omega \,|\, \gamma(S^1) \cap W \neq \varnothing\}$$

of Ω. We introduce a smooth function $g : \Omega \to [0,1]$ with the property that

$$\begin{cases} g(\gamma) = 1 & \text{if } \operatorname{dist}(\gamma(S^1), W) \leq \frac{1}{2}\kappa^{\frac{1}{2}}, \\ g(\gamma) = 0 & \text{if } \operatorname{dist}(\gamma(S^1), W) > \frac{3}{2}\kappa^{\frac{1}{2}}. \end{cases}$$

Then the flow ϕ_t of $-g\nabla E$ is defined for all times $t \geq 0$ and coincides with the negative gradient flow for curves in K. Two crucial observations about the set K are the following: firstly, for all $\bar\kappa < \kappa$ the set $K \cap E^{-1}([0,\bar\kappa])$ is compact. Secondly, if $\phi_t(\gamma) \in K$ for some $\gamma \in \Omega$ and some time $t \geq 0$, we already have $\gamma \in K$ as the flow ϕ_t respects the convex sets $C_\pm$. From this it follows:

ii) Let $0 < \kappa_0 < \kappa_0 + \varepsilon < \kappa$. Let V denote a neighborhood of the closed geodesics in K of energy κ_0. Suppose there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0 + \varepsilon])$. Then there exists a time $\tau > 0$, such that

$$\phi_\tau\left(E^{-1}\left([0,\kappa_0+\varepsilon]\right)\right) \cap K \subset E^{-1}([0,\kappa_0)) \cup V.$$

This is just the deformation lemma; for a proof see for instance [**71**, Lemma 3.4]. We are now set to complete the proof of the theorem. Define the set of homotopies

$$\Pi := \left\{\beta : [0,1] \to \Omega \text{ continuous} \,|\, \beta_0 \in P_n^-,\, \beta_1 \in P_n^+\right\}.$$

Note that Π is not empty, as $h \in \Pi$. Furthermore, $\phi_t \circ \beta \in \Pi$ for all $\beta \in \Pi$ and all $t \geq 0$ as the flow respects the convex sets $C_\pm$ and therefore $\phi_t(\beta_0) \in P_n^-$ and $\phi_t(\beta_1) \in P_n^+$ for all $t \geq 0$. Define now

$$\kappa_0 := \inf_{\beta \in \Pi} \max_{\substack{t \in [0,1] \\ \beta_t \in K}} E(\beta_t)\,.$$

By definition of κ, one has $\kappa_0 < \kappa$. For every $\beta \in \Pi$ for time $t_0 := \min\{t \in [0,1] \,|\, \beta_t \notin P_n^-\}$ it holds that $\beta_{t_0} \in K$ and $E(\beta_{t_0}) \geq A$ (as E and β are continuous and there exists a sequence $(t_k) \nearrow t_0$ such that $\beta_{t_k} \in P_n^-$ and $\operatorname{dist}(\beta_{t_k}, W) < \delta$). Consequently, we get $\kappa_0 \geq A$. Since $\kappa_0 < \kappa$, for $\varepsilon > 0$ small enough, the subset $K \cap E^{-1}([0,\kappa_0+\varepsilon])$ is compact and there are only finitely many S^1-orbits of closed geodesics inside (we assumed every orbit to be isolated). Let $\{S^1 \cdot d_j\}_{1 \leq j \leq l}$ denote the critical circles of energy κ_0 in K. By definition of A and by using i) when d_j is a power of some γ_i (otherwise this is true according to the remark just before the beginning of the proof), there exist disjoint neighborhoods U_j of the $S^1 \cdot d_j$'s such that $U_j \cap E^{-1}([0,\kappa_0))$ is connected for all j. Since $\partial C_\pm$ do not contain any closed geodesic, we know that the d_j's are not contained in ∂K and we can assume that $U_j \subset K$. Now because there are only finitely many closed geodesics in $K \cap E^{-1}([0,\kappa_0+\varepsilon])$ for $\varepsilon > 0$ small enough, one can take $\varepsilon > 0$ such that there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0+\varepsilon])$. By the definition of κ_0 there exists a homotopy $\beta \in \Pi$ satisfying $E(\beta_t) \leq \kappa_0 + \varepsilon$ for all $t \in [0,1]$ such that $\beta_t \in K$. Choose neighborhoods V_j of $S^1 \cdot d_j$ such that $\overline{V_j} \subset \operatorname{int}(U_j)$ and use property ii) on the neighborhood $V := \bigcup_{k=1}^{l} V_j$ of closed geodesics of energy κ_0 in K to obtain a $\tau > 0$ with the property that for the homotopy $\phi_\tau \circ \beta$ we have that $(\phi_\tau \circ \beta)_t \in K$ implies $E((\phi_\tau \circ \beta)_t) < \kappa_0$ or $(\phi_\tau \circ \beta)_t \in V$. Now $(\phi_\tau \circ \beta)^{-1}(V) = \bigcup_{r=1}^{m}(t_r, t_r')$ and by our choice of the V_j we have $(\phi_\tau \circ \beta)([t_r, t_r']) \subset U_j$ and for the endpoints $(\phi_\tau \circ \beta)_{t_r}, (\phi_\tau \circ \beta)_{t_r'} \in U_j \cap E^{-1}([0,\kappa_0))$ for some $j \in \{1,\dots,l\}$ (which is why we applied ii) only to V and not to $\bigcup_{j=1}^{l} U_j$ directly). Now, by using i) if d_j a power of some γ_i (otherwise it is true by the remark just before the beginning of the proof), we know that $U_j \cap E^{-1}([0,\kappa_0))$ is connected and consequently we can replace $(\phi_\tau \circ \beta)|_{[t_r, t_r']}$ by a path in $E^{-1}([0,\kappa_0))$ with the same endpoints. After m steps we obtain a homotopy $\hat\beta : [0,1] \to \Omega$ such that $E(\hat\beta_t) < \kappa_0$ when $\hat\beta_t \in K$. Since $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin K$ it follows that $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin \bigcup_{j=1}^{l} U_j$ and therefore $\hat\beta_0 \in P_n^-$, $\hat\beta_1 \in P_n^+$, hence $\hat\beta \in \Pi$. This contradicts the minimality of κ_0. $\square$

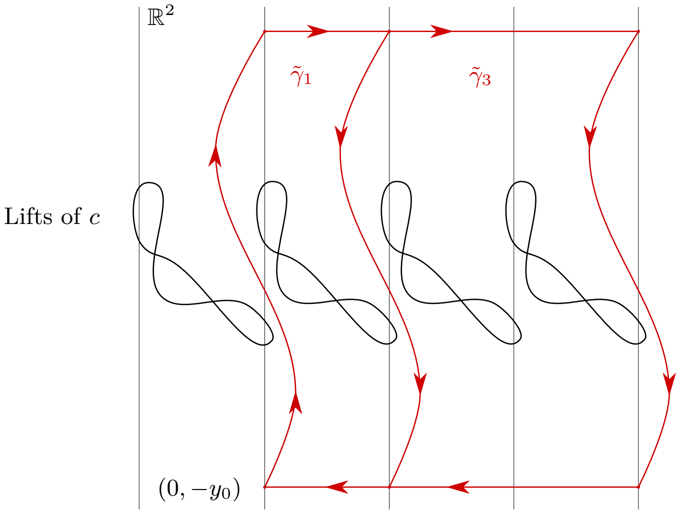

Figure 5. The family of loops $(\tilde{\gamma}_n)$

4. Contractible and intersecting closed geodesics

Here M still denotes a complete Riemannian cylinder. We assume that there exists a contractible closed geodesic $c \in \Lambda M$. Let us consider the unbounded components of $M \setminus c(S^1)$. Since $c(S^1)$ is bounded, there are at most two distinct unbounded components. If there are two distinct unbounded components C_- and C_+, one can assume that C_- is a neighborhood of $-\infty$ and C_+ is a neighborhood of $+\infty$. By $C_\pm$ we will mean any of these two neighborhoods. Then $\partial C_\pm$ is a broken geodesic with angles strictly less than π inside $C_\pm$ since c is a closed geodesic (see Figure 6 for an instance of ∂C_+). Hence $C_\pm$ is locally convex. Moreover if the boundary were totally geodesic, then $\partial C_\pm$ would be parametrised by c which is impossible for c is contractible. We can thus apply Theorem 6.7 in this case.

We now assume that $M \setminus c(S^1)$ has only one unbounded component C. Let us identify M with $S^1 \times \mathbb{R}$ in the remaining of this proof in order to fix notation. Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be the universal cover of $S^1 \times \mathbb{R}$. By compactness of $c(S^1)$, there exists $A > 0$ such that $c(S^1) \subset S^1 \times (-A, A)$. Let $y_0 > A$, since $S^1 \times (-\infty, -A)$ and $S^1 \times (A, +\infty)$ belong to the same component of $M \setminus c(S^1)$, there exists a smooth path $\alpha : [0,1] \to M \setminus c(S^1)$ such that $\alpha(0) = (0, -y_0)$ and $\alpha(1) = (0, y_0)$. Let β_0 be the smooth lift of α in $\mathbb{R}^2$ such that $\beta_0(0) = (0, -y_0)$ and $\beta_0(1) = (n_0, y_0)$ for some $n_0 \in \mathbb{Z}$ that we can take equal to $n_0 = 0$ by chaining α with $t \mapsto (tn_0 \bmod 1, y_0)$. Let $\delta_{n,\pm} : [0,1] \to \mathbb{R}^2$ be the path $t \mapsto (nt, \pm y_0)$ and $\beta_n : [0,1] \to \mathbb{R}^2$ be the family of lifts $\beta_n := (n, 0) + \beta_0$, $n \in \mathbb{N}$. We define the family of loops $\tilde{\gamma}_n \in \Lambda\mathbb{R}^2$ by

$$\tilde{\gamma}_n := \beta_0 \cdot \delta_{n,+} \cdot \beta_n^{-1} \cdot \delta_{n,-}^{-1}.$$

They project to $\gamma_n := \pi \circ \tilde{\gamma}_n$ in $M \setminus c(S^1)$. Let $q_0 \in \mathbb{R}^2$ be a lift of some point of $c(S^1)$ and define $q_n := q_0 + (n, 0)$. Then the first homology group $H_1(\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}})$ is the free abelian group with generators $(g_n)_{n\in\mathbb{Z}}$, and by construction the class of $\tilde{\gamma}_n$ is $g_1 + g_2 + \cdots + g_n$. The covering transformations of $\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}} \to S^1 \times \mathbb{R} \setminus \pi(q_0)$, which form a group isomorphic to $\mathbb{Z}$, acts on the first homology group by $k \cdot g_i = g_{i+k}$. Therefore, for natural integers $n \neq m$ and integers $k, l \in \mathbb{Z}^*$, the fact that

$$k(g_{1+a} + g_{2+a} + \cdots + g_{n+a}) \neq l(g_{1+b} = g_{2+b} + \cdots + g_{m+b}), \quad \forall a, b \in \mathbb{Z},$$

implies that the iterated loops γ_n^k and γ_m^l are not freely homotopic in $M \setminus \pi(q_0)$ and hence in the unbounded component C of $M \setminus c(S^1)$. For $\gamma \in \Lambda C$, let us denote by $[\gamma]$ the free homotopy class of γ. For $m \geq 2$, let us consider the infimum

$$e_m := \inf_{\substack{\gamma \in \Lambda C \\ [\gamma] = [\gamma_m]}} E(\gamma).$$

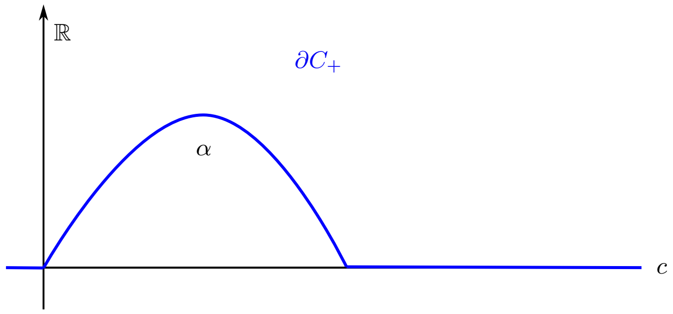

FIGURE 6. Construction of the locally convex neighborhood C_+ (α could also have self-intersections)

Let $K \subset M$ be a compact set that contains $c(S^1)$ such that $M \setminus K$ has two distinct unbounded component. Since any $\gamma \in \Lambda C$ that is freely homotopic to γ_m must intersect K, one can restrict the domain of the infimum to those γ which image is inside the compact set $L \subset M$ of points that are at distance at most $\sqrt{e_m}$ of K (which is compact by completeness of the metric on M). Indeed, if γ were a loop of length $\geq 2\sqrt{e_m}$ then $E(\gamma) \geq 4e_m$ by Cauchy-Schwarz inequality. By compactness of L, we can use a finite-dimensional approximation to get a closed geodesic c_m on $C \cup c(S^1)$ with $E(c_m) = e_m$ that is a limit of broken geodesics on C freely homotopic to γ_m. By uniqueness of the Cauchy problem, if c_m intersect $c(S^1)$, the closed geodesic must be a power of c (up to a translation of the parametrisation). This is impossible since the powers of c are not in the closure of $\{\gamma \in \Lambda C \mid [\gamma] = [\gamma_m]\}$ for $m \geq 2$ (such γ's must intersect every line joining both ends $\pm\infty$ of M). Therefore, the above infimum is reached by the closed geodesic $c_m \in \Lambda C$. We thus get a family of closed geodesics (c_m) such that $[c_m^k] \neq [c_n^l]$ for all $k, l \in \mathbb{Z}^*$ and $m \neq n$. Therefore the closed geodesics (c_m) are geometrically distinct. They all intersect the compact set K. As a local minimum, every c_m is homologically visible.

Now that Theorem 6.1 is proved under hypothesis 1, in order to prove it when there is one self-intersecting closed geodesic c or two intersecting ones c_1 and c_2, one can assume that these geodesics are not contractible. Therefore, in both respective cases, $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ has exactly two unbounded connected components C_- and C_+, which are locally convex by construction. The intersection hypothesis then implies that none of the boundaries $\partial C_\pm$ is totally geodesic. Hence the conclusion follows by applying Theorem 6.7.

5. Closed geodesic of non-zero average index

We assume that there exists a closed geodesic $c \in \Lambda M$ of average index $\overline{\mathrm{ind}}(c) > 0$. If c is contractible or self-intersecting, we already know that there are infinitely many homologically visible closed geodesics. Let us assume that c is an embedded curve generating $\pi_1(M) \simeq \mathbb{Z}$. By a slight abuse of notation, we identify the loop $c : S^1 \to M$ with its lift $\mathbb{R} \to M$.

LEMMA 6.9. *There exist a non-zero Jacobi field $J : \mathbb{R} \to c^*TM$ of c and $\delta > 0$ such that $J(s) \neq 0$ for all $s \in (0, \delta)$ and $J(0) = J(\delta) = 0$.*

PROOF. Since $\overline{\mathrm{ind}}(c) > 0$, Bott's iteration inequality (6.2) and the concavity bound (6.3) imply that there exists $k \in \mathbb{N}^*$ such that

$$\mathrm{ind}_\Omega(c^k) \geq 1.$$

Let us fix such a $k \geq 1$. The conclusion is now a direct application of the Morse index theorem (6.4) to the geodesic path c^k. □

In order to fix notation, let us identify the image of the loop c to $S^1 \times \{0\}$, with $c(s) = (s, 0)$ for $s \in S^1$, so that $M \setminus c(S^1)$ is the disjoint union of the neighborhood $S^1 \times (-\infty, 0)$ of $-\infty$ and the neighborhood $S^1 \times (0, +\infty)$ of $+\infty$ (we only need this identification to be a homeomorphism). Let $J : \mathbb{R} \to c^*TM$ and $\delta > 0$ be the Jacobi field and the positive number given by Lemma 6.9. Let $\varepsilon > 0$ and $I := (-\varepsilon, \delta + \varepsilon)$. Since there exists a smooth family $(\beta_s)_{s \in (-1,1)}$ of geodesic paths $I \to M$ such that $J|_I = \frac{\partial \beta_s}{\partial s}|_{s=0}$, it implies that there exists a geodesic path $\alpha : [0, 1] \to S^1 \times [0, +\infty)$ intersecting c (transversally) only at its endpoints.

By construction, the unbounded component C_+ of $S^1 \times (0, +\infty) \setminus \alpha([0, 1])$ has a boundary which is a broken geodesic with angles strictly less than π. By symmetry, we get two disjoint neighborhoods of $+\infty$ and $-\infty$ respectively which are locally convex and whose boundaries are not totally geodesic, we can thus apply Theorem 6.7.

6. Two homologically visible closed geodesics

Here M denotes a complete Riemannian cylinder. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We assume that there exists two geometrically distinct and homologically visible closed geodesics. We suppose by contradiction that for any compact set $K \subset M$ only a finite number of geometrically distinct homologically visible closed geodesics intersect K. By the previous cases of Theorem 6.1, every prime closed geodesic of M must be embedded, non-contractible, without intersections with another closed geodesic, and of zero average index. Thus the images of closed geodesics of $M \simeq S^1 \times \mathbb{R}$ with a homologically visible iterate are naturally ordered by their smallest intersection with $* \times \mathbb{R}$ where $*$ denotes any point of S^1. The order is independent of the choice of $* \in S^1$. We will say that two closed geodesics are consecutive if they are so with respect to this order. Since only a finite number of geometrically distinct homologically visible closed geodesic intersect a given compact set, one can talk about the next and the previous one with respect to this order.

LEMMA 6.10. *There exists two closed embedded geodesics c_1 and c_2 of M with degree $[c_1] = [c_2] = 1$ bounding a compact locally convex cylinder $C \simeq S^1 \times [0, 1]$ such that*

(1) c_1 is a local minimum of $E|_{\Lambda C}$,
(2) c_2 is not a local minimum of $E|_{\Lambda C}$,
(3) c_1 and c_2 are the only closed geodesics of M inside C that have homologically visible iterates.

PROOF. We first show that two consecutive closed geodesics among closed geodesics that possess homologically visible iterates cannot be both local minima of $E|_{\Lambda C'}$ if C' is the compact cylinder that they bound. By contradiction, let us assume so and let us call γ_0 and γ_1 these two geodesics. Up to a change of parametrization, one can assume that $[\gamma_0] = [\gamma_1]$ and thus that these two geodesics are homotopic in $\Lambda C'$. Let

$$\Pi := \{h : [0, 1] \to \Lambda C' \text{ continuous} \mid h(0) = \gamma_0 \text{ and } h(1) = \gamma_1\}$$

denote the set of homotopy of loops in C' starting at γ_0 and ending at γ_1. We consider the following min-max:

$$\tau = \inf_{h \in \Pi} \max E \circ h.$$

By compactness of C', $E|_{\Lambda C'}$ satisfies Palais-Smale (alternatively, one can work in the compact finite-dimensional manifold of k-broken-geodesics of energy $\leq c + \varepsilon$ for a large $k \in \mathbb{N}$ and $\varepsilon > 0$). Let $e := \max(E(\gamma_0), E(\gamma_1))$. Since the critical orbits $S^1 \cdot \gamma_0$ and $S^1 \cdot \gamma_1$ are isolated local minima of $E|_{\Lambda C'}$ that satisfies Palais-Smale, $\tau > e$. By local convexity of C', the $(-\nabla E)$-flow preserves $\Lambda C'$. By the minimax principle, τ is thus a critical value of $E|_{\Lambda C'}$ and there exists a homologically visible closed geodesic $\gamma \in \Lambda C'$ of energy τ. Hence γ_0 and γ_1 are not consecutive, a contradiction.

By a similar argument, we show that one out of two consecutive closed geodesics among those that possess homologically visible iterates is a local minimum of $E|_{\Lambda C'}$. Indeed, otherwise one has that

$$\inf_{\substack{\gamma \in \Lambda C' \\ [\gamma]=1}} E(\gamma) < \min(E(\gamma_0), E(\gamma_1)),$$

and this infimum is reached for some closed geodesic in C' by compactness and local convexity of C' (and this is not a point since $E(\gamma) \geq (2r)^2$ for all $\gamma \in \Lambda C'$ of degree $[\gamma] = 1$ where $r > 0$ denotes the injectivity radius of the compact Riemannian manifold with boundary C'). This new closed geodesic is a local minimum of E by definition and thus homologically visible.

The requirements of the lemma are thus fulfilled by taking any two consecutive closed geodesics among those with homologically visible iterates. □

PROOF OF THEOREM 6.1. Let c_1 and c_2 be closed geodesics of M satisfying Lemma 6.10. We will reach a contradiction by finding a homologically visible geodesic which is not c_1 or c_2 and arbitrarily close to C.

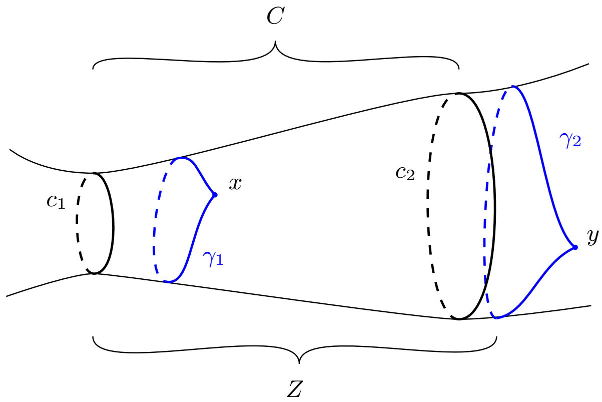

Figure 7. Construction of cylinder Z

Let $x \in \mathrm{Int}(C)$ be outside the image of the isolated set of closed geodesics and let $\gamma_1 \in \Lambda C$ be the loop of degree $[\gamma_1] = 1$ based at x of minimal length. It exists by local convexity and compactness of C. The loop γ_1 is not a periodic geodesic (this is a geodesic as a path $[0, 1] \to C$ but not as a loop $S^1 \to C$) since there is no local minimum of $E|_{\Lambda C}$ but c_1. This loop lies inside $\mathrm{Int}(C)$ so that either the connected component of $C \setminus \gamma_1(S^1)$ containing c_1 or the connected component containing c_2 is locally convex – depending on the angle of γ_1 at $\gamma_1(0) = \gamma_1(1) = x$. If the connected component containing c_2 were locally convex, then the infimum of E among loops of degree one lying inside the locally convex compact cylinder bounded by γ_1 and c_2 would give a closed geodesic loop $\neq c_1$ which would be a local minimum. Thus the connected component of $C \setminus \gamma_1(S^1)$ containing c_1 is a locally convex compact cylinder. Hence the unbounded component of $M \setminus \gamma_1(S^1)$ containing c_1 is a locally convex neighborhood of $-\infty$ which is not totally geodesic since γ_1 is not a closed geodesic.

Let c_3 be the homologically visible closed geodesic succeeding c_2 if it exists. Let C' be either the compact cylinder that c_2 and c_3 bound or the infinite cylinder $\simeq S^1 \times [0, +\infty)$ with boundary c_2 and ending at $+\infty$, depending on the existence of c_3 (so that $C \cap C' = c_2(S^1)$ in both cases). Let $y \in \mathrm{Int}(C')$ be outside the image of any closed geodesic and let $\gamma_2 \in \Lambda C'$ be a loop of degree $[\gamma_2] = 1$ based at y of minimal length. Since C' is complete and locally convex, γ_2 exists. It cannot be a closed geodesic for c_3 succeeds c_2. One of the two unbounded components of $M \setminus \gamma_2(S^1)$ is thus locally convex, depending on the angle of γ_2 at $\gamma_2(0) = \gamma_2(1) = y$. If the neighborhood of $+\infty$ was the locally convex one, by Theorem 6.7 applied to the locally convex neighborhood of $-\infty$ defined above with γ_1 and this neighborhood of $+\infty$, there would be infinitely many homologically visible and geometrically distinct closed geodesics intersecting some compact set of M. Thus the neighborhood of $-\infty$ is the locally convex unbounded component of $M \setminus \gamma_2(S^1)$. Restricting this neighborhood to the compact cylinder $C \cup C'$, one gets a compact locally convex cylinder Z intersecting only two geodesics c_1 and c_2 that possess homologically visible iterates, moreover $c_1(S^1) \subset \partial Z$ and $c_2(S^1) \subset \mathrm{Int}(Z)$.

Let $k \in \mathbb{N}^*$ be such that $\mathrm{C}_*(S^1 \cdot c_2^k) \neq 0$. Let $\Lambda_h \subset \Lambda Z$ be the connected component of loops $\gamma \in \Lambda Z$ of degree $[\gamma] = h$. For all $m \in \mathbb{N}^*$, let $\psi_m : \Lambda_k \to \Lambda_{km}$ be the iteration map $\psi_m(\gamma) := \gamma^m$. According to the Bangert-Klingenberg theorem (6.7), there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$ there exists $e_m > m^2 E(c_2^k)$ such that the composition

$$\mathrm{C}_*(S^1 \cdot c_2^k) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\{E|_{\Lambda_{km}} < e_m\}, \{E|_{\Lambda_{km}} < m^2 E(c_2^k)\}\right)$$

is zero. According to the Gromoll-Meyer theorem (6.6), since $\overline{\mathrm{ind}}(c_2^k) = k\,\overline{\mathrm{ind}}(c_2) = 0$, there exist infinitely many m such that

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c_2^k) \to \mathrm{C}_*(S^1 \cdot c_2^{km})$$

is an isomorphism. Let $m \geq m_0$ be such an integer, then the inclusion induces a zero map

$$\mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\{E|_{\Lambda_{km}} < e_m\}, \{E|_{\Lambda_{km}} < m^2 E(c_2^k)\}\right),$$

which contradicts the fact that c_2^{km} is the homologically visible critical point of $E|_{\Lambda_{km}}$ of maximal value. Critical points of $E|_{\Lambda_{km}}$ are closed geodesics of Z of degree km. Thus $S^1 \cdot c_1^{km}$ and $S^1 \cdot c_2^{km}$ are the only homologically visible critical circle of $E|_{\Lambda_{km}}$ (and $E(c_2^{km}) > E(c_1^{km})$ since c_1 is the only local minimum in C). Since Z is locally convex, the $(-\nabla E)$-flow preserves ΛZ. Moreover Z is compact and has only isolated closed geodesics, we can thus apply Morse theoretical arguments since $E|_{\Lambda_{km}}$ has isolated critical circles and satisfies Palais-Smale or, alternatively, one can restrict E to the finite-dimensional subspace of j-broken-geodesics of Λ_{km} of energy less than $e_m + \varepsilon$ for some large $j \in \mathbb{N}$ and some $\varepsilon > 0$. Thus, if $S^1 \cdot c_2^{km}$ were the only homologically visible critical circle of energy $\geq m^2 E(c_2^k)$, Morse deformation lemma would imply inc_* to be an isomorphism. $\square$

7. The case of the Möbius band

Assuming Theorem 6.1 concerning complete cylinders, we deduce Corollary 6.3.

Let (M, g) be a complete Möbius band and let us denote by $\pi : \widetilde{M} \to M$ its connected double cover. Hence, $(\widetilde{M}, \tilde{g})$ with $\tilde{g} := \pi^* g$ is a complete cylinder. Let us denote by $E : \Lambda M \to \mathbb{R}$ and $\widetilde{E} : \Lambda\widetilde{M} \to \mathbb{R}$ the respective energy functionals of M and $\widetilde{M}$. Any closed geodesic of M is covered by one or two closed geodesics of $\widetilde{M}$. The proof would be obvious if the homological visibility of one of the iterates of the geodesic on M were equivalent to the homological visibility of one of the iterates of the covering geodesics. However, it is not clear whether it is the case or not when only one closed geodesic covers the closed geodesic on M. We will see that Smith inequality (5.4) will give us the equivalence over the field $R = \mathbb{F}_2$.

In the statement of both next lemmas, we use the above notation M, $\widetilde{M}$, E, $\widetilde{E}$ where $\pi : \widetilde{M} \to M$ denote any Riemannian cover of some Riemannian manifold M.

Lemma 6.11. *Let $\tilde{c} \in \Lambda\widetilde{M}$ be a closed geodesic and let $c := \pi \circ \tilde{c}$. Then the map $\pi_\sharp : \Lambda\widetilde{M} \to \Lambda M$, $\tilde{\gamma} \mapsto \pi \circ \tilde{\gamma}$, induces an isomorphism*

$$\mathrm{C}_*(S^1 \cdot \tilde{c}) \xrightarrow{\simeq} \mathrm{C}_*(S^1 \cdot c).$$

Moreover $\mathrm{ind}(\tilde{c}) = \mathrm{ind}(c)$.

Proof. Since π is a covering map, the map $\pi_\sharp$ is a diffeomorphism in a small neighborhood $\widetilde{U}$ of $S^1 \cdot \tilde{c}$ by the uniqueness of the lift to $\widetilde{U}$ of a loop belonging to the neighborhood $U := \pi_\sharp(\widetilde{U})$ of $S^1 \cdot c$. Since $\widetilde{E} = E \circ \pi_\sharp$, the Morse indices $\mathrm{ind}(c)$ and $\mathrm{ind}(\tilde{c})$ are equal. The conclusion now follows from the local property (6.5) of the local homologies of $S^1 \cdot \tilde{c}$ and $S^1 \cdot c$. $\square$

Let us recall the Smith inequality which was already used in Part 1. Given a group G acting on a space X, let $X^G \subset X$ be the set of fixed points of G. According to the Smith inequality,

$$\dim H_*\big(X; \mathbb{F}_p\big) \geq \dim H_*\big(X^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{F}_p\big), \tag{6.8}$$

where X is locally compact space or pair such that $H_*(X; \mathbb{F}_p)$ is finitely generated, a space on which acts the group $\mathbb{Z}/p\mathbb{Z}$ with p prime. Here $\dim H_*$ means the total dimension $\sum_k \dim H_k$. The following lemma is a counterpart of a result of Çineli-Ginzburg relating the local homologies of a Hamiltonian orbit and its p-iterate [**29**].

Lemma 6.12. *For all isolated closed geodesic $c \in \Lambda M$ and all prime number p,*

$$\dim \mathrm{C}_*(S^1 \cdot c^p; \mathbb{F}_p) \geq \dim \mathrm{C}_*(S^1 \cdot c; \mathbb{F}_p).$$

Let us notice that this last inequality is an equality when p is large enough according to Gromoll-Meyer theory [**47**, Theorem 3] (the coefficients field $\mathbb{F}_p$ can be replaced by any ring R in this case).

Proof. Since the local homology of $S^1 \cdot c$ only depends on a small neighborhood of $S^1 \cdot c$ (local property (6.5)), one can assume that M is a closed manifold. Let $X \subset \Lambda M$ be the topological pair

$$X := \left(\{E < E(c^p)\} \cup S^1 \cdot c^p, \{E < E(c^p)\}\right).$$

This pair retracts on a locally compact pair by using a finite-dimensional approximation. According to the Gromoll-Meyer theory, the homology group $H_*(X; \mathbb{F}_p) = \mathrm{C}_*(S^1 \cdot c)$ is finitely generated (see

Section 2). By seeing $\mathbb{Z}/p\mathbb{Z}$ as the subgroup of p-th roots of unity, $\mathbb{Z}/p\mathbb{Z} \subset S^1$ acts on ΛM. This action preserves the sublevel sets of E so it preserves X and $\gamma \mapsto \gamma^p$ induces a homeomorphism

$$\big(\{E < E(c)\} \cup S^1 \cdot c, \{E < E(c)\}\big) \xrightarrow{\simeq} X^{\mathbb{Z}/p\mathbb{Z}}.$$

This is now a direct consequence of the Smith inequality (6.8). □

Proof of Corollary 6.3. Let $\pi : \widetilde{M} \to M$ be the connected double cover of the complete Möbius band M. Let us identify $H_1(M;\mathbb{Z})$ and $H_1(\widetilde{M};\mathbb{Z})$ with $\mathbb{Z}$, so that the induced morphism $\pi_* : H_1(\widetilde{M};\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is the multiplication by 2. Given a closed geodesic $\gamma \in \Lambda M$, we denote by $[\gamma] \in \mathbb{Z}$ its homology class. By the lifting property of covers, there exists $\tilde{\gamma} \in \Lambda\widetilde{M}$ such that $\gamma = \pi \circ \tilde{\gamma}$ if and only if $[\gamma]$ is even (we recall that $\pi_1(M) \simeq H_1(M;\mathbb{Z})$ for M and for $\widetilde{M}$ as well).

If hypothesis 1, 2 or 3 is satisfied on M, then it is also satisfied on $\widetilde{M}$ by considering covering closed geodesics. If hypothesis 4 is satisfied on M, let $c \in \Lambda M$ be a closed geodesic with $\overline{\mathrm{ind}}(c) > 0$. Now $[c^2] = 2[c]$ is even so there exists a closed geodesic $\tilde{\gamma} \in \Lambda\widetilde{M}$ such that $c^2 = \pi \circ \tilde{\gamma}$. According to Lemma 6.11, $\overline{\mathrm{ind}}(\tilde{\gamma}) = \overline{\mathrm{ind}}(c^2) = 2\,\overline{\mathrm{ind}}(c) > 0$ so hypothesis 4 is also satisfied on $\widetilde{M}$. Finally if hypothesis 5 is satisfied on M and $c_1, c_2 \in \Lambda M$ denote the two closed geodesics that are homologically visible over $\mathbb{F}_2$, c_1^2 and c_2^2 are also homologically visible over $\mathbb{F}_2$ by Lemma 6.12 and one can apply Lemma 6.11 as before to get that hypothesis 5 is satisfied on $\widetilde{M}$ over $\mathbb{F}_2$.

According to Theorem 6.1, in any of the above cases $\widetilde{M}$ contains infinitely many closed geodesics intersecting some common compact set $\widetilde{K}$ that are homologically visible over $\mathbb{F}_2$. By Lemma 6.11, the projection of these closed geodesics gives infinitely many closed geodesics intersecting the compact set $\pi(\widetilde{K})$ that are homologically visible over $\mathbb{F}_2$. □

8. The case of the plane

Let $M \simeq \mathbb{R}^2$ be a complete Riemannian plane with isolated closed geodesics. Using what we have seen in the previous sections, we now give the proof of Theorem 6.5.

Proof of Theorem 6.5. When hypothesis 1, 2 or 3 is assumed, the conclusion follows from the same argument as in the case of the cylinder: by construction of an open neighborhood $C \neq M$ of infinity. More precisely, this neighborhood C is the unbounded component of $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ if c is self-intersecting or c_1 and c_2 are intersecting closed geodesics. In the case when there exists a closed geodesic c of non-zero average index, C is constructed by "integrating a Jacobi field" along c as was done in Section 5.

Now, let us assume that all the closed geodesics of M are without self-intersection, with zero average index and do not intersect any other closed geodesic. Moreover, let us assume that only finitely many (geometrically distinct) closed geodesics intersect any given compact set among homologically visible closed geodesics. Let us show by contradiction that it cannot occur whenever M possesses at least one homologically visible closed geodesic. Let c be a simple closed geodesic that has a homologically visible iterate and such that there is not any homologically visible closed geodesic inside the disk D bounded by c. Let $G = \bigcup_\gamma \gamma(S^1) \subset M$ be the union of the images of the closed geodesics γ of M. Let U be the connected component of $M \setminus (D \cup G)$ that contains $c(S^1)$ in its boundary. Since U contains loops that are not contractible in $\mathbb{R}^2 \setminus D$ (by taking loops close to the boundary $c(S^1)$), U is not simply connected. Let $y \in U$ and let $\gamma \in \Lambda\overline{U}$ be a loop minimizing the length among the loop of $\overline{U}$ based at y that are freely homotopic to c (it exists since $\overline{U}$ is complete). Since ∂U is a disjoint union of closed geodesics, γ lies in the interior of U and is a geodesic path. Depending on the angle that γ makes at y, either the unbounded component of $M \setminus \gamma(S^1)$ is locally convex and not totally geodesic or the bounded component containing c is locally convex. In the first case, one can apply Bangert's theorem to get a contradiction.

Let us now apply argument similar to [**8**, Theorem 3] in order to conclude. We can thus assume that c lies in the interior of a compact and locally convex subset $K \subset M$ and that some powers of c are the only homologically visible closed geodesics of K. Since $\overline{\mathrm{ind}}(c) = 0$, the local homology groups $\mathrm{C}_d(S^1 \cdot c^m)$ are trivial in degrees $d \geq 4$ for all $m \in \mathbb{N}$. Let $d \in \{0, 1, 2, 3\}$ be the maximal degree such that $\mathrm{C}_d(S^1 \cdot c^m) \neq 0$ for some $m \in \mathbb{N}^*$. Let $k \in \mathbb{N}^*$ be such that $\mathrm{C}_d(S^1 \cdot c^k) \neq 0$. According to Gromoll-Meyer theory, there exist infinitely many $m \in \mathbb{N}^*$ such that the map induced by the iteration map

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c^k) \to \mathrm{C}_*(S^1 \cdot c^{km})$$

is an isomorphism. As above, according to the Bangert-Klingenberg theorem (6.7), there exists $m_0 \in \mathbb{N}^*$ such that, for all such $m \in \mathbb{N}^*$ greater than m_0, the inclusion of sublevel sets of $E|_{\Lambda K}$ induces the zero map

$$\mathrm{C}_*(S^1 \cdot c^{km}) \xrightarrow{\mathrm{inc}_*} H_* \left(\left\{ E|_{\Lambda K} < e_m \right\}, \left\{ E|_{\Lambda K} < m^2 E(c^k) \right\} \right),$$

for some $e_m > m^2 E(c^k)$. Thus, for such an m, the long exact sequence of the triple

$$\left(\left\{ E|_{\Lambda K} < e_m \right\}, \left\{ E|_{\Lambda K} < m^2 E(c^k) \right\} \cup S^1 \cdot c^{km}, \left\{ E|_{\Lambda K} < m^2 E(c^k) \right\} \right)$$

implies that

$$H_{d+1} \left(\left\{ E|_{\Lambda K} < e_m \right\}, \left\{ E|_{\Lambda K} < m^2 E(c^k) \right\} \cup S^1 \cdot c^{km} \right) \neq 0.$$

Therefore, by the Morse deformation lemma applied to the C^2 function $E|_{\Lambda K}$ which satisfies the Palais-Smale condition and whose anti-gradient flow preserves ΛK (by compactness and local convexity of K), there must be a closed geodesic $\gamma \in \Lambda K$ such that $\mathrm{C}_{d+1}(S^1 \cdot \gamma) \neq 0$ (see for instance [**25**, Theorems 4.2 and 4.3 p. 35-36] where one can replace isolated critical points by isolated critical S^1-orbits *verbatim*). By maximality of d, γ and c are geometrically distinct. But c is the only homologically visible closed geodesic of K, a contradiction. □

CHAPTER 7

The growth rate of geodesic chords

1. Introduction

Let M be a forward complete Finsler manifold of infinite fundamental group (every manifold M will be assumed to be connected). In this chapter, we are interested in the growth rate of geodesics joining two arbitrarily given points $p, q \in M$, and especially in asymptotic properties that only involve the topology of M. For $\ell > 0$, we denote by $n(\ell; p, q)$ the number of geometrically distinct geodesics between p and q of length $\leq \ell$. It is well known that for $\pi_1(M)$ "large enough" this number tends to infinity without any further assumption. A precise statement is the following:

Proposition 7.1. *Let M be a manifold such that $\pi_1(M)$ has a polynomial growth of degree $d > 1$. For each forward complete Finsler metric on M, there exist continuous functions $a : M \to (0, +\infty)$ and $b : M \to \mathbb{R}$ such that*

$$n(\ell; p, q) \geq a(q)\ell^{d-1} + b(q), \quad \forall p, q \in M.$$

For the reader's convenience, we add the proof of this result, which is certainly well known to the experts. We are interested in the remaining case in which the growth rate of $\pi_1(M)$ is linear. In fact, we show the following general result:

Theorem 7.2. *Let M be a manifold of infinite fundamental group $\pi_1(M)$ and not homotopy-equivalent to S^1. Then, given any forward complete Finsler metric on M,*

$$n(\ell; p, q) \to +\infty \quad as \quad \ell \to +\infty, \quad \forall p, q \in M.$$

Of course, the assertion of Theorem 7.2 does not hold for the flat cylinders $S^1 \times \mathbb{R}^n$, which are homotopy-equivalent to S^1. We can be more specific when $H_1(M; \mathbb{Z})$ has non-zero rank:

Theorem 7.3. *Let M be a closed manifold not homotopy-equivalent to S^1 (that is any closed M of dimension ≥ 2) and with first Betti number $\beta_1(M; \mathbb{Z}) \geq 1$. Then, given any Finsler metric on M, there exist $a > 0$ and $b \in \mathbb{R}$ such that*

$$n(\ell; p, q) \geq a \log \ell + b, \quad \forall \ell > 0, \forall p, q \in M.$$

Theorem 7.4. *Let M be a manifold not homotopy-equivalent to S^1 and with first Betti number $\beta_1(M; \mathbb{Z}) \geq 1$. Then, given any forward complete Finsler metric on M, there exists a continuous function $b : M \to \mathbb{R}$ such that*

$$n(\ell; p, q) \geq \frac{\log(\log \ell)}{2 \log 2} + b(q), \quad \forall \ell > 0, \forall p, q \in M.$$

When the universal cover of M is not contractible (that is M is not an Eilenberg-MacLane space), Theorems 7.2, 7.3 and 7.4 are deduced from a min-max argument inspired by Bangert-Hingston [**11**]. When M has a contractible universal cover, the estimate is even stronger, since the growth is at least linear:

Lemma 7.5. *Let M be a manifold not homotopy-equivalent to S^1 and with a contractible universal cover. Then, $\pi_1(M)$ has at least a quadratic growth rate.*

We notice that any closed manifold of dimension ≥ 2 with a contractible universal cover satisfies the above condition.

In Section 2, we fix the notation and the conventions on the objects that we will use throughout the chapter, and we briefly recall the variational theory of geodesics for a Finsler manifold. In Section 3, we give a proof of Proposition 7.1 and Lemma 7.5. In Section 4, we prove Theorems 7.2, 7.3 and 7.4.

2. Preliminaries

2.1. Definitions and conventions on path spaces. Let M be a connected manifold (every manifold will be assumed to be connected). We fix, once for all, an auxiliary complete Riemannian metric g_0 on M. By H^1-path, we mean an absolutely continuous function $\gamma : [0,1] \to M$ such that the integral $\int_0^1 g_0(\gamma', \gamma')\mathrm{d}t$ is finite. For $p, q \in M$ let $\Omega_{p,q}$ be the set of H^1-paths $\gamma : [0,1] \to M$ with end-points $\gamma(0) = p$ and $\gamma(1) = q$ and $\Omega_p := \Omega_{p,p}$. For $\gamma, \delta \in \Omega_{p,q}$, we write $\gamma \approx \delta$ if γ and δ belong to the same path-connected component of $\Omega_{p,q}$. For $\gamma \in \Omega_{p,q}$ and $\delta \in \Omega_{q,r}$, we denote by $\gamma \cdot \delta \in \Omega_{p,r}$ the chained path $t \mapsto \gamma(2t)$ for $t \in [0, 1/2]$ and $t \mapsto \delta(2t-1)$ for $t \in [1/2, 1]$. We denote $a \cdot b \cdot c = (a \cdot b) \cdot c$ so that $a \cdot b \cdot c \approx a \cdot (b \cdot c)$. For $\gamma \in \Omega_{p,q}$, let $\gamma^{-1} \in \Omega_{q,p}$ be the reversed path $t \mapsto \gamma(1-t)$, so that $\gamma \cdot \gamma^{-1} \approx \bar{p}$ where $\bar{p} \in \Omega_p$ denotes the constant path. If $\gamma \in \Omega_q$ for some $q \in M$, $[\gamma]_{\pi_1} \in \pi_1(M, q)$ denotes its class in the fundamental group, or simply $[\gamma]$ if there is no ambiguity on notation.

Let $p, q, p', q' \in M$, $\alpha \in \Omega_{p,p'}$ and $\beta \in \Omega_{q,q'}$, then $f : \Omega_{p,q} \to \Omega_{p',q'}$ and $g : \Omega_{p',q'} \to \Omega_{p,q}$ defined by $f(\gamma) = \alpha^{-1} \cdot \gamma \cdot \beta$ and $g(\gamma) = \alpha \cdot \gamma \cdot \beta^{-1}$ are homotopy inverses, thus $\Omega_{p,q}$ and $\Omega_{p',q'}$ are homotopy-equivalent spaces. For all $h \in \pi_1(M, q)$ let Ω_q^h be the path-connected component such that for all $\gamma \in \Omega_q^h$, $[\gamma] = h$. We fix an arbitrary $\alpha \in \Omega_{p,q}$ once for all and we define $\Omega_{p,q}^h := \{\gamma \in \Omega_{p,q} \mid [\alpha^{-1} \cdot \gamma] = h\}$.

2.2. Background on Finsler geodesics. Let us recall some basic notion from Finsler geometry. For a general reference, see [**12**].

Let M be a manifold, TM be its tangent bundle and $\pi : TM \to M$ be the base projection. A continuous function $F : TM \to [0, +\infty)$ is a Finsler metric if

- it is smooth on $TM \setminus 0$, where $0 \subset TM$ denotes the 0-section,
- it is fiberwise positively homogeneous of degree 1, *i.e.* $F(\lambda v) = \lambda F(v)$ for $v \in TM$ and $\lambda > 0$,
- its square F^2 is fiberwise strongly convex, that is the fundamental tensor
$$g_u(v, w) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(u + tv + sw)\bigg|_{t=s=0}, \quad \forall v, w \in T_{\pi(u)}M,$$
is positive definite for every $u \in TM \setminus 0$.

A Finsler metric F on M induces a length on $\Omega_{p,q}$, given by
$$\mathrm{length}(\gamma) := \int_0^1 F(\gamma'(t))\mathrm{d}t, \quad \forall \gamma \in \Omega_{p,q}.$$
and a (not necessarily symmetric) distance d on M, given by:
$$d(p, q) := \inf_{\gamma \in \Omega_{p,q}} \mathrm{length}(\gamma), \quad \forall p, q \in M.$$
Since $F(-v) = F(v)$ does not necessarily hold, d is not necessarily symmetric. A sequence (x_i) in M is called a forward Cauchy sequence if, for all $\varepsilon > 0$, there exists a positive integer N such that $N \leq i < j$ implies $d(x_i, x_j) < \varepsilon$. The Finsler manifold (M, F) is said to be *forward complete* if every forward Cauchy sequence converges in M.

Similarly to the Riemannian case, where F is simply the associated Riemannian norm, *geodesics* are the curves whose small portions are length minimizing. Moreover, they satisfy a differential equation inducing an exponential map between a neighborhood of $p \in M$ and a neighborhood of T_pM. If F is forward complete, the Hopf-Rinow Theorem from Riemannian geometry remains true in the Finsler setting: the exponential map is onto and, for all $p, q \in M$, there exists a geodesic in $\Omega_{p,q}$ minimizing the length.

Throughout this part, all the geodesics γ will be considered parametrized with constant speed equal to $F(\gamma')$ and we will often identify geodesics and reparametrized geodesics when writing "$\delta \cdot \gamma$ is a geodesic". For $p, q \in M$, geodesics in $\Omega_{p,q}$ are then exactly the critical points of the energy functional:
$$E(\gamma) := \int_0^1 F^2(\gamma'(t))\mathrm{d}t, \quad \forall \gamma \in \Omega_{p,q}.$$
If the Finsler metric is forward complete, then $E : \Omega_{p,q} \to [0, +\infty)$ satisfies the Palais-Smale condition (see for example [**24**, Section 3]). Given $\gamma \in \Omega_{p,q}$ critical, we denote by $\mathrm{ind}(\gamma)$ its index,

which is the non-negative integer computed in the same way as in the Riemannian case using Jacobi fields (6.4). The index of a geodesic chord shares properties similar to the Riemannian case. In particular, for two geodesic chords $\gamma \in \Omega_{p,q}$ and $\delta \in \Omega_{q,r}$:

(i) if $\mathrm{ind}(\gamma) = 0$, then for $s \in (0,1)$, $\gamma|_{[0,s]}$ reparametrized by constant speed on $[0,1]$ is a local minimum of $E : \Omega_{p,\gamma(s)} \to \mathbb{R}$,
(ii) If $\gamma \cdot \delta$ is a geodesic, then $\mathrm{ind}(\gamma \cdot \delta) \geq \mathrm{ind}(\gamma) + \mathrm{ind}(\delta)$.

However, E is only of class $C^{1,1}$ in general and it can be very technical to make this functional fit into the Morse *apparatus*. To overcome this issue, we can retract $\{E < \lambda\}$ to a finite dimensional subspace B of broken geodesics joining p and q. We briefly recall the construction of $B \subset \{E < \lambda\}$ and its retraction (r_s) (a comprehensive reference for the Riemannian case is [**61**, Part III, §16]). Let $k \geq 1$ be large so that for all $\gamma \in \{E < \lambda\}$ there exists a unique minimizing geodesic joining $\gamma(s)$ and $\gamma(t)$ for $|s - t| \leq 1/k$. Let

$$B := \left\{\gamma \in \{E < \lambda\} \mid \gamma|_{[i/k,(i+1)/k]} \text{ is a geodesic}, 0 \leq i \leq k-1\right\}$$

be the subspace of k-broken geodesics $\subset \{E < \lambda\}$. It is a finite dimensional manifold since it is diffeomorphic to an open subset of the $(k-1)$-fold product $M \times \cdots \times M$ *via* $\gamma \mapsto (\gamma(1/k), \dots, \gamma((1-k)/k))$. The retraction homotopy $r_s : \{E < \lambda\} \to \{E < \lambda\}$, with $r_0 = \mathrm{id}$ and $\mathrm{im}(r_1) = B$, is defined as follows. For each $\gamma \in \{E < \lambda\}$, $r_s(\gamma)$ coincides with γ everywhere except on intervals of the form $[i/k, (i+s)/k]$, $0 \leq i \leq k-1$, and the restrictions of $r_s(\gamma)$ to such intervals are minimizing geodesics. This retraction has the following properties:

(a) $\forall s \in [0,1]$, $E \circ r_s \leq E$,
(b) if $\gamma \in \{E < \lambda\}$ is a geodesic, $r_s(\gamma) \equiv \gamma$,
(c) critical points of $E|_B$ are exactly the critical points of $E|_{\{E<\lambda\}}$, $E|_B$ is smooth in their neighborhood and their Morse index is equal to their index defined with Jacobi fields.

3. growth rate of geodesic chords and growth rate of $\pi_1(M)$

Throughout this section, M is a forward complete Finsler manifold.

3.1. Growth rate of geodesic chords and growth of the fundamental group. We suppose that $\pi_1(M,q)$ is a finitely generated group and denote by $e \in \pi_1(M,q)$ its neutral element. Let $S \subset \pi_1(M,q)$ be a finite set of generators. We recall that the word length of $g \in \pi_1(M,q)$ associated with S is

$$|g| := \min\left\{m \in \mathbb{N} \mid \exists g_1, \dots, g_m \in S \cup S^{-1} \cup \{e\},\ g = g_1 \cdots g_m\right\} \in \mathbb{N}$$

and we denote the associated ball of radius $r \in \mathbb{N}$ by $B_r := \{g \in \pi_1 \mid |g| \leq r\}$. We will say that $\pi_1(M,q)$ has at least a polynomial growth rate of degree $d > 0$ if there exists some $a > 0$ such that $\#B_r \geq ar^d$. This notion is indeed independent of the choice of S.

For $h \in \pi_1(M,q)$, we fix an arbitrary $\gamma_h \in \Omega^h_{p,q}$ minimizing the length, it gives us a family of homotopically (but not geometrically) distinct geodesics $(\gamma_h)_{h\in\pi_1}$ (where $\pi_1 = \pi_1(M,q)$ by a slight abuse of notation).

Proof of Proposition 7.1. We suppose $\pi_1(M,q)$ has at least a polynomial growth rate of degree $d > 0$. We take a finite generating part $S := \{s_1, \dots, s_n\}$, which is symmetric: $S = S^{-1}$ and contains the neutral element, and define the balls $B_r \subset X_S$ as above. Let $c_1, \dots, c_n \in \Omega_q$ be such that $[c_i] = s_i$ and c_i is minimizing length in its homotopy class. We will first give a lower bound on the counting number $N(\ell; p, q)$ of geodesics between p and q not necessarily geometrically distinct.

Given $r \in \mathbb{N}$, let $g \in B_r$. There exist $i_1, \dots, i_r \in \{1, \dots, n\}$ such that $g = s_{i_1} \cdots s_{i_r}$, so that $[\alpha^{-1} \cdot \gamma_g] = [c_{i_1} \cdots c_{i_r}]$ (recall that $\alpha \in \Omega_{p,q}$). Since γ_g is minimizing length in its homotopy class,

$$\mathrm{length}(\gamma_g) \leq \mathrm{length}(\alpha \cdot c_{i_1} \cdots c_{i_r}) \leq \mathrm{length}(\alpha) + r \max(\mathrm{length}(c_j)).$$

Therefore, since $(\gamma_g)_{g\in B_r}$ is a family of distinct geodesics, there exists $a > 0$ depending only on the growth rate of $\pi_1(M,q)$ such that

$$N(\ell; p, q) \geq a \left(\frac{\ell - \mathrm{length}(\alpha)}{\max(\mathrm{length}(c_i))}\right)^d, \quad \forall \ell > 0.$$

We remark that there exists some positive number $b(p) > 0$ depending only on the Finsler metric on M such that any k-iterate closed geodesic containing p has length $\geq b(p)k$ (one can take

$b(p)$ to be twice the injectivity radius at p). Since a geodesic in $\Omega_{p,q}$ whose image appears multiple times can be uniquely written $d \cdot c^k$ with a primitive closed geodesic $c \in \Omega_q$ and a specific choice of geodesic chord $d \in \Omega_{p,q}$ (look at the definition of a primitive geodesic chord in Section 4.1 for the precise statement),

$$n(\ell; p, q) \geq \frac{b}{2\ell} N(\ell; p, q) \geq a' \ell^{d-1} + b', \quad \forall \ell > 0,$$

where $a' > 0$ and $b' \in \mathbb{R}$ depend only on the metric and the growth rate of $\pi_1(M)$ and can be made continuous in $p \in M$. $\square$

3.2. Growth rate of the fundamental group of $K(\pi_1, 1)$ closed manifolds. Let M be a $K(\pi_1, 1)$ manifold with an infinite fundamental group and a contractible universal cover. According to Smith's theorem (see for instance [**54**, Theorem 16.1, page 287]), $\pi_1(M)$ is torsion-free. We suppose that every finitely generated subgroup of $\pi_1(M)$ grows strictly less than any quadratic polynomial. Then according to a deep result of Gromov [**48**], $\pi_1(M)$ must be virtually isomorphic to $\mathbb{Z}$: that is $\pi_1(M)$ has a subgroup of finite index which is isomorphic to $\mathbb{Z}$. Alternatively, the reader can find an elementary proof of this statement in the finitely generated case in [**82**]. For a precise proof of the case where $\pi_1(M)$ is *a priori* infinitely generated, see [**58**, Theorem 2] (with notation of this theorem, since G is torsion-free, L is trivial and G is virtually $N \simeq \mathbb{Z}$).

The following algebraic lemma is certainly well known, but we add its proof here for the reader's convenience, as we could not find it in the literature.

LEMMA 7.6. *Let G be a torsion-free group, if G is virtually isomorphic to $\mathbb{Z}$ (*i.e.* there exists a subgroup of finite index $H < G$ which is isomorphic to $\mathbb{Z}$) then $G \simeq \mathbb{Z}$.*

PROOF. Since G is virtually isomorphic to a finitely generated group, G is a finitely generated group. Let $S = \{s_1, \ldots, s_n\}$ be a set of generators of G and let $H < G$ be a subgroup of finite index isomorphic to $\mathbb{Z}$. If $C(s) := \{g \in G \mid gs = sg\}$ denotes the centralizer of $s \in G$, then $C(s) \cap H$ is not trivial. Indeed, $C(s)$ is infinite (it contains $\langle s \rangle$ which is infinite by hypothesis for $s \neq e$ and $C(e) = G$) and there exists a finite sequence (g_i) in G such that $G = \bigcup_i g_i H$, thus there exists some $g = g_i$ such that $C(s) \cap gH$ is infinite. Let $c, c' \in C(s) \cap gH$ be distinct. There are $h, h' \in H$ such that $c = gh$ and $c' = gh'$, then $c^{-1}c' = h^{-1}h' \neq 1$ is in $C(s) \cap H$.

Since a finite intersection of non-trivial subgroups of $H \simeq \mathbb{Z}$ has a finite index, $\bigcap_i C(s_i) \cap H$ has a finite index in H. Thus the centralizer $Z = \bigcap_i C(s_i)$ of G has a finite index in G. According to a theorem of Schur, it implies that the commutator subgroup $D := [G, G]$ is finite [**66**, Theorem 5.32]. Since G is torsion-free, D is trivial and G is abelian. An abelian torsion-free finitely generated group is isomorphic to $\mathbb{Z}^r$ and $\mathbb{Z}$ is the only one virtually isomorphic to $\mathbb{Z}$. $\square$

PROOF OF LEMMA 7.5. Let M be a $K(\pi_1, 1)$ manifold. We assume that $\pi_1(M)$ grows less than a quadratic polynomial, so that it is virtually isomorphic to $\mathbb{Z}$. Since $\pi_1(M)$ is torsion-free, Lemma 7.6 implies that $\pi_1(M) \simeq \mathbb{Z}$. Since M is a $K(\mathbb{Z}, 1)$ manifold, it is homotopy-equivalent to S^1. $\square$

4. Growth rate when the universal cover is not contractible

Throughout this section, M is a forward complete Finsler manifold with an infinite fundamental group and a non-contractible universal cover.

4.1. A sequence of min-max geodesics. Since the universal cover of M is not contractible, there is some $n > 1$ such that $\pi_n(M, q) \neq 0$, we fix such an $n > 1$. For all $h \in \pi_1(M, q)$, we recall that $\gamma_h \in \Omega^h_{p,q}$ denotes a minimizing geodesic. We take a non-zero class $\nu \in \pi_{n-1}(\Omega_q, \bar{q}) \simeq \pi_n(M, q)$ and, for all $h \in \pi_1(M, q)$, let $\nu_h := (\gamma_h)_* \nu$ be the induced non-zero class of $\pi_{n-1}(\Omega^h_{p,q}, \gamma_h)$.

To be more precise on the definition of ν_h, let $x_0 \in \mathbb{S}^{n-1}$ be the base point and d be the round distance on $\mathbb{S}^{n-1}$. Let $f : (\mathbb{S}^{n-1}, x_0) \to (\Omega_q, \bar{q})$ be a smooth function in the class ν such that $f(s) = \bar{q}$ for all $s \in \mathbb{S}^{n-1}$ such that $d(x_0, s) < 1$. For $h \in \pi_1(M, q)$, let $f_h : (\mathbb{S}^{n-1}, x_0) \to (\Omega^h_{p,q}, \gamma_h)$ be essentially defined by $f_h(s) := \gamma_h \cdot f(s)$ for all $s \in \mathbb{S}^{n-1}$. To be more specific, in order to have $f_h(x_0) = \gamma_h$, let $f_h(s) := \gamma_h \cdot f(s)$ for all $s \in \mathbb{S}^{n-1}$ such that $d(x_0, s) \geq 1$ and

$$\forall t \in [0,1], \quad f_h(s)(t) := \begin{cases} \gamma_h(t/\lambda(s)), & \text{if } t \in [0, \lambda(s)], \\ q, & \text{otherwise,} \end{cases} \tag{7.1}$$

for all $s \in \mathbb{S}^{n-1}$ such that $d(x_0, s) < 1$, with $\lambda(s) = 1 - \frac{d(x_0,s)}{2} \in [1/2, 1]$. Then we define ν_h as the homotopy class of f_h.

We suppose that $E : \Omega_{p,q} \to \mathbb{R}$ has a discrete set of critical points (otherwise the conclusions of Theorems 7.2, 7.3 and 7.4 are clearly true) and consider the min-max:

$$\tau_h = \inf_{f \in \nu_h} \max_{s \in \mathbb{S}^{n-1}} E(f(s)). \tag{7.2}$$

Then τ_h is a critical value of E and there exists a critical point $\delta_h \in \Omega_{p,q}$ of value τ_h which is not a local minimum and satisfies $\mathrm{ind}(\delta_h) \leq n-1$. This is a classical result if E is C^2 in the neighborhood of its critical points and satisfies Palais-Smale (see for instance [**25**, Chapter II]). Even though E is not C^2 in any neighborhood of its critical points for a general Finsler metric, we can apply a retraction (r_s) of $\{E < \lambda\}$ for some $\lambda > \tau_h$ to a finite dimensional subspace B of broken geodesics, as explained above. We thus see that τ_h satisfies (7.2) restricted to the finite dimensional subspace B, according to property (a) of (r_s). Now property (c) allows us to find $\delta_h \in \Omega_{p,q}$ among the critical points of $E|_B$ which is C^2 in the neighborhood of its critical points.

The following estimate will be useful in Section 4.2:

LEMMA 7.7. *There exists a constant $C > 0$ such that, for all $h \in \pi_1(M, q)$,*

$$\mathrm{length}(\gamma_h)^2 \leq \tau_h \leq 2\,\mathrm{length}(\gamma_h)^2 + C.$$

PROOF. Given $f_h \in \nu_h$, for all $s \in \mathbb{S}^{n-1}$, $f_h(s) \approx \gamma_h$ and γ_h is minimizing length with a constant velocity, thus $E(\gamma_h) \leq E(f_h(s))$ which gives $\mathrm{length}(\gamma_h)^2 \leq \tau_h$.

Let $x_0 \in \mathbb{S}^{n-1}$ be the base point and d be the round distance on $\mathbb{S}^{n-1}$. Let $f : (\mathbb{S}^{n-1}, x_0) \to (\Omega_q, \bar{q})$ be a smooth function in the class ν such that $f(s) = \bar{q}$ for all $s \in \mathbb{S}^{n-1}$ such that $d(x_0, s) < 1$ and define $f_h \in \nu_h$ by (7.1). Then, for all $s \in \mathbb{S}^{n-1}$,

$$E(f_h(s)) \leq 2E(\gamma_h) + 2E(f(s)),$$

thus $\tau_h \leq 2E(\gamma_h) + C$ with $C := 2 \max E \circ f$ independent of h. ☐

We recall that a geodesic loop $c \in \Omega_q$ is *primitive* if there does not exist any geodesic loop $c_0 \in \Omega_q$ and any positive integer $k > 1$ such that $c = c_0^k$. We will say that a geodesic chord $d \in \Omega_{p,q}$ is *primitive* if there does not exist any geodesic loop $c \in \Omega_q$ such that $\Im(d) = \Im(c)$ or if it is a primitive geodesic loop (which is only possible in the case $p = q$). For all geodesic chord $\beta \in \Omega_{p,q}$, there exists a unique primitive geodesic chord $d \in \Omega_{p,q}$ such that $\beta = d \cdot c^k$, where $c \in \Omega_q$ is either $\bar{q}$ or the primitive geodesic loop containing d and $k \in \mathbb{N}$. We will say that the geodesic chord $\beta \in \Omega_{p,q}$ *carries* the primitive chord $d \in \Omega_{p,q}$. Thus if the family (δ_h) carry m distinct primitive chords $d_1, \ldots, d_m$ and $p \neq q$, then $d_1, \ldots, d_m$ are geometrically distinct geodesic chords joining p and q. In the special case $p = q$, it is possible that $d_r = d_s^{-1}$ for some $r \neq s$ so that at least $\lceil m/2 \rceil$ of them are geometrically distinct.

We now study the number of times chords carrying the same primitive chord $d \in \Omega_{p,q}$ can appear in the infinite family $(\delta_h)_{h \in \pi_1}$. Let $d \in \Omega_{p,q}$ be a primitive geodesic chord. Let $(\delta'_i)_{1 \leq i \leq N} := (\delta_{h_i})$, $N \geq 2$, be a sequence included in (δ_h), $h \in \pi_1$, such that each δ'_i carries d. Since $N \geq 2$, the primitive chord d is included in some primitive geodesic loop $c \in \Omega_q$ and there exists a sequence of non-negative integers $(k_i)_{1 \leq i \leq N}$ such that $\delta'_i = d \cdot c^{k_i}$. Since each δ'_i belongs to a different path-connected component $\Omega_{p,q}^{h_i} \subset \Omega_{p,q}$, the k_i's are distinct.

Now we remark that $\mathrm{ind}(\delta'_i) \geq 1$ for $i = 1$ or $i = 2$: otherwise if one supposes $k_2 > k_1$ then δ'_1 must be a local minimum of E, according to property (i) of Section 2.2. If one supposes $k_2 > k_1$, then $\mathrm{ind}(d \cdot c^{k_2}) \geq 1$ implies that $\mathrm{ind}(d \cdot c^{n(k_2+1)}) \geq n$ (property (ii) of Section 2.2) thus $k_i < n(k_2 + 1)$ for all i.

PROOF OF THEOREM 7.2. The case of a contractible universal cover is a consequence of Lemma 7.5.

In our present setting, the theorem follows from the fact that a same primitive chord joining p and q can only be carried a finite number of time in the infinite family $(\delta_h)_{h \in \pi_1}$. Indeed, let $(\delta'_i)_{1 \leq i \leq N}$ be a sequence inside the family with $N \geq 2$ possibly infinite, each δ'_i carrying the same primitive chord and let (k_i) be the associated injective sequence in $\mathbb{N}$. Since the k_i's are distinct, $N \leq n(\max(k_1, k_2) + 1)$. ☐

4.2. Logarithmic growth when $\beta_1(M;\mathbb{Z}) \geq 1$. Let M be a forward complete Finsler manifold with first Betti number $\beta_1(M;\mathbb{Z}) \geq 1$ and which is not $K(\pi_1, 1)$.

Let $h \in \pi_1(M, q)$ be such that its image under the Hurewicz map $\pi_1(M, q) \to H_1(M;\mathbb{Z})$ is of infinite order (in particular the order of h is also infinite). Here, for $m \in \mathbb{Z}$, $\Omega^m_{p,q} := \{\gamma \in \Omega_{p,q} \mid [\alpha^{-1} \cdot \gamma]_{\pi_1} = h^m\}$, where $\alpha \in \Omega_{p,q}$ is fixed once for all. Let $\gamma_m := \gamma_{h^m}$ be a global minimizer of E on $\Omega^m_{p,q}$ and $\delta_m := \delta_{h^m}$.

LEMMA 7.8. *If M is closed, there are $a, a' > 0$ and $b, b' \in \mathbb{R}$ such that*

$$\forall m \in \mathbb{N}, \quad am + b \leq \operatorname{length}(\delta_m) \leq a'm + b'.$$

The inequality $\operatorname{length}(\delta_m) \leq a'm + b'$ *still holds when M is not closed.*

PROOF. According to Lemma 7.7, it suffices to prove these inequalities for $\operatorname{length}(\gamma_m)$ instead of $\operatorname{length}(\delta_m) = \sqrt{\tau_{h^m}}$.

Let $c := \alpha^{-1} \cdot \gamma_1 \in \Omega_q$, then $[c^m] = [\alpha^{-1} \cdot \gamma_1]^m = h^m = [\alpha^{-1} \cdot \gamma_m]$ so $\alpha \cdot c^m \in \Omega^m_{p,q}$ and, since γ_m is minimizing the length, $\operatorname{length}(\gamma_m) \leq \operatorname{length}(c^m) + \operatorname{length}(\alpha) = m \operatorname{length}(c) + \operatorname{length}(\alpha)$.

For the lower bound, it comes from the fact that $\operatorname{length}(\alpha^{-1} \cdot \gamma_m) \geq m\|[h]\|_s$, where $\|[h]\|_s > 0$ is the stable norm of $[h] \in H_1(M, \mathbb{R})$. To show that directly, one can take a 1-form ω such that $\langle \omega, [h] \rangle \neq 0$ (it exists since $[h] \neq 0$ on $H_1(M, \mathbb{R})$ by hypothesis) and remark that

$$m |\langle \omega, [h] \rangle| = |\langle \omega, [h^m] \rangle| = \left| \int_{\alpha^{-1} \cdot \gamma_m} \omega \right| \leq \left(\sup_{x \in M} \|\omega_x\| \right) \operatorname{length}(\alpha^{-1} \cdot \gamma_m).$$

Since M is closed, $\sup_{x \in M} \|\omega_x\|$ is finite. □

Thanks to the estimate on $\operatorname{length}(\delta_m)$ given by Lemma 7.8, we can be more specific on the number of times a same primitive chord can be carried in (δ_m):

LEMMA 7.9. *If M is closed, there exist some $a, b > 0$ such that if δ_{m_1} and δ_{m_2} carry the same primitive chord d for some $m_1 < m_2$, then for all $m > am_2 + b$ the chord δ_m does not carry d.*

PROOF. Let $(\delta'_i) = (d \cdot c^{k_i})_i$ be the sub-sequence (δ_{m_i}) of (δ_m) carrying the primitive chord d with (m_i) being increasing and let $\kappa := \max(k_1, k_2) \geq 1$. We have seen that $\operatorname{ind}(d \cdot c^\kappa) \geq 1$, thus $\operatorname{ind}(d \cdot c^{n(\kappa+1)}) \geq n$ which implies that the finite sequence (k_i) is bounded by $n(\kappa + 1)$. According to Lemma 7.8, for $i = 1$ and $i = 2$, $\operatorname{length}(\delta'_i) = \operatorname{length}(d) + k_i \operatorname{length}(c) \leq a'm_i + b' \leq a'm_2 + b'$ so $\kappa \leq \frac{a'm_2 + b'}{\operatorname{length}(c)}$.

Then the lower bound of Lemma 7.8 together with $k_i \leq n(\kappa + 1)$ implies that

$$m_i \leq \frac{n}{a} \kappa \operatorname{length}(c) + \frac{n \operatorname{length}(c) + \operatorname{length}(d) - b}{a}.$$

Since κ is non-zero, $\operatorname{length}(c) + \operatorname{length}(d) \leq a'm_2 + b'$ and finally $m_i \leq 2\frac{a'}{a} n m_2 + 2n\frac{b'}{a} - \frac{b}{a}$. □

PROOF OF THEOREM 7.3. Let A_N be the number of distinct primitive chords carried in (δ_m) for $m \in \{0, \ldots, N\}$. According to Lemma 7.9, there exist $a > 0$ and $b \in \mathbb{R}$ such that $A_{aN+b} \geq A_N + 1$. Let $a' > a$, then for sufficiently large N, $a'N > aN + b$ and $A_{a'N} \geq A_N + 1$. Thus, for all $k \geq 1$, $A_{(a')^k N} \geq A_N + k$ and there exists $c_0 > 0$ such that

$$A_m \geq \frac{\log m}{\log a'} - c_0, \quad \forall m \in \mathbb{N}.$$

Paths in (δ_m) for $m \in \{0, \ldots, N\}$ have length $\leq cN + d$ for some $c > 0$ and $d \in \mathbb{R}$ according to Lemma 7.8 (and they are longer than the primitive chords they carry) therefore,

$$2n(\ell; p, q) \geq A_{\lfloor (\ell - d)/c \rfloor} \geq \frac{\log \ell}{\log a'} - c_1, \quad \forall \ell > 0,$$

for some constant $c_1 > 0$ (in the case $p = q$, a primitive chord and its inverse are geometrically identical, hence the factor of 2). □

We go back to the general case where M is not assumed to be closed.

LEMMA 7.10. *There exists a quadratic polynomial $P \in \mathbb{R}[X]$ such that if δ_{m_1} and δ_{m_2} carry the same primitive chord d for some $m_1 < m_2$, then for all $m > P(m_2)$ the chord δ_m does not carry d. Coefficients of P can be made continuous in the base point $q \in M$.*

PROOF. Let $(\delta'_i) = (d \cdot c^{k_i})_i$ be the sub-sequence (δ_{m_i}) of (δ_m) carrying the primitive chord d with (m_i) being increasing and let $\kappa := \max(k_1, k_2)$. Similarly to the proof of Lemma 7.9, the finite sequence (k_i) is bounded by $n(\kappa+1)$ with $\kappa \leq \frac{am_2+b}{\mathrm{length}(c)}$, where $a, b \in \mathbb{R}$ are given by the upper-bound of Lemma 7.8. We fix any linear projection $H_1(M,\mathbb{Z}) \to \langle h \rangle \simeq \mathbb{Z}$, $\beta \mapsto [\beta]$. By definition of m_i, $[\alpha^{-1} \cdot d] + k_i[c] = m_i$. Let $u := [\alpha^{-1} \cdot d] \in \mathbb{Z}$ and $v := [c] \in \mathbb{Z}$. Since $(k_2 - k_1)v = m_2 - m_1$ and $m_1 < m_2$, one has $|v| \leq m_2$ and thus $|u| \leq m_2 + k_2 m_2$. Finally,

$$
\begin{aligned}
m_i &\leq |u| + k_i|v| \leq m_2 + \kappa m_2 + n(\kappa+1)m_2 \\
&\leq m_2 + \frac{am_2+b}{r} + n\left(\frac{am_2+b}{r} + 1\right) m_2 =: P(m_2),
\end{aligned}
$$

where $r := \inf_{\gamma \in \Omega_q,\ [\gamma] \neq 0} \mathrm{length}(\gamma) > 0$ depends continuously in $q \in M$. □

PROOF OF THEOREM 7.4. This is the same proof as for Theorem 7.3 but now, for sufficiently large N, there exists some $a > 1$ such that $A_{aN^2} \geq A_N + 1$ with the same notation. Thus $A_{(aN)^{2^k}} \geq A_N + k$ and there exists $c_0 > 0$ such that

$$
\forall m \in \mathbb{N}, \quad A_m \geq \frac{\log(\log m)}{\log 2} - c_0.
$$

Since the upper-bound given by Lemma 7.8 is still true, we can conclude similarly. □

APPENDIX A

Integration along the fiber

In this appendix, we recall some well known properties of the morphism of integration along fiber (see [**27**, Section A.2] and references therein). Since we cannot find any reference concerning the proof of the composition property, we give a proof of this key property used to prove Proposition 2.5.

Let G be a group. Throughout this appendix, $H_*(X)$ and $H^*(X)$ will denote respectively the singular homology and the singular cohomology of the topological space or pair X with coefficients in G. If we want to put another group of coefficients G', we will write down explicitly $H_*(X;G')$ or $H^*(X;G')$ so that for instance $H_*(X;H_d(Y)) = H_*(X;H_d(Y;G))$ where Y is a topological space or pair.

Let us assume that $\pi : X \to B$ is a Serre fibration with fiber F which has the type of a CW complex of dimension d (throughout this section, we will simply write that the fibers of π have dimension d). In order to simplify the statements, we will always assume that $\pi_1(B)$ acts trivially on $H_*(F)$. Then the Serre spectral sequence in homology $(E^r_{p,q})$ of π satisfies $E^2_{p,q} \simeq H_p(B;H_q(F)) = 0$ for $q > d$. Hence, it gives natural morphisms $E^2_{p,d} \to E^\infty_{p,d} \to H_{p+d}(X)$ for all p. Let $\pi^* : H_*(B;H_q(F)) \to H_{*+d}(X)$ denote the composition of the Serre isomorphism $H_*(B;H_d(F)) \simeq E^2_{*,d}$ with this map $E^2_{*,d} \to H_{*+d}(X)$. Dually, one can define a morphism $\pi_* : H^*(X) \to H^{*-d}(B;H^d(F))$. In de Rham cohomology, the map π_* can be easily defined on compact smooth fiber bundles as induced by the integration of differential forms along the fibers.

In the special case $F \simeq \mathbb{S}^n$, the map π_* corresponds to the Gysin morphism. This definition extends directly to relative fibrations $\pi : (X,X') \to (B,B')$ and the induced maps commute with the long exact sequences of pair and triple by naturality of the Serre spectral sequence. Given a commuting square of (possibly relative) fibrations

$$\begin{tikzcd} X \arrow[r, "\tilde f"] \arrow[d, "F"', "p"] & Y \arrow[d, "F'"', "q"] \\ B \arrow[r, "f"] & C \end{tikzcd},$$

with fibers F and F' of the same dimension d, the naturality of the Serre spectral sequence induces the commutative square

$$\begin{tikzcd} H_{*+d}(X) \arrow[r, "\tilde f_*"] & H_{*+d}(Y) \\ H_*(B;H_d(F)) \arrow[u, "p^*"] \arrow[r, "f_*"] & H_*(B;H_d(F')) \arrow[u, "q^*"'] \end{tikzcd}, \tag{A.1}$$

where f_* sends a class $[\sigma \otimes h]$, $\sigma \in C_*(B;\mathbb{Z})$ and $h \in H_d(F)$, on $[f_*\sigma \otimes \tilde f_* h]$.

Finally, it satisfies the following composition property. Let $\pi_1 : Y \to X$ and $\pi_2 : X \to B$ be (possibly relative) Serre fibrations of respective fibers F_1 and F_2 of dimension d_1 and d_2. Then $\pi := \pi_2 \circ \pi_1$ is a fibration whose fiber F is a fibration over F_2 with fibers F_1. According to the Serre spectral sequence, it has dimension $d_1 + d_2$ and $H_{d_1+d_2}(F)$ is naturally isomorphic to $H_{d_2}(F_2;H_{d_1}(F_1))$.

Proposition A.1. *The following diagram commutes*

$$\begin{tikzcd}
H_{*+d_2}(X;H_{d_1}(F_1)) \arrow[r, "\pi_1^*"] & H_{*+d_1+d_2}(Y) \\
H_*(B;H_{d_2}(F_2;H_{d_1}(F_1))) \arrow[u, "\pi_2^*"] \arrow[r, "\simeq"] & H_*(B;H_{d_1+d_2}(F)) \arrow[u, "\pi^*"']
\end{tikzcd}$$

where the bottom isomorphism is induced by the isomorphism $H_{d_2}(F_2;H_{d_1}(F_1)) \simeq H_{d_1+d_2}(F)$ between the groups of coefficients.

In Chapter 2, the fibers F_1, F_2 and F are naturally oriented so that there are canonical isomorphisms between their top degree homology groups and the coefficient group and one does not have to bother with changes of coefficient group. However, if one wants to avoid the change of coefficients in the naturality statement (A.1) when $H_d(F) \simeq G$ and $H_d(F') \simeq G$, the map $\tilde{f}_* : H_d(F) \to H_d(F')$ must send preferred generator to preferred generator.

Proof of Proposition A.1. Without loss of generality, one can assume that B and X are actual CW complexes and that π_2 is a locally trivial cellular fibration [**14**]. We denote by E, E_1 and E_2 Serre spectral sequences of π, π_1 and π_2 respectively, E_2 having $H_{d_1}(F_1)$ coefficients. Let B^p and X^p denote respectively the p-skeleton of B and the p-skeleton of X. Let $X_p := \pi_2^{-1}(B^p)$, $Y_p := \pi^{-1}(B^p)$ and $Y_{1;p} := \pi_1^{-1}(X^p)$ denote the filtration of the spaces X and Y associated with E_2, E and E_1 respectively. Therefore, for instance the first page of E_2 is given by $E^1_{2;p,q} := H_{p+q}(X_p, X_{p-1})$.

Since π_2 is a cellular fibration with fibers of dimension d_2, $X_p = \pi_2^{-1}(B^p)$ is included in X^{p+d_2}. Hence, $Y_p \subset Y_{1;p+d_2}$ and this inclusion between filtrations induces a morphism of spectral sequences (with a shift in degree) $E^r_{p,q} \to E^r_{1;p+d_2,q-d_2}$. Therefore, one gets the following commutative diagram

$$\begin{tikzcd}
 & H_{*+d_1+d_2}(Y) \arrow[r, equal] & H_{*+d_1+d_2}(Y) & \\
 & E^\infty_{*,d_1+d_2} \arrow[u] \arrow[r] & E^\infty_{1;*+d_2,d_1} \arrow[u] & \\
H_*(B;H_{d_1+d_2}(F)) \arrow[uur, "\pi^*"] & E^2_{*,d_1+d_2} \arrow[u] \arrow[r] \arrow[l, "\simeq"] & E^2_{1;*+d_2,d_1} \arrow[u] \arrow[r, "\simeq"] & H_{*+d_2}(X;H_{d_1}(F_1)) \arrow[uul, "\pi_1^*"']
\end{tikzcd},$$

where both the left and the right "squares" are the ones defining morphisms π^* and π_1^*. The bottom row allows us to define a morphism $f : H_*(B;H_{d_1+d_2}(F)) \to H_{*+d_2}(X;H_{d_1}(F_1))$. According to the last diagram, it is enough to prove that $f = \pi_2^*$ under the identification $H_{d_1+d_2}(F) \simeq H_{d_2}(F_2;H_{d_1}(F_1))$ to conclude.

We recall that the Serre isomorphism $E^2_{*,d_1+d_2} \simeq H_*(B;H_{d_1+d_2}(F))$ is induced by a chain isomorphism between the chain complex $E^1_{*,d_1+d_2} = H_{*+d_1+d_2}(Y_*, Y_{*-1})$ and the chain complex of the cellular filtration (B^p) of B that is $H_*(B^*, B^{*-1}; H_{d_1+d_2}(F))$. We denote by $\Psi : H_{*+d_1+d_2}(Y_*, Y_{*-1}) \to H_*(B^*, B^{*-1}; H_{d_1+d_2}(F))$ the chain isomorphism associated with π and by Ψ_1 and Ψ_2 the chain isomorphisms associated with π_1 and π_2 respectively. Since these chain isomorphisms are natural (this is included in the proof of the naturality of the Serre spectral sequence), one gets the following commutative diagram of chain complexes:

$$\begin{tikzcd}
H_{*+d_1+d_2}(Y_*, Y_{*-1}) \arrow[r] \arrow[d, "\simeq"', "\Psi"] & H_{*+d_1+d_2}(Y_{1;*+d_2}, Y_{1;*+d_2-1}) \arrow[d, "\simeq"', "\Psi_1"] \\
H_*(B^*, B^{*-1};\mathbb{Z}) \otimes H_{d_1+d_2}(F) \arrow[d, "\simeq"', "\mathrm{id}\otimes u"] & H_{*+d_2}(X^{*+d_2}, X^{*+d_2-1};\mathbb{Z}) \otimes H_{d_1}(F_1) \\
H_*(B^*, B^{*-1};\mathbb{Z}) \otimes H_{d_2}(F_2;H_{d_1}(F_1)) & H_{*+d_2}(X_*, X_{*-1}) \otimes H_{d_1}(F_1) \arrow[l, "\simeq", "\Psi_2"'] \arrow[u]
\end{tikzcd},$$

where u denotes the natural isomorphism $H_{d_1+d_2}(F) \to H_{d_2}(F_2; H_{d_1}(F_1))$. By passing to homology, one gets the following commutative diagram:

$$\begin{tikzcd}
E^2_{*,d_1+d_2} \arrow[r] \arrow[d, "\Psi_*", "\simeq"'] & E^2_{1;*+d_2,d_1} \arrow[d, "(\Psi_1)_*", "\simeq"'] & \\
H_*(B; H_{d_1+d_2}(F)) \arrow[r, "f"] \arrow[d, "(\mathrm{id}\otimes u)_*", "\simeq"'] & H_{*+d_2}(X; H_{d_1}(F_1)) & \\
H_*(B; H_{d_2}(F_2; H_{d_1}(F_1))) & E^2_{2;*,d_2} \arrow[l, "\simeq", "(\Psi_2)_*"'] \arrow[u] \arrow[r] & E^\infty_{2;*,d_2} \arrow[ul]
\end{tikzcd},$$

where only the commutativity of the triangle is not a direct consequence of the previous diagram. The commutativity of the triangle is a consequence of the naturality of the morphism induced by the inclusion of filtrations $X_p \subset X^{p+d_2}$ between the associated spectral sequences. Indeed, since (X^{p+d_2}) is a cellular filtration, the associated spectral sequence whose first page is $(H_{p+q}(X^{p+d_2}, X^{p+d_2-1}))_{p,q}$ degenerates at the second page. The bottom part of the diagram shows that f is indeed π_2^* under the identification induced by u. $\square$

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Abstract

In this thesis, we are studying topological and dynamical conditions imposing infinitely many periodic orbits for some dynamical systems. In a first part, we elaborate on theories of Givental and Théret based on generating functions in order to study the case of complex projective spaces. We find recent results back without appealing to the theory of J-holomorphic curves. In particular, we prove Shelukhin theorem showing a homology version of the Hofer-Zehnder conjecture.

In a second part, we study the geodesic flow and show new results bringing examples of topological and dynamical conditions imposing infinitely many closed geodesics or geodesic chords. We give conditions under which the existence of one or two closed geodesics on a complete Riemannian plane, cylinder or Möbius band impose the existence of infinitely many closed geodesics. In particular, we show that a complete Riemannian cylinder (whose closed geodesics are isolated) has zero, one or infinitely many homologically visible closed geodesics ; it solves a conjecture of Alberto Abbondandolo. We also prove that every complete Finsler manifold with an infinite fundamental group that is not homotopy equivalent to a circle possesses infinitely many geometrically distinct geodesic chords joining any given pair of points. Results of this part are partially joint with Tobias Soethe.

Résumé

Dans cette thèse, nous nous intéressons aux conditions dynamiques ou topologiques imposant l'existence d'un nombre infini de trajectoires périodiques pour certains types de systèmes hamiltoniens. Dans une première partie, nous prolongeons les théories de Givental et Théret basées sur les fonctions génératrices afin d'étudier le cas des espaces projectifs complexes ; nous retrouvons ainsi des résultats très récents sans faire appel à la théorie J-holomorphe. Nous montrons, en particulier, le théorème de Shelukhin démontrant une version homologique de la conjecture de Hofer-Zehnder.

Dans une seconde partie, nous nous intéressons aux flots géodésiques et démontrons de nouveaux résultats apportant des exemples de telles conditions dynamiques ou topologiques. Nous énonçons des conditions sous lesquelles la présence d'une ou deux géodésiques fermées géométriquement distinctes sur un plan, un cylindre ou un ruban de Möbius riemannien complet impose la présence d'une infinité de géodésiques fermées géométriquement distinctes. En particulier, nous montrons qu'un cylindre riemannien complet (dont les géodésiques fermées sont isolées) admet zéro, une ou une infinité de géodésiques fermées homologiquement distinctes ; cela répond à une question d'Alberto Abbondandolo. On prouve aussi que toute variété de Finsler complète de groupe fondamental infini et non homotopiquement équivalente à un cercle possède une infinité de géodésiques géométriquement distinctes joignant n'importe quelle paire de points. Les résultats de cette seconde partie sont partiellement issus d'une collaboration avec Tobias Soethe.