EXAMPLE OF MORSE-THEORETIC STUDY OF A FUNCTION

Let $F:\mathbb{R}^2\to\mathbb{R}$ be the function given by

$$F(x,y)=(y-2x^2)(y-x^2)(y+x^2)(y+2x^2).$$

The derivatives of F are

$$\partial_x F(x,y)=32x^7-20x^3y^2=32x^3(x^4-\tfrac{5}{8}y^2),$$
$$\partial_y F(x,y)=-10x^4y+4y^3=-10y(x^4-\tfrac{2}{5}y^2).$$

From this we readily see that the origin is the only critical point of F, and its critical value is 0.

We claim that F satisfies the Palais-Smale condition. This can be seen as follows. We need to fix a large constant $r>0$ and a small one $\epsilon>0$: as we will see, it will be enough to choose them so that the four quantities labeled by $(*)$ here below be positive. We have two cases two consider:

- Assume that $|y|\geq r$. If $|x^4-\frac{2}{5}y^2|>\epsilon$, then
$$|\nabla F(x,y)|\geq|\partial_y F(x,y)|>10\,r\epsilon.$$
Otherwise, $|x^4-\frac{2}{5}y^2|\leq\epsilon$, and therefore $|x|\geq(\frac{2}{5}y^2-\epsilon)^{1/4}\geq(\frac{2}{5}r^2-\epsilon)^{1/4}$ and
$$|x^4-\tfrac{5}{8}y^2|\geq\tfrac{9}{40}y^2-|x^4-\tfrac{2}{5}y^2|\geq\tfrac{9}{40}r^2-\epsilon,$$
which in turn implies
$$|\nabla F(x,y)|\geq|\partial_x F(x,y)|\geq 32\,\underbrace{(\tfrac{2}{5}r^2-\epsilon)}_{(*)}{}^{3/4}\,\underbrace{(\tfrac{9}{40}r^2-\epsilon)}_{(*)}.$$

- Now, assume that $|x|\geq r$. If $|x^4-\frac{5}{8}y^2|>\epsilon$, then
$$|\nabla F(x,y)|\geq|\partial_x F(x,y)|>32\,r^3\epsilon.$$
Otherwise, $|x^4-\frac{5}{8}y^2|\leq\epsilon$, and therefore $|y|\geq[\frac{8}{5}(x^4-\epsilon)]^{1/2}\geq[\frac{8}{5}(r^4-\epsilon)]^{1/2}$ and
$$|x^4-\tfrac{2}{5}y^2|=|\tfrac{9}{40}y^2+x^4-\tfrac{5}{8}y^2|\geq\tfrac{9}{40}y^2-|x^4-\tfrac{5}{8}y^2|\geq\tfrac{9}{40}\cdot\tfrac{8}{5}(r^4-\epsilon)-\epsilon,$$
which in turn implies
$$|\nabla F(x,y)|\geq|\partial_y F(x,y)|\geq 10[\tfrac{8}{5}\underbrace{(r^4-\epsilon)}_{(*)}]^{1/2}\,\underbrace{[\tfrac{9}{40}\cdot\tfrac{8}{5}(r^4-\epsilon)-\epsilon]}_{(*)}.$$

Summing up, we have showed that outside of any compact neighborhood $K\subset\mathbb{R}^2$ of the origin, the norm $|\nabla F|$ is bounded from below by a constant depending only on K. This implies that F satisfies the Palais-Smale condition.

The Hessian of F at his critical point vanishes, i.e.

$$\mathrm{d}^2F(0,0) = \left(\begin{array}{cc} \partial^2_{xx}F(0,0) & \partial^2_{xy}F(0,0) \\ \partial^2_{yx}F(0,0) & \partial^2_{yy}F(0,0) \end{array} \right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right).$$

Therefore the Morse index $\mathrm{ind}(F,(0,0))$ vanishes, whereas the nullity $\mathrm{nul}(F,(0,0))$ is equal to 2. The origin is a totally degenerate isolated critical point of F.

We can easily draw the open sublevel set $\{F<0\}$. By doing this, we see that it consists of four contractible connected components. The union $\{F<0\}\cup\{(0,0)\}$ is path-connected and actually contractible. We consider the local homology of F at $(0,0)$

$$L_*(F,(0,0)) := H_d(\{F<0\}\cup\{(0,0)\},\{F<0\}).$$

Let us look at the long exact sequence of the pair $\{F<0\}\subset\{F<0\}\cup\{(0,0)\}$

$$\ldots \longrightarrow H_d(\{F<0\}\cup\{(0,0)\}) \longrightarrow L_d(F,(0,0)) \xrightarrow{\partial_*} H_{d-1}(\{F<0\}) \longrightarrow H_{d-1}(\{F<0\}\cup\{(0,0)\}) \longrightarrow \ldots$$

The group $H_d(\{F<0\})$ is non-trivial only in degree $d=0$, where it is isomorphic to $\mathbb{Z}^4$ (here we are considering homology groups with coefficients in $\mathbb{Z}$). Analogously, the group $H_d(\{F<0\}\cup\{(0,0)\})$ is non-trivial only in degree $d=0$, where it is isomorphic to $\mathbb{Z}$. Thus, the only non-trivial portion of the above long exact sequence is the following

$$0 \longrightarrow L_1(F,(0,0)) \xrightarrow{\partial_*} \mathbb{Z}^4 \xrightarrow{\iota_*} \mathbb{Z} \longrightarrow L_0(F,(0,0)) \longrightarrow 0,$$

and the homomorphism $\iota_* : H_0(\{F<0\}) \to H_0(\{F<0\}\cup\{(0,0)\})$ is clearly surjective. This implies that

$$L_d(F,(0,0)) = \begin{cases} 0 & \text{if } d \neq 1, \\ \mathbb{Z}^3 & \text{if } d = 1. \end{cases}$$