MORSE-GROMOLL-MEYER LEMMA

Let U be an open neighborhood of the origin in a (finite or infinite dimensional) separable Hilbert space $\mathbb{E}$. We consider a C^3 function $F : U \to \mathbb{R}$ that satisfies the Palais-Smale condition and has a critical point at the origin. We denote by A the self-adjoint bounded linear operator on $\mathbb{E}$ associated to the Hessian $\mathrm{d}^2F(0)$, i.e.

$$\mathrm{d}^2F(0)[v,w] = \langle Av, w\rangle, \qquad \forall v,w \in \mathbb{E}.$$

Notice that A is precisely $\mathrm{d}(\nabla F)(0) : \mathbb{E} \to \mathbb{E}$.

We assume that A is a Fredholm operator, which means that $\mathbb{E}_0 := \ker A$ is finite dimensional. We denote by $\mathbb{E}_1$ the orthogonal complement of $\mathbb{E}_0$. Notice that

$$\langle Av, w\rangle = \langle v, \underbrace{Aw}_{=0}\rangle = 0, \qquad \forall w \in \mathbb{E}_0,\ v \in \mathbb{E}_1,$$

and therefore $Av \in \mathbb{E}_1$ as well. This shows that the restriction $A_1 := A|_{\mathbb{E}_1}$ is a self-adjoint injective bounded linear operator on $\mathbb{E}_1$. Since $\mathrm{coker}(A_1) = \ker(A_1) = \{0\}$, the operator A_1 is actually bijective and, by the open mapping theorem, it is an isomorphism (i.e. its inverse is also a bounded operator).

Theorem 0.1 (Morse-Gromoll-Meyer Lemma)**.** *There exist open neighborhoods of the origin $V_0 \subset \mathbb{E}_0$ and $V_1 \subset \mathbb{E}_1$, a map $\Phi : V_0 \times V_1 \to U$ that is a homeomorphism onto a neighborhood of the origin and that fixes the origin, and a C^2 function $F_0 : V_0 \to \mathbb{R}$ with a totally degenerate critical point at the origin, such that*

$$F \circ \Phi(v_0, v_1) = F_0(v_0) + \tfrac{1}{2}\langle A_1 v_1, v_1\rangle, \qquad \forall (v_0,v_1) \in V_0 \times V_1.$$

Proof. Let us write points in U as (w_0, w_1), where $w_0 \in U \cap \mathbb{E}_0$ and $w_1 \in U \cap \mathbb{E}_1$. Notice that

$$\mathrm{d}^2F(0) = \begin{pmatrix} \partial^2_{w_0w_0}F(0,0) & \partial^2_{w_0w_1}F(0,0) \\ \partial^2_{w_1w_0}F(0,0) & \partial^2_{w_1w_1}F(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}.$$

In particular $\partial^2_{w_1w_1}F(0,0) = A_1$ is an isomorphism on the Hilbert space $\mathbb{E}_1$. Therefore, by the implicit function theorem, there exist open neighborhoods of the origin $U_0 \subset U \cap \mathbb{E}_0$ and $U_1 \subset U \cap \mathbb{E}_1$, and a C^2 map $\psi : U_0 \to U_1$ such that $\psi(0) = 0$ and

$$\partial_{w_1}F(w_0, \psi(w_0)) = 0, \qquad \forall w_0 \in U_0.$$

By differentiating this relation at the origin, we obtain

$$0 = \partial^2_{w_0w_1}F(0,\psi(0)) + \partial^2_{w_1w_1}F(0,\psi(0)) \circ \mathrm{d}\psi(0) = A_1 \circ \mathrm{d}\psi(0).$$

Since A_1 is an isomorphism, we infer that

$$\mathrm{d}\psi(0) = 0.$$

We define a map $\Psi : U_0 \times U_1 \to U$ by

$$\Psi(w_0, w_1) = (w_0, \psi(w_0) + w_1), \qquad \forall (w_0, w_1) \in U_0 \times U_1.$$

This map is a C^2 diffeomorphism onto a neighborhood of the origin, with inverse

$$\Psi^{-1}(z_0, z_1) = (z_0, -\psi(z_0) + z_1).$$

We will employ Ψ as a local chart around the origin. The function $\tilde F := F \circ \Psi$ is C^2 and satisfies

$$\partial_{w_1} \tilde F(w_0, 0) = 0, \qquad \forall w_0 \in U_0.$$

Since $\partial_{w_1 w_1} \tilde F(0,0) = A_1$ is an isomorphism, up to shrinking the neighborhoods U_0 and U_1 around the origin, the operator $\partial_{w_1 w_1} \tilde F(w_0, w_1)$ is an isomorphism for all $(w_0, w_1) \in U_0 \times U_1$. Namely, for each $w_0 \in U_0$, the function $w_1 \mapsto \tilde F(w_0, w_1)$ has a non-degenerate critical point at the origin. Moreover, since $\mathrm{d}\psi(0) = 0$, the function $w_0 \mapsto \tilde F_0(w_0) := \tilde F(w_0, 0)$ has a totally degenerate critical point at the origin, that is

$$\mathrm{d}^2 \tilde F_0(0) = \mathrm{d}^2 \tilde F(0,0)|_{\mathbb{E}_0} = 0.$$

From now on we will simply write F for $\tilde F$, F_0 for $\tilde F_0$, and U for the preimage $\Psi^{-1}(U)$.

Now, let us denote $F_1(w_1) := \frac{1}{2}\langle A_1 w_1, w_1\rangle$. This is a quadratic function with linear gradient $\nabla F_1(w_1) = A_1 w_1$. We denote by λ_t the partial flow of the vector field

$$V(w_1) := -\frac{\nabla F_1(w_1)}{\|\nabla F_1(w_1)\|} = -\frac{A w_1}{\|A w_1\|}.$$

Notice that this vector field is well defined and smooth outside the origin. Moreover, it has norm $\|V\| \equiv 1$. Therefore, the trajectories of its flow λ_t move with speed 1. Since a trajectory is well defined until it hits the origin, we conclude that $\lambda_t(w_1)$ is well defined provided $w_1 \neq 0$ and $|t| < \|w_1\|$.

We set $\delta := \frac{1}{2}\|A_1^{-1}\|^{-1}$. By the C^2-continuity of F, up to shrinking the neighborhoods U_0 and U_1 around the origin we have

$$\|\partial_{w_1 w_1} F(w_0, w_1) - \partial_{w_1 w_1} F(0,0)\| = \|\partial_{w_1 w_1} F(w_0, w_1) - A_1\| \leq \delta.$$

If we further shrink U_0 and U_1, for all $(w_0, w_1) \in U_0 \times U_1$ and $t \in \big(-\|w_1\|, \|w_1\|\big)$ we actually have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F(w_0, \lambda_t(w_1)) &= -\partial_{w_1} F(w_0, \lambda_t(w_1)) \frac{A_1 \lambda_t(w_1)}{\|A_1 \lambda_t(w_1)\|} \\
&= -\int_0^1 \frac{1}{\|A_1 \lambda_t(w_1)\|} \langle \partial_{w_1 w_1} F(w_0, s\,\lambda_t(w_1)) \lambda_t(w_1), A_1 \lambda_t(w_1)\rangle \,\mathrm{d}s \\
&\leq -\frac{1}{\|A_1 \lambda_t(w_1)\|} \langle A_1 \lambda_t(w_1), A_1 \lambda_t(w_1)\rangle + \delta \|\lambda_t(w_1)\| \\
&= -\|A_1 \lambda_t(w_1)\| + \delta \|\lambda_t(w_1)\| \\
&\leq -\|A_1^{-1}\|^{-1} \cdot \|\lambda_t(w_1)\| + \delta \|\lambda_t(w_1)\| \\
&= -\delta \|\lambda_t(w_1)\|.
\end{aligned} \tag{0.1}$$

Since the last quantity is always negative (remember that $\lambda_t(w_1)$ lies outside the origin), we have shown that F is a negative Lyapunov function for the flow λ_t, that is, it strictly decreases along its orbits.

Now, let us fix a quantity $\epsilon \in (0, \frac{3}{4}\delta)$. Let us quantify how much the function F_1 decreases along its reparametrized anti-gradient flow λ_t in time $t_1 = \pm\frac{1}{2}\|w_1\|$:

$$\begin{aligned} |F_1(\lambda_{t_1}(w_1)) - F_1(w_1)| &= \int_0^{|t_1|} \|A_1\lambda_s(w_1)\| \,\mathrm{d}s \\ &\geq 2\delta \int_0^{|t_1|} \|\lambda_s(w_1)\| \,\mathrm{d}s \\ &\geq 2\delta \int_0^{|t_1|} \Big(\|w_1\| - s\Big) \,\mathrm{d}s \\ &\geq 2\delta \left(\|w_1\| \cdot |t_1| - \frac{t_1^2}{2}\right) \\ &= 2\delta \left(\frac{1}{2} - \frac{1}{8}\right) \|w_1\|^2 \\ &> \epsilon\|w_1\|^2. \end{aligned}$$

Up to shrinking one last time the neighborhoods U_0 and U_1 around the origin, we have

$$\begin{aligned} &|F(w_0, w_1) - F(w_0, 0) - F_1(w_1)| \\ &\quad= \left|\int_0^1 \partial_{w_1} F(w_0, sw_1)w_1 \,\mathrm{d}s - F_1(w_1)\right| \\ &\quad= \left|\int_0^1\int_0^1 s\Big(\langle\partial_{w_1w_1}F(w_0, rsw_1)w_1, w_1\rangle - \langle Aw_1, w_1\rangle\Big) \,\mathrm{d}r \,\mathrm{d}s\right| \\ &\quad\leq \epsilon\|w_1\|^2. \end{aligned}$$

If we put together the last two groups of estimates, for $t_1 = \frac{1}{2}\|w_1\|$ we get

$$F_1(\lambda_{t_1}(w_1)) \leq F(w_0, w_1) - F(w_0, 0) \leq F_1(\lambda_{-t_1}(w_1)).$$

This shows that there exists a unique intermediate value $\theta(w_0, w_1) \in (-\frac{1}{2}\|w_1\|, \frac{1}{2}\|w_1\|)$ such that, for all $(w_0, w_1) \in U_0 \times U_1$ outside the origin, we have the equality

$$F(w_0, w_1) = F(w_0, 0) + F_1(\lambda_{\theta(w_0,w_1)}(w_1)).$$

Analogously, there exists a value $\phi(v_0, v_1) \in (-\|v_1\|, \|v_1\|)$ such that

$$F(v_0, \lambda_{\phi(v_0,v_1)}(v_1)) = F(v_0, 0) + F_1(v_1).$$

and, by (0.1), this value is also unique. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(F(w_0, w_1) - F(w_0, 0) - F_1(\lambda_t(w_1))\Big) = -\frac{\mathrm{d}}{\mathrm{d}t}F_1(\lambda_t(w_1)) > 0,$$

we can apply the implicit function theorem and conclude that θ is a C^2 function outside the origin. Analogously, ϕ is a C^2 function outside the origin. Moreover, θ and ϕ can be continuously (but not necessarily differentiably) extended to the origin by setting

$$\theta(0,0) = \phi(0,0) = 0.$$

We define the map $\Theta : U_0 \times U_1 \to \mathbb{E}$ by

$$\Theta(w_0, w_1) = (w_0, \lambda_{\theta(w_0,w_1)}(w_1)).$$

This map is a homeomorphism, and its inverse $\Phi : \Theta(U_0 \times U_1) \to U_0 \times U_1$, which is given by

$$\Phi(v_0, v_1) = (v_0, \lambda_{\phi(v_0,v_1)}(v_1)),$$

is our desired chart. ■