SOLUTIONS OF THE EXAM – March 23, 2018

Exercise 1. Consider a compact cylinder $M = [-1,1] \times \mathbb{R}/\mathbb{Z}$ equipped with the flat Riemannian metric $g = dx \otimes dx + d\theta \otimes d\theta$ (here, x is the variable on $[0,1]$, whereas θ is the variable on $\mathbb{R}/\mathbb{Z}$). Is the X-ray transform

$$I_0 : C^\infty(M) \to C^\infty(\partial_+ SM), \qquad I_0(f) = \int_0^{\tau_g(x,v)} f(\exp_x(tv))\, dt$$

injective?

Hint. Solve first the analogous exercise for $M = [-1,1]$ with the Euclidean Riemannian metric $g = dx \otimes dx$.

Solution. The X-ray transform I_0 is not injective. Indeed, let $h \in C^\infty([-1,1])$ be any odd function, for instance $h(x) = \sin(x)$. We see it as a function $f \in C^\infty(M)$, by setting $f(x,\theta) := h(x)$. For each $q = (x,\theta) \in \partial M$ and $v = (v_1, v_2) \in \partial_+ S_x M$, the corresponding geodesic is the straight line $\exp_q(tv) = (x + tv_1, \theta + tv_2)$, which is defined for $t \in [0, 2/|v_1|]$. Since h is odd, the X-ray transform $I_0(f)$ vanishes, for

$$I_0(f)(q,v) = \int_0^{2/|v_1|} h(x + tv_1)\, dt = |v_1|^{-1} \int_{-1}^{1} h(y)\, dy = 0.$$

Exercise 2. For $i = 0, 1$, let (M_i, g_i) be a closed Riemannian surface with associated geodesic vector field X_i, geodesic flow ϕ_i^t, and Liouville 1-form α_i on SM_i. Assume that there exists a diffeomorphism $\psi : SM_0 \to SM_1$ such that

$$\psi \circ \phi_0^t = \phi_1^t \circ \psi, \qquad \forall t \in \mathbb{R}.$$

(i) For each $s \in [0, 1]$, set $\beta_s := s\psi^*\alpha_1 + (1 - s)\alpha_0$, $\dot\beta_s := \frac{d}{ds}\beta_s$, and compute

$$(\psi^*\alpha_1)(X_0), \quad X_0 \lrcorner\, d(\psi^*\alpha_1), \quad X_0 \lrcorner\, \dot\beta_s \wedge d\beta_s, \quad \dot\beta_s \wedge d\beta_s, \quad \frac{d}{ds}\int_{SM_0} \beta_s \wedge d\beta_s.$$

(ii) Do we necessarily have $\mathrm{Vol}(M_0, g_0) = \mathrm{Vol}(M_1, g_1)$?

Solution. (i) By differentiating the members of the identity $\psi \circ \phi_0^t = \phi_1^t \circ \psi$ with respect to t, we find $d\psi(x, v)X_0(x, v) = X_1(\psi(x, v))$. Therefore

$$\psi^*\alpha_1(X_0) = \psi^*(\alpha_1(X_1)) \equiv 1,$$
$$X_0 \lrcorner\, d(\psi^*\alpha_1) = X_0 \lrcorner\, \psi^* d\alpha_1 = \psi^*(X_1 \lrcorner\, d\alpha_1) \equiv 0.$$

This implies, for each $s \in [0, 1]$,

$$\beta_s(X_0) \equiv 1, \qquad X_0 \lrcorner\, d\beta_s \equiv 0.$$

Therefore $\dot\beta_s(X_0) = 0$, and

$$X_0 \lrcorner\, \dot\beta_s \wedge d\beta_s = 0.$$

Since $\dot\beta_s \wedge d\beta_s$ is a 3-form on the 3-dimensional manifold SM_0, and since the geodesic vector field X_0 is nowhere vanishing, we infer that

$$\dot\beta_s \wedge d\beta_s = 0.$$

Finally,

$$\begin{aligned}
\frac{d}{ds}\int_{M_1} \beta_s \wedge d\beta_s &= \int_{SM_1} \dot\beta_s \wedge d\beta_s + \int_{M_1} \beta_s \wedge d\dot\beta_s \\
&= \int_{SM_1} \beta_s \wedge d\dot\beta_s \\
&= \int_{SM_1} \Big(- d(\beta_s \wedge \dot\beta_s) + d\beta_s \wedge \dot\beta_s \Big) \\
&= -\int_{SM_1} d(\beta_s \wedge \dot\beta_s) \\
&= 0.
\end{aligned}$$

(ii) Yes. Indeed,

$$\begin{aligned}
\mathrm{Vol}(M_0, g_0) &= \frac{1}{2\pi}\int_{SM_0} \alpha_0 \wedge (d\alpha_0)^{d-1} = \frac{1}{2\pi}\int_{SM_0} \beta_0 \wedge (d\beta_0)^{d-1} \\
&= \frac{1}{2\pi}\int_{SM_0} \beta_1 \wedge (d\beta_1)^{d-1} = \frac{1}{2\pi}\int_{SM_0} \psi^*(\alpha_1 \wedge (d\alpha_1)^{d-1}) \\
&= \frac{1}{2\pi}\int_{SM_1} \alpha_1 \wedge (d\alpha_1)^{d-1} = \mathrm{Vol}(M_1, g_1).
\end{aligned}$$

Exercise 3. Let (B^2, g) be an oriented simple Riemannian ball, with associated geodesic vector field X on SB^2, vertical vector field V on SB^2, and X-ray transform I. Consider the vector field $X_\perp := [X, V]$, and a smooth function $f \in C^\infty(M)$ such that $I(X_\perp f) = 0$ and $f|_{\partial M} \equiv c \in \mathbb{R}$. Determine the function f.

Solution. We recall that $X_\perp$ acts on functions $f \in C^\infty(M)$ as

$$X_\perp f(x, v) = XVf(x, v) - VXf(x, v) = -VXf(x, v) = -V(df)(x, v) = -df(x)Jv,$$

where J is the complex structure of (B^2, g), i.e. Jv is the tangent vector obtained by rotating v positively by an angle $\pi/2$. This shows that $X_\perp f$ is a 1-tensor, and therefore $I_1(X_\perp f) = I(X_\perp f) = 0$. By the s-injectivity of the X-ray transform I_1, if $I_1(X_\perp f) = 0$ then there exists $h \in C^\infty(M)$ such that $h|_{\partial M} \equiv 0$ and $dh(x)v = X_\perp f(x, v) = -df(x)Jv$ for all $(x, v) \in SB^2$. Therefore, $f + ih : B^2 \to \mathbb{C}$ is a J-holomorphic function, and $\Delta_g f = \Delta_g h \equiv 0$. Notice that the constant function $f' \equiv c$ satisfies $\Delta_g f' = 0$ and $f'|_{\partial B^2} = f|_{\partial B^2}$. Since harmonic functions are completely determined by their boundary values, we conclude that $f = f' \equiv c$.

Exercise 4. Let (B^2, g) be an oriented simple Riemannian ball, with associated geodesic vector field X on SB^2, and vertical vector field V on SB^2. Consider a smooth function $u \in C^\infty(SM)$ such that $u|_{\partial SM} \equiv 0$, and set $f := Xu$. We write the fiberwise Fourier decompositions of u and f as

$$u = \sum_{k\in\mathbb{Z}} u_k, \qquad f = \sum_{k\in\mathbb{Z}} f_k,$$

where $u_k, f_k \in \ker(-iV - k)$. Assume that $f_k \equiv 0$ for all $k < 0$. Compute u_k, for all $k \leq 0$.

Hint. Set $w := u - iHu = u_0 + 2\sum_{k<0} u_k$, where H is the Hilbert transform. Compute $I(Xw_{\text{even}})$ and $I(Xw_{\text{odd}})$, where I is the X-ray transform. Compute the expression of Xw by means of Pestov-Uhlmann's relation. Compute $f - iHf$. Then compute Xw_{even} and Xw_{odd}. Use the s-injectivity of the X-ray transform on 0-tensors and your result of Exercise 3.

Solution. Since $u|_{\partial SM} = 0$, we have that $w_{\text{even}}|_{\partial SM} = w_{\text{odd}}|_{\partial SM} = 0$ as well, and therefore

$$I(Xw_{\text{even}}) = I(Xw_{\text{odd}}) = 0. \tag{1}$$

By Pestov-Uhlmann's relation $HXu - XHu = X_\perp u_0 + (X_\perp u)_0$, we have

$$Xw = Xu - iHXu + iX_\perp u_0 + i(X_\perp u)_0 = f - iHf + iX_\perp u_0 + i(X_\perp u)_0.$$

Since $f_k \equiv 0$ for all $k < 0$, we have $f - iHf = f_0$, and therefore

$$Xw = f_0 + iX_\perp u_0 + i(X_\perp u)_0.$$

Since the vector fields X and $X_\perp$ transform even function to odd functions and viceversa, the previous equation can be decomposed in even and odd part, and gives

$$Xw_{\text{even}} = iX_\perp u_0, \qquad Xw_{\text{odd}} = f_0 + i(X_\perp u)_0.$$

Equation (1) thus implies $I_1(X_\perp u_0) = 0$ and $I_0(f_0 + i(X_\perp u)_0) = 0$. The equality $I_1(X_\perp u_0) = 0$, together with $u_0|_{\partial M} \equiv 0$ and the result of Exercise (3), implies that $u_0 \equiv 0$. Therefore, $Xw_{\text{even}} = 0$, and since $w_{\text{even}}|_{\partial SM} \equiv 0$ we have $w_{\text{even}} \equiv 0$. The equality $I_0(f_0 + i(X_\perp u)_0) = 0$, together with the injectivity of I_0, implies that $Xw_{\text{odd}} = f_0 + i(X_\perp u)_0 \equiv 0$, and since $w_{\text{odd}}|_{\partial SM} \equiv 0$ we have $w_{\text{odd}} \equiv 0$. Summing up, we proved that $w \equiv 0$, which means that $u_k = 0$ for all $k \leq 0$.