SANTALÓ'S FORMULA

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0.1. **Contact geometry of the unit tangent bundle.** Let M be a smooth manifold. The cotangent bundle T^*M admits a canonical exact symplectic form $d\lambda$, which is an exact 2-form on T^*M defined by $\lambda_{(x,p)} := p \circ d\pi(x,p)$. Here $\pi : TM \to M$, $\pi(x,v) = x$ is the base projection of the tangent bundle. Let us express λ in suitable local coordinates. On tangent and cotangent bundles, the charts that one employs are usually induced by charts on the base manifold. If $x^1, ..., x^n$ are local coordinates on M, the corresponding local coordinates $x^1, ..., x^n, v^1, ..., v^n$ on TM identify the point $(x,v) \in TM$, where $x \in M$ has coordinates $(x^1, ..., x^n)$, and $v = v^1 \frac{\partial}{\partial x^1} + ... + v^n \frac{\partial}{\partial x^n} \in T_x M$. Analogously, the local coordinates $x^1, ..., x^n, p_1, ..., p_n$ on T^*M identify the point $(x,p) \in TM$, where $x \in M$ is as before, and $p = p_1\, dx^1 + ... + p_n\, dx^n \in T_x^*M$. Notice that, in these local coordinates,

$$\mathrm{d}\pi(x,v)\tfrac{\partial}{\partial x^i} = \tfrac{\partial}{\partial x^i}, \qquad \mathrm{d}\pi(x,v)\tfrac{\partial}{\partial v^i} = 0.$$

Therefore, λ and $d\lambda$ can be expressed in local coordinates as

$$\lambda = p_1\, dx^1 + ... + p_n\, dx^n, \qquad d\lambda = dp_1 \wedge dx^1 + ... + p_n \wedge dx^n.$$

The symplectic form $d\lambda$ is non-degenerate, meaning that $d\lambda(w,\cdot) \neq 0$ whenever $w \neq 0$ (more generally, a symplectic form on a manifold is a closed, non-degenerate, 2-form). Its primitive λ is called the **Liouville form** of T^*M. The maximal exterior product $(d\lambda)^n = d\lambda \wedge ... \wedge d\lambda$ is a volume form on T^*M, as can be easily seen from its expression in local coordinates

$$(d\lambda)^n = n!\big(dp_1 \wedge dx^1 + ... + dp_n \wedge dx^n\big).$$

The tangent bundle TM does not admit a canonical symplectic structure. However, a Riemannian metric g on M provides a bundle isomorphism

$$\flat : TM \to T^*M, \qquad (x,v) \mapsto (x, v^\flat),$$

where $v^\flat := g_x(v,\cdot)$. By means of this isomorphism, we can pull-back λ to a 1-form Λ on TM, which we will still call the Liouville form. Clearly, Λ depends on the Riemannian metric g, for it is given by

$$\Lambda_{(x,v)}(w) = g_x(v, d\pi(x,v)w), \qquad \forall w \in T_{(x,v)}TM.$$

In local coordinates, if we write

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j,$$

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we can write Λ and the symplectic form $d\Lambda$ as

$$\Lambda = \sum_{i,j=1}^{n} g_{ij}\, v^i dx^j, \qquad d\Lambda = \sum_{i,j=1}^{n} \Big(g_{ij}\, dv^i \wedge dx^j + v^i\, dg_{ij} \wedge dx^j \Big). \tag{0.1}$$

We now introduce the unit tangent bundle

$$SM := \big\{ (x,v) \in TM \;\big|\; g_x(v,v) = 1 \big\},$$

which is a smooth hypersurface of the tangent bundle TM. The symplectic structure of TM induces a co-called contact structure on SM. In order to explain this, consider the radial deformation $\rho_t : TM \to TM$, $\rho_t(x,v) = (x, e^t v)$, which is generated by the vector field $R := \frac{d}{dt}\big|_{t=0} \rho_t$. In local coordinates, R is given by

$$R = v^1 \tfrac{\partial}{\partial v^1} + ... + v^n \tfrac{\partial}{\partial v^n}. \tag{0.2}$$

Proposition 0.1. *We have that $\Lambda \wedge (\mathrm{d}\Lambda)^{n-1} = \frac{1}{n} R \lrcorner (\mathrm{d}\Lambda)^n$. In particular, the restriction of the Liouville form Λ to SM is a contact form, meaning that $\Lambda \wedge (d\Lambda)^{n-1}$ restricts to a volume form on SM.*

Proof. Equations (0.1) and (0.2) readily imply that $R \lrcorner d\Lambda = \Lambda$, and Cartan's formula allows to compute the Lie derivative $\mathcal{L}_R d\Lambda$ as

$$\mathcal{L}_R d\Lambda = d(R \lrcorner d\Lambda) + R \lrcorner dd\Lambda = d(R \lrcorner d\Lambda) = d\Lambda.$$

In symplectic geometry, a vector field R satisfying this latter property is called a Liouville vector field. Clearly, R is transverse to the unit tangent bundle SM. Therefore, the $(n-1)$-form $R \lrcorner (d\Lambda)^n$ restricts to a volume form on SM. Since

$$\Lambda \wedge (\mathrm{d}\Lambda)^{n-1} = (R \lrcorner d\Lambda) \wedge (\mathrm{d}\Lambda)^{n-1} = \tfrac{1}{n} R \lrcorner (d\Lambda)^n,$$

we conclude that $\Lambda \wedge (d\Lambda)^{n-1}$ restricts to a volume form on SM. ■

From now on, we will consider the Liouville form Λ and its exterior derivative $d\Lambda$ as differential forms on SM, without denoting the restriction. Since Λ is nowhere vanishing, its kernel is a vector subbundle $\ker(\Lambda) \subset TSM$ of rank $2(n-1)$. In contact geometry, $\ker(\Lambda)$ is called a contact distribution. The fact that $\Lambda \wedge (d\Lambda)^{n-1}$ is a volume form is equivalent to the requirement that $d\Lambda$ be non-degenerate on $\ker(\Lambda)$, namely, $d\Lambda(w,\cdot) \neq 0$ for all $w \in \ker(\Lambda)$. We are now ready to define the **geodesic vector field** X on SM.

Proposition 0.2. *There exists a unique vector field X on SM, called the geodesic vector field, satisfying $\Lambda(X) \equiv 1$ and $X \lrcorner d\Lambda \equiv 0$.*

Proof. Since Λ is nowhere vanishing, we can find a vector field Y on SM such that $\Lambda(Y)$ is a nowhere vanishing function. Therefore we have a splitting $TSM = \mathrm{span}\{Y\} \oplus \ker(\Lambda)$, and we can write any vector field X on SM as $X = fY + Z$, where $f : SM \to \mathbb{R}$ is a smooth function and Z takes values in $\ker(\Lambda)$. The condition $\Lambda(X) \equiv 1$ is equivalent to $f = \Lambda(Y)^{-1}$, whereas $X \lrcorner d\Lambda \equiv 0$ is equivalent to $Z \equiv 0$. ■

The flow ϕ_t of X, which is defined by the O.D.E. $\frac{d}{dt}\phi_t = X \circ \phi_t$, is called the **geodesic flow**. The following statement shows that ϕ_t is an example of a so-called strict contactomorphism.

Proposition 0.3. *For each $t \in \mathbb{R}$ and $(x,v) \in SM$ so that $\phi_t(x,v)$ is well defined, we have $(\phi_t^*\Lambda)_{(x,v)} = \Lambda_{(x,v)}$.*

Proof. On the domain of ϕ_t, we have

$$\tfrac{d}{dt}\phi_t^*\Lambda = \phi_t^*\mathcal{L}_X\Lambda = \phi_t^*(X \lrcorner d\Lambda + d(\Lambda(X))) = 0. \qquad \blacksquare$$

Let us now justify the terminology for the geodesic flow.

Proposition 0.4. *The flow lines of the geodesic flow are of the form $\phi_t(\gamma(0),\dot\gamma(0)) = (\gamma(t),\dot\gamma(t))$, where $\gamma : [0,t] \to M$ is a unit-speed geodesic of (M,g).*

Proof. We fix $(x,v) \in SM$, and consider the geodesic $\gamma(t) := \exp_x(tv)$. All we have to show is that $\dot\Gamma(0) = X(\Gamma(0))$, where Γ denotes the lifted geodesic $\Gamma(t) := (\gamma(t),\dot\gamma(t)) = (\exp_x(tv), d\exp_x(tv)v)$. One simple way to verify this is by employing the geodesic normal coordinates $x^1,...,x^n$ centered at x. Namely, we fix a g-orthonormal basis $e_1,...,e_n$ of the tangent space T_xM, and we write $(x^1,...,x^n)$ for the coordinates of the point $\exp_x(x^1e_1 + ... + x^ne_n) \in M$. If we write our Riemannian metric in these local coordinates as $g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j$, its coefficients at x satisfy $g_{ij}(x) = \delta_{ij}$ and $dg_{ij}(x) = 0$. Indeed,

$$g_{ij}(x) = g_x(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}) = g_x(d\exp_x(0)e_i, d\exp_x(0)e_j) = g_x(e_i,e_j) = 0,$$

and, by Gauss Lemma, we have

$$dg_{ij}(x)e_i = \tfrac{d}{dt}\big|_{t=0} g(d\exp(te_i)e_i, d\exp(te_i)e_j) = \tfrac{d}{dt}\big|_{t=0} g(e_i,e_j) = 0.$$

Therefore, in these coordinates, Equation (0.1) gives

$$\Lambda_{(x,v)} = \sum_{i=1}^n v^i dx^i, \qquad d\Lambda_{(x,v)} = \sum_{i=1}^n dv^i \wedge dx^i,$$

and therefore $X(\Gamma(0)) = X(x,v) = v^1\frac{\partial}{\partial x^1} + ... + v^n\frac{\partial}{\partial x^n}$. For $t \in \mathbb{R}$ sufficiently close to 0, $\Gamma(t)$ is given in local coordinates by $(tv^1,...,tv^n,v^1,...,v^n)$, and its derivative $\dot\Gamma(t)$ is thus given in local coordinates by $(v^1,...,v^n,0,...,0)$. This shows that $\dot\Gamma(0) = v^1\frac{\partial}{\partial x^1} + ... + v^n\frac{\partial}{\partial x^n}$. $\blacksquare$

0.2. **Santaló's formula.** Consider the Liouville form Λ on SM, which we introduced in Section 0.1. We denote by m_g the measure on SM obtained by integrating the contact volume form $\frac{1}{(n-1)!}\Lambda \wedge (d\Lambda)^{n-1}$, i.e.

$$\int_{SM} F(x,v)\, dm_g(x,v) = \frac{1}{(n-1)!}\int_{SM} F\,\Lambda \wedge (d\Lambda)^{n-1}, \qquad \forall F \in C^0(SM).$$

We will refer to this measure as to the **Liouville measure** on SM.

We fix an arbitrary orientation on M, and denote by vol_g the Riemannian volume form compatible with this orientation. We recall that vol_g is the unique n-form such that $\mathrm{vol}_g(e_1,...,e_n) = 1$ for each oriented orthonormal basis $e_1,...,e_n$ of a tangent space T_xM. In oriented local coordinates $x^1,...,x^n$ around $x \in M$, if we write $g = \sum_{i,j} g_{ij}dx^i \otimes dx^j$, vol_g is given by

$$\mathrm{vol}_g = \det(g_{ij})^{1/2} dx^1 \wedge ... \wedge dx^n.$$

The evaluation g_x of the Riemannian metric at some $x \in M$ gives an inner product on T_xM. We now treat the tangent space T_xM as a manifold itself, and see g_x as a flat Riemannian metric on it. If $x^1, ..., x^n, v^1, ..., v^n$ are the induced local coordinates on TM, the Riemannian volume form vol_{g_x} can be written in local coordinates as

$$\mathrm{vol}_{g_x} = \det(g_{ij}(x))^{1/2} dv^1 \wedge ... \wedge dv^n.$$

Let R be the radial vector field to SM introduced in Equation (0.2).

Lemma 0.5. *The vector field R is a unit normal to $S_xM \subset T_xM$ with respect to the Riemannian metric g_x, i.e. $\|R(x,v)\|_{g_x} = 1$ and $g_x(R(x,v), w) = 0$ for each $x \in M$ and $x \in T_v(S_xM)$.*

Proof. The fact that R has unit norm along SM follows from

$$\|R(x,v)\|_{g_x} = \|v\|_g, \qquad \forall (x,v) \in TM.$$

We fix $x \in M$, and notice that $T_v(S_xM) = \ker(r)$, where $r : S_xM \to \mathbb{R}$ is the squared norm $r(v) = \|v\|^2_{g_x}$. Since

$$dr(v)w = \tfrac{d}{dt}\big|_{t=0} \|R(x,v) + tw\|^2_{g_x} = 2g_x(R(x,v), w),$$

we conclude that R is orthogonal to S_xM. ■

We now denote by $\iota_x : S_xM \hookrightarrow T_xM$ the inclusion, and we introduced the Riemannian metric $h_x := \iota_x^* g_x$ on S_xM. Notice that, with the suitable orientation on S_xM, the Riemannian volume form associated to h_x is given by

$$\mathrm{vol}_{h_x} := R \lrcorner \mathrm{vol}_{g_x}.$$

The Riemannian measure m_g can be suitably disintegrated as follows.

Lemma 0.6. *For each $F \in C^0(SM)$, we have*

$$\int_{SM} F(x,v)\, dm_g = \int_M \left(\int_{S_xM} F(x,\cdot)\, \mathrm{vol}_{g_x} \right) \mathrm{vol}_g$$

Proof. We saw in Proposition 0.1 that $\Lambda \wedge (\mathrm{d}\Lambda)^{n-1} = \frac{1}{n} R \lrcorner (d\Lambda)^n$. By the local coordinate expression (0.1), we readily compute

$$\begin{aligned} \frac{1}{n}(d\Lambda)^n &= \frac{n!}{n} \det(g_{ij})\, dv^1 \wedge ... \wedge dv^n \wedge dx^1 \wedge ... \wedge dx^n \\ &= (n-1)! \underbrace{\det(g_{ij})^{1/2}\, dv^1 \wedge ... \wedge dv^n}_{(*)} \wedge \underbrace{\det(g_{ij})^{1/2}\, dx^1 \wedge ... \wedge dx^n}_{(**)}. \end{aligned}$$

Notice that $(*)$ and $(**)$ are the local coordinates expressions for the Riemannian volume forms vol_g and vol_{h_x} respectively. Therefore, we obtain the desired desintegration

$$\int_{SM} F(x,v)\, dm_g = \frac{1}{(n-1)!} \int_{SM} F\, \Lambda \wedge (d\Lambda)^{n-1} = \int_M \left(\int_{S_xM} F(x,\cdot)\, \mathrm{vol}_{g_x} \right) \mathrm{vol}_g.$$

■

We now consider the geodesic vector field X on SM, which was introduced with Proposition 0.2. if $\iota : \partial M \to M$ is the inclusion, we denote by $h := \iota^* g$ the induced Riemannian metric on the boundary ∂M.

Lemma 0.7. *We have the disintegration*

$$\frac{1}{(n-1)!}\int_{\partial SM} F\,(d\Lambda)^{n-1} = \int_{\partial M}\left(\int_{S_xM} F(x,\cdot)\,g(\nu(x),\cdot)\,\mathrm{vol}_{h_x}\right)\mathrm{vol}_h,$$
$$\forall F\in C^0(\partial SM).$$

Proof. Since $\Lambda(X)\equiv 1$, $X \lrcorner\, d\Lambda = 0$ and $\Lambda\wedge(\mathrm{d}\Lambda)^{n-1} = \frac{1}{n}R \lrcorner\,(d\Lambda)^n$, we have

$$(d\Lambda)^{n-1} = X \lrcorner\,\Lambda\wedge(d\Lambda)^{n-1} = \tfrac{1}{n}X \lrcorner\,(R \lrcorner\,(d\Lambda)^n). \tag{0.3}$$

We now fix $(x,v)\in\partial SM$, and proceed as in the proof of Proposition 0.4. We consider geodesic normal coordinates $x^1,...,x^n$ centered at x, and the corresponding coordinates $x^1,...,x^n,v^1,...,v^n$ on ∂SM, so that

$$X(x,v) = \sum_{i=1}^n v^i\tfrac{\partial}{\partial x^i},$$
$$R(x,v) = \sum_{i=1}^n v^i\tfrac{\partial}{\partial v^i},$$
$$\tfrac{1}{n}(d\Lambda)^n = (n-1)!\,dv^1\wedge...\wedge dv^n\wedge dx^1\wedge...\wedge dx^n.$$

In this coordinates, we can write (0.3) at (x,v) as

$$\begin{aligned}(d\Lambda)^{n-1}_{(x,v)} &= (n-1)!R(x,v) \lrcorner \big(dv^1\wedge...\wedge dv^n\big)\wedge X(x,v) \lrcorner \big(dx^1\wedge...\wedge dx^n\big)\\ &= (n-1)!\,\underbrace{R(x,v) \lrcorner \big(dv^1\wedge...\wedge dv^n\big)}_{(*)}\wedge\underbrace{\sum_{i=1}^n v^i\tfrac{\partial}{\partial x^i} \lrcorner \big(dx^1\wedge...\wedge dx^n\big)}_{(**)}.\end{aligned}$$

Here, $(*)$ is the local coordinates expression for $R \lrcorner\,\mathrm{vol}_{g_x} = \mathrm{vol}_{h_x}$, whereas $(**)$ is the local coordinates expression for $v \lrcorner\,\mathrm{vol}_g = g(\nu(x),v)\nu(x) \lrcorner\,\mathrm{vol}_g = g(\nu(x),v)\mathrm{vol}_h$. This implies the desired disintegration. ∎

Proposition 0.7 implies that $\frac{1}{(n-1)!}(d\Lambda)^n$ restricts to a volume form on the non-tangential boundary $\partial SM\setminus S\partial M$, which is positive on $\partial_{\mathrm{in}}SM$ and negative on $\partial_{\mathrm{out}}SM$. We denote by $m_{g,\nu}$ the associated measure on $\partial_{\mathrm{in}}SM$, i.e.

$$\int_{\partial_{\mathrm{in}}SM} F(x,v)\,dm_{g,\nu}(x,v) = \frac{1}{(n-1)!}\int_{\partial_{\mathrm{in}}SM} F\,(d\Lambda)^{n-1},$$
$$\forall F\in C^0(\partial_{\mathrm{in}}SM).$$

We will briefly call $m_{g,\nu}$ the **Liouville measure at the boundary**. It is worthwhile to stress the following immediate corollary of Proposition 0.7.

Corollary 0.8. *The Liouville measure at the boundary $m_{g,\nu}$ is completely determined by the values of the Riemannian metric g at the boundary of M. Namely, if g and g' are two Riemannian metrics on M such that $g_x = g'_x$ for all $x\in\partial M$, then $m_{g,\nu} = m_{g',\nu}$.*

We are finally ready to prove the following important theorem due to Santaló, which gives a further disintegration of the Liouville measure.

Theorem 0.9 (Santaló's formula)**.** *For each $F \in C^0(SM)$, we have*

$$\int_{SM} F(x,v)\, dm_g(x,v) = \int_{\partial_{\mathrm{in}} SM} I_g F(x,v)\, dm_{g,\nu}(x,v), \qquad \forall F \in C^0(SM),$$

where $I_g : C^0(SM) \to \mathbb{R}$ is the X-ray transform of (M,g), i.e.

$$I_g F(x,v) = \int_0^{\tau_g(x,v)} F \circ \phi_t(x,v)\, dt.$$

Proof. Let us make a change of variables by means of the diffeomorphism

$$\psi : U \to SM \setminus S\partial M, \qquad \psi(x,v,t) = \phi_t(x,v),$$

where $U = \big\{(x,v,t) \;\big|\; (x,v) \in \partial_{\mathrm{in}} SM,\ t \in [0, \tau_g(x,v)]\big\} \subset \partial_{\mathrm{in}} SM \times [0,\infty)$. The differential of ψ is given by

$$d\psi(x,v,t) = d\phi_t(x,v) \circ \pi_1 + X(\phi_t(x,v))dt,$$

where $\pi_1 : T_{(x,v)}SM \times T_t\mathbb{R} \to T_{(x,v)}\partial_{\mathrm{in}} SM$ is the projection onto the first factor. Since $X \lrcorner\, d\Lambda \equiv 0$, and since the geodesic flow ϕ_t preserves the Liouville form Λ (Proposition 0.3), we have

$$\begin{aligned}\psi^*(\Lambda \wedge (d\Lambda)^{n-1}) &= (\pi_1^*\phi_t^*\Lambda + \Lambda(X)\, dt) \wedge \pi_1^*\phi_t^*(d\Lambda)^{n-1}\\ &= dt \wedge \pi_1^*(d\Lambda)^{n-1}.\end{aligned}$$

This provides the desired disintegration

$$\begin{aligned}\int_{SM} F(x,v)\, dm_g &= \frac{1}{(n-1)!} \int_{\partial_{\mathrm{in}} SM} \left(\int_0^{\tau_g(x,v)} F \circ \phi_t(x,v)\, dt \right) (d\Lambda)^{n-1},\\ &= \int_{\partial_{\mathrm{in}} SM} I_g F(x,v)\, dm_{g,\nu}(x,v). \qquad \blacksquare\end{aligned}$$

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