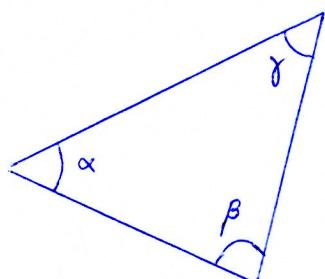


## GAUSS-BONNET THEOREM

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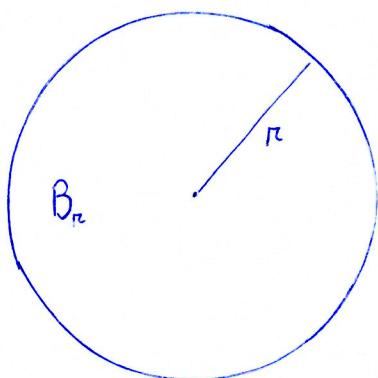
This is one of the most remarkable theorems on Riemannian surfaces. It asserts that suitable integral geometric quantities, which a priori are real valued, are integer valued.

We already know two special instances of this theorem:



In any triangle in the Euclidean plane, the sum of the interior angles is

$$\alpha + \beta + \gamma = \pi$$



For any round ball  $B$  in the Euclidean plane, we have

$$\frac{\text{length}(\partial B)}{\text{radius}(B)} = 2\pi$$

Before stating the theorem, we need some preliminaries on plane topology.

Consider an embedded, closed,  $C^1$ -curve in  $\mathbb{R}^2$ , i.e.

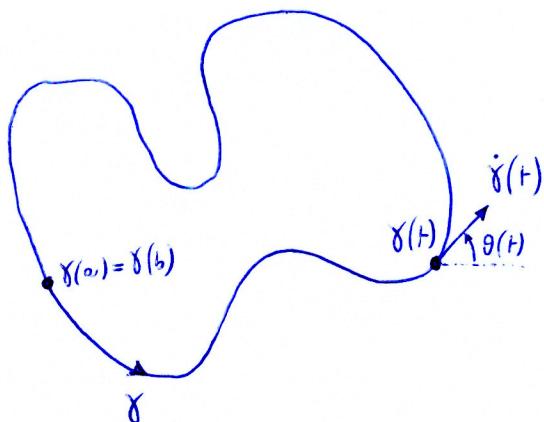
$$\gamma : [a, b] \xrightarrow{C^1} \mathbb{R}^2 \cong \mathbb{C}$$

such that  $\gamma(a) = \gamma(b)$   
 we identify  
 $\mathbb{R}^2$  with  $\mathbb{C}$   
 as usual:  
 $(x, y) \equiv x + iy$

$$\dot{\gamma}(a) = \dot{\gamma}(b)$$

$\gamma|_{(a,b)}$  is injective

$$\|\dot{\gamma}(t)\| = 1 \quad \forall t \in [a, b]$$



$$\dot{\gamma}(t) = e^{i\theta(t)},$$

where  $\theta : [a, b] \xrightarrow{C^1} \mathbb{R}$

Theorem (Hopf Umlaufsatz)

$$\theta(b) - \theta(a) = 2\pi$$

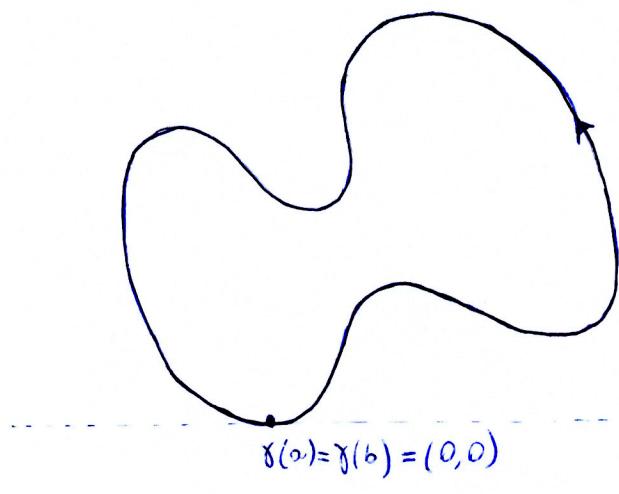
Proof.

Without loss of generality, we can assume

$$\gamma(a) = 0,$$

$$\gamma(t) \in \mathbb{R} \times [0, \infty) \quad \forall t \in [a, b]$$

$$\theta(a) = 0$$

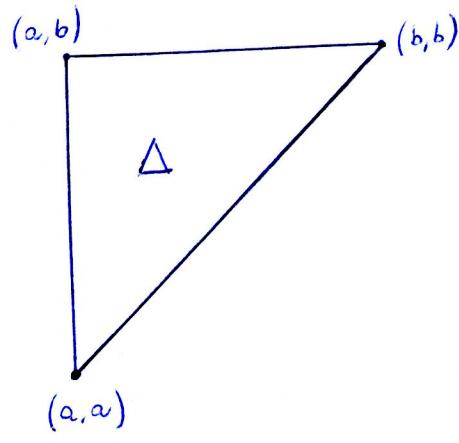


This can be achieved by shifting the curve, and by choosing its parametrization so that  $\gamma(a) = \gamma(b) = 0$  is a point of tangency with the axis  $\{y=0\}$ , while  $\gamma$  is contained in the upper half-plane  $\{y \geq 0\}$ .

$$\Delta = \{ (t_1, t_2) \in \mathbb{R}^2 \mid a \leq t_1 \leq t_2 \leq b \}$$

$$F: \Delta \xrightarrow{\text{C}} S^1 \subset \mathbb{C}$$

$$F(t_1, t_2) = \begin{cases} \gamma(t_1) & \text{if } t_1 = t_2, \\ -\gamma(a) & \text{if } t_1 = a \text{ and } t_2 = b, \\ \frac{\gamma(t_2) - \gamma(t_1)}{\|\gamma(t_2) - \gamma(t_1)\|} & \text{otherwise.} \end{cases}$$



$$p: \mathbb{R} \rightarrow S^1, \quad p(\theta) = e^{i\theta} \quad \text{universal covering map of } S^1$$

Since  $\Delta$  is simply connected (it is homeomorphic to a compact 2-ball),  
F admits a lift to the universal cover of  $S^1$ :

$$\exists \tilde{F}: \Delta \xrightarrow{\text{C}} \mathbb{R} \text{ such that } \tilde{F}(a, a) = 0, \quad p \circ \tilde{F} = F$$

By the assumptions that we made on  $\gamma$  in the first point, we have

$$\tilde{F}(a, t) \in [0, \pi] \quad \forall t \in [a, b];$$

Since  $F(a, a) = -F(a, b)$ , we must have  $\tilde{F}(a, b) = \pi$ ;

$$\tilde{F}(t, b) \in [\pi, 2\pi] \quad \forall t \in [a, b];$$

Since  $F(a, b) = -F(b, b)$ , we must have  $\tilde{F}(b, b) = 2\pi$

Notice that  $\vartheta(t) = \tilde{F}(t, t)$ , and recall that we are assuming  $\vartheta(a) = 0$ .

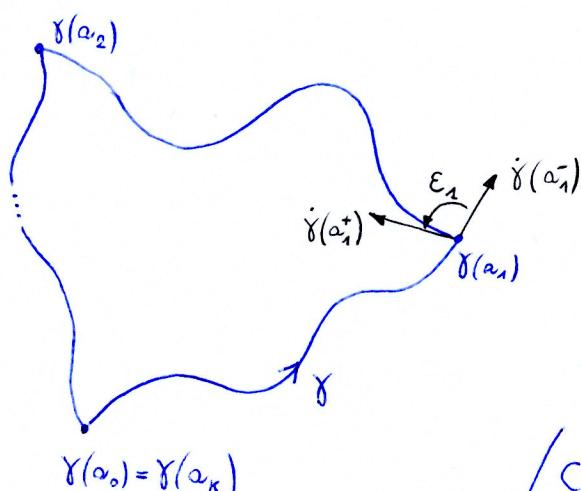
Therefore

$$\vartheta(b) - \vartheta(a) = \vartheta(b) = \tilde{F}(b, b) = 2\pi.$$

□

The theorem can be extended to piecewise- $C^1$  curves, up to adding a suitable correction:

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  as before, but only piecewise  $C^1$



$$\begin{cases} a_0 = a \\ a_k = b \end{cases}$$

(\*)

$$\dot{\gamma}(t) = e^{i\theta(t)} \quad \forall t \in [a, b] \setminus \{a_0, \dots, a_k\}$$

$\theta|_{(a_j, a_{j+1})}$  is continuous

$\epsilon_j := \text{angle between } \dot{\gamma}(a_j^-) \text{ and } \dot{\gamma}(a_j^+)$

$$\epsilon_j \in [-\pi, \pi]$$

Careful here:

- if then  $\epsilon_j = \pi$

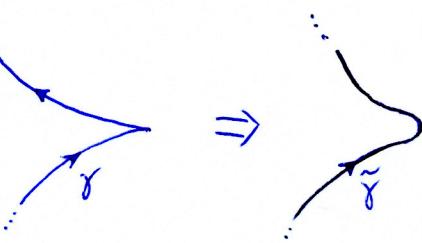
- if then  $\epsilon_j = -\pi$

Definition

$$\text{Rot}(\gamma) := \sum_{j=0}^{k-1} (\theta(a_{j+1}^-) - \theta(a_j^+) + \epsilon_{j+1})$$

Thm  $\text{Rot}(\gamma) = 2\pi$

Proof If we perturb  $\gamma$  by "rounding its corners", we obtain a  $C^1$  embedded closed curve  $\tilde{\gamma}$  such that  $\text{Rot}(\gamma) = \text{Rot}(\tilde{\gamma})$



By Hopf Umlaufatz, we know that  $\text{Rot}(\tilde{\gamma}) = 2\pi$

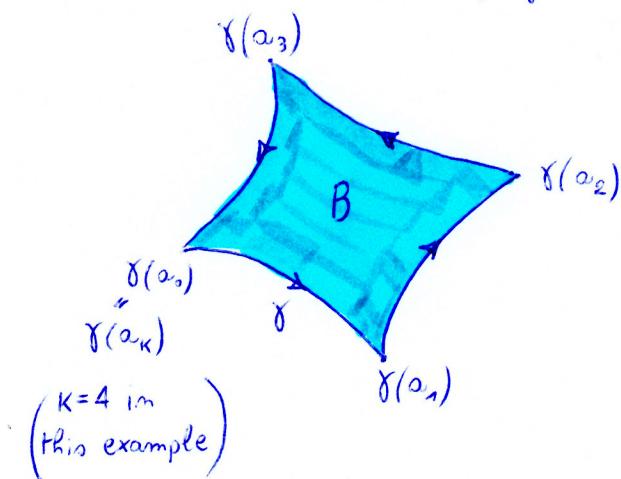
□

The same result holds for polygonal curves in Riemannian surfaces:

Let  $(M, g)$  be an oriented Riemannian surface ( $\dim M = 2$ )

$B \subset M$  compact subset with piecewise  $C^1$  boundary  $\gamma$ , and diffeomorphic to a compact 2-ball

$\gamma$  is oriented as the boundary of  $B$ , with the counterclockwise convention:



we parametrize  $\gamma$  so that

$$\gamma: [a_0, a_K] \rightarrow \partial B$$

$\gamma|_{(a_j, a_{j+1})}$  smooth embedding  
 $\forall j = 0, \dots, K-1$

Since  $B$  is diffeomorphic to a ball, it is contained in the domain of an oriented chart, which gives local coordinates  $x, y$  on  $B$

We choose a function  $\theta: [a_0, a_K] \rightarrow \mathbb{R}$  such that  $\theta|_{(a_j, a_{j+1})}$  in  $C^1$   $\forall j$

$$\text{and } g\left(\dot{\gamma}(t), \frac{\partial}{\partial x}\right) = \|\dot{\gamma}(t)\|_g \cdot \left\|\frac{\partial}{\partial x}\right\|_g \cos \theta(t) \quad \forall t \in [a_0, a_K]$$

(i.e.  $\theta(t)$  is the oriented angle from  $\frac{\partial}{\partial x}$  to  $\dot{\gamma}(t)$ )

$\varepsilon_j \in [-\pi, \pi]$  oriented angle from  $\dot{\gamma}(a_j^-)$  to  $\dot{\gamma}(a_j^+)$

$$(i.e. g\left(\dot{\gamma}(a_j^-), \dot{\gamma}(a_j^+)\right) = \|\dot{\gamma}(a_j^-)\|_g \cdot \|\dot{\gamma}(a_j^+)\|_g \cos \varepsilon_j)$$

$\varepsilon_j > 0$  if  $\dot{\gamma}(a_j^-), \dot{\gamma}(a_j^+)$  is oriented basis of  $T_{\gamma(a_j)} M$

$\varepsilon_j < 0$  "  $-\dot{\gamma}(a_j^-), \dot{\gamma}(a_j^+)$  " " " "

if  $\dot{\gamma}(a_j^-) = -\dot{\gamma}(a_j^+)$  then the sign of  $\varepsilon_j$  is established as in (\*) on page 12

$$\underline{\text{Definition}} \quad \text{Rot}_g(\gamma) = \sum_{j=1}^{k-1} (\vartheta(t_{j+1}) - \vartheta(t_j) + \varepsilon_{j+1})$$

$$\underline{\text{Proposition}} \quad \text{Rot}_g(\gamma) = 2\pi$$

Proof.

- Since  $\text{Rot}_g(\gamma)$  is the "total rotation angle" of the closed curve  $\gamma$ , we must have  $\text{Rot}_g(\gamma) \in 2\pi\mathbb{Z} = \{2\pi k \mid k \in \mathbb{Z}\}$
- Consider the Euclidean metric on  $B$  defined by means of the local coordinates  $x, y$

$$g_0 := dx \otimes dx + dy \otimes dy$$

By Hopf Umlaufsatz, we have  $\text{Rot}_{g_0}(\gamma) = 2\pi$

- $\forall s \in [0, 1] \quad g_s := sg + (1-s)g_0$  is a Riemannian metric on  $B$
- $f: [0, 1] \rightarrow \mathbb{R} \quad f(s) = \text{Rot}_{g_s}(\gamma)$  is clearly a continuous function

Since  $f(s) \in 2\pi\mathbb{Z} \quad \forall s \in [0, 1]$ , and since  $2\pi\mathbb{Z}$  is a discrete set,  $f$  must be constant. Therefore:

$$\text{Rot}_g(\gamma) = f(1) = f(0) = \text{Rot}_{g_0}(\gamma) = 2\pi.$$

□