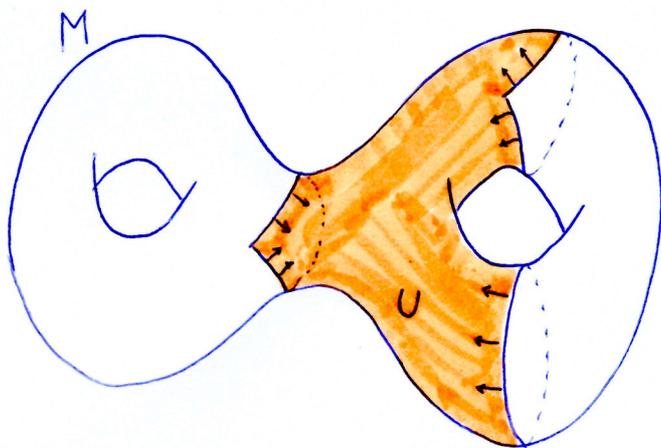


GAUSS-BONNET THEOREM 2/2

(M, g) oriented Riemannian surface

$U \subset M$ open subset with piecewise smooth boundary ∂U

N unit normal vector field to ∂U , pointing inside U



N is only defined at those $x \in \partial U$ where ∂U is smooth,

$$N(x) \perp_g T_x \partial U$$

$$\|N(x)\|_g = 1$$

$N(x)$ points inside U

$\gamma \subset \partial U$ connected component

we fix a parametrization for γ by arclength:

$$\gamma: \mathbb{R}/\ell\mathbb{Z} \hookrightarrow M \text{ piecewise smooth embedding}$$

$$\|\dot{\gamma}(t)\|_g = 1 \quad \forall t \text{ where } \gamma \text{ is smooth}$$

$$(\ell = \text{length}(\gamma))$$

We say that γ is POSITIVELY ORIENTED as boundary of U when

$\dot{\gamma}(t), N(\gamma(t))$ is an oriented basis of $T_{\gamma(t)}M \quad \forall t$ where γ is smooth

from now on we just write $N(t)$

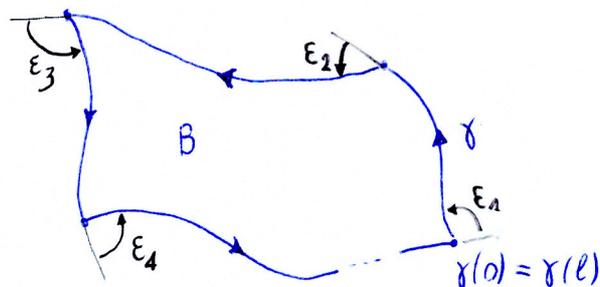
Definition The SIGNED CURVATURE of γ is $K_N(t) := g(\nabla_t \dot{\gamma}, N(t))$

Remark $\nabla_t \dot{\gamma} = K_N(t) N(t)$ (indeed $g(\nabla_t \dot{\gamma}, \dot{\gamma}) = \frac{1}{2} \frac{d}{dt} \|\dot{\gamma}(t)\|_g^2 = 0$)

therefore γ is a piecewise geodesic if and only if $K_N \equiv 0$

Theorem (GAUSS-BONNET FORMULA)

If $B \subset M$ is an open 2-ball with piecewise smooth, positively oriented boundary $\gamma = \partial B$ with angles $\epsilon_1, \dots, \epsilon_r \in [-\pi, \pi]$



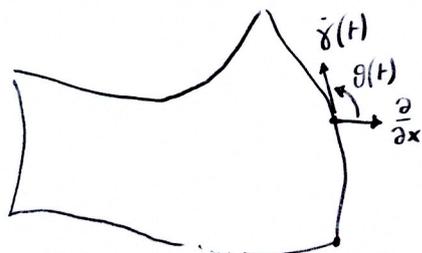
Then

$$\int_B K_g \text{vol}_g + \int_0^l K_N(t) dt + \sum_{i=1}^r \epsilon_i = 2\pi$$

(where $K_g: M \rightarrow \mathbb{R}$ is the Gaussian curvature of (M, g))

Proof.

- $[0, l] = I_1 \cup I_2 \cup \dots \cup I_r$ such that $\gamma|_{I_i}$ is smooth



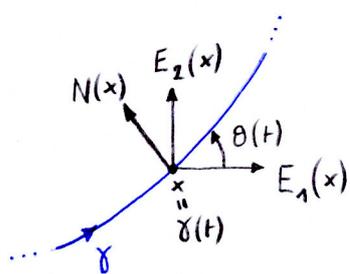
we fix local coordinates x, y in a neighborhood of B , and define the angle $\vartheta(t) \forall t$ where γ is smooth

We already proved that $2\pi = \text{Rot}_g(\gamma) = \sum_{i=1}^r \epsilon_i + \sum_{i=1}^r \int_{I_i} \dot{\vartheta}(t) dt$ (*)

- We define two vector fields E_1, E_2 on \bar{B} :

$$E_1 := \frac{\partial_x}{\|\partial_x\|_g}, \quad E_2 \text{ positive orthonormal to } E_1$$

$$\left(\begin{array}{l} \|E_1\|_g = \|E_2\|_g \equiv 1, \quad g(E_1, E_2) \equiv 0 \\ E_1(x), E_2(x) \text{ oriented basis of } T_x M, \quad \forall x \in \bar{B} \end{array} \right)$$



$$\dot{\gamma}(t) = \cos \theta(t) E_1(\gamma(t)) + \sin \theta(t) E_2(\gamma(t))$$

$$N(t) = -\sin \theta(t) E_1(\gamma(t)) + \cos \theta(t) E_2(\gamma(t))$$

$$\begin{aligned} \Rightarrow \nabla_t \dot{\gamma} &= -(\sin \theta(t) \dot{\theta}(t) E_1 + (\cos \theta(t) \dot{\theta}(t) E_2 \\ &\quad + \cos \theta(t) \nabla_{\dot{\gamma}} E_1 + \sin \theta(t) \nabla_{\dot{\gamma}} E_2 \\ &= \dot{\theta}(t) N(t) + \cos \theta(t) \nabla_{\dot{\gamma}} E_1 \\ &\quad + \sin \theta(t) \nabla_{\dot{\gamma}} E_2 \end{aligned}$$

- We define a 1-form ω on \bar{B} by

$$\omega(X) := -g(\nabla_X E_1, E_2) = g(E_1, \nabla_X E_2)$$

since

$$0 = X g(E_1, E_2) = g(\nabla_X E_1, E_2) + g(E_1, \nabla_X E_2)$$

Since $0 = X \|E_i\|_g^2 = 2 g(\nabla_X E_i, E_i)$, we have

$$\nabla_X E_1 = -\omega(X) E_2$$

$$\nabla_X E_2 = \omega(X) E_1$$

- We compute the curvature of γ :

$$\begin{aligned}
 K_N(t) &= g(\nabla_t \dot{\gamma}, N(t)) = \dot{\vartheta}(t) g(N(t), N(t)) + \cos \vartheta(t) g(\nabla_{\dot{\gamma}} E_1, N(t)) \\
 &\quad + \sin \vartheta(t) g(\nabla_{\dot{\gamma}} E_2, N(t)) \\
 &= \dot{\vartheta} - \underbrace{\cos \vartheta(t) \cdot \omega(\dot{\gamma})}_{\cos \vartheta(t)} g(E_2, N) + \underbrace{\sin \vartheta(t) \cdot \omega(\dot{\gamma})}_{-\sin \vartheta(t)} g(E_1, N) \\
 &= \dot{\vartheta}(t) - \omega(\dot{\gamma}(t))
 \end{aligned}$$

Therefore, formula (*) on page 16 becomes:

$$2\pi = \sum_{i=1}^k \varepsilon_i + \int_0^{\ell} K_N(t) dt + \underbrace{\int_0^{\ell} \omega(\dot{\gamma}(t)) dt}_{\int_{\gamma} \omega = \int_B d\omega}$$

Stokes

We are left to show that $d\omega = K_g \text{vol}_g$:

$$K_g \text{vol}_g(E_1, E_2) = K_g = R(E_1, E_2, E_2, E_1)$$

$$= g\left(\underbrace{\nabla_{E_1} \nabla_{E_2} E_2}_{\omega(E_2)E_1} - \underbrace{\nabla_{E_2} \nabla_{E_1} E_2}_{\omega(E_1)E_1} - \underbrace{\nabla_{[E_1, E_2]} E_2}_{\omega([E_1, E_2])E_1}, E_1\right)$$

$$= E_1 \omega(E_2) - E_2 \omega(E_1) - \omega([E_1, E_2]) = d\omega(E_1, E_2)$$

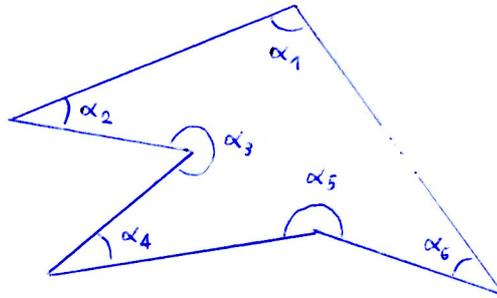
since $g(\nabla_x E_1, E_1) = 0$

□

Remark

Gauss-Bonnet formula readily implies a classical formula from plane geometry:

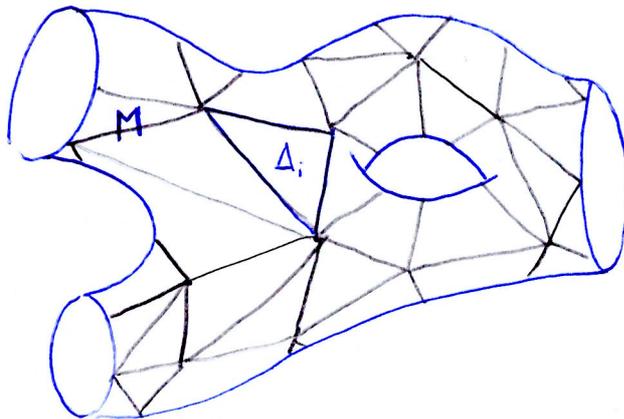
\forall m -gon in the Euclidean \mathbb{R}^2 with inner angles $\alpha_1, \dots, \alpha_m$, we have $\alpha_1 + \dots + \alpha_m = (m-2)\pi$



(apply Gauss-Bonnet's formula, with $K_g \equiv 0$, $K_N \equiv 0$, $\alpha_i = \pi - \epsilon_i$)

We now wish to extend Gauss-Bonnet's formula to arbitrary compact oriented surfaces with boundary. For this, we need two notions from topology:

M compact surface, possibly with boundary ∂M

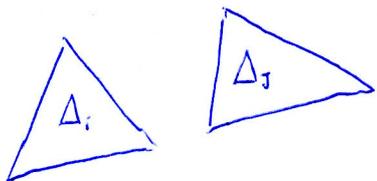


A TRIANGULATION of M is a \dots of M as $M = \bigcup_{i=1}^n \Delta_i$

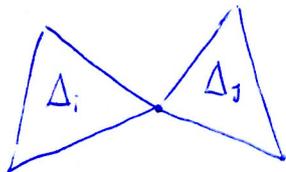
where each Δ_i is a compact embedded triangle (i.e. $\partial \Delta_i$ is smooth except at 3 points, called vertices), and

$$\Delta_i \cap \Delta_j = \begin{cases} \emptyset \\ \text{or} \\ \text{a vertex} \\ \text{or} \\ \text{an edge} \end{cases}$$

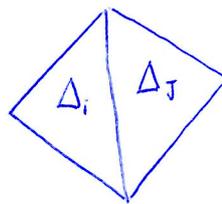
empty intersection



intersection on a vertex



intersection on an edge



Remark The boundary ∂M must be a union of edges of some triangles

Theorem Every compact surface (with or without boundary) admits a triangulation

(we do not prove it)

Definition The EULER CHARACTERISTIC of M is

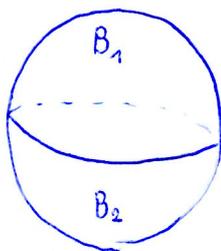
$$\chi(M) = \# \text{ vertices} - \# \text{ edges} + \# \text{ faces}$$

Thm $\chi(M)$ is independent of the chosen triangulation

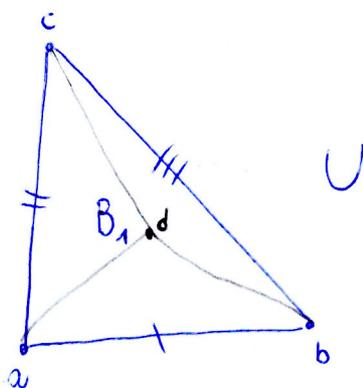
(we do not prove it)

example

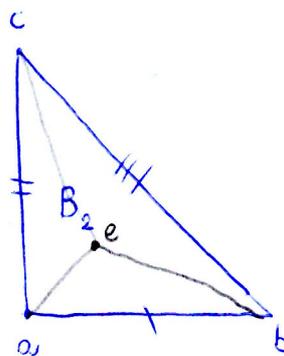
• S^2



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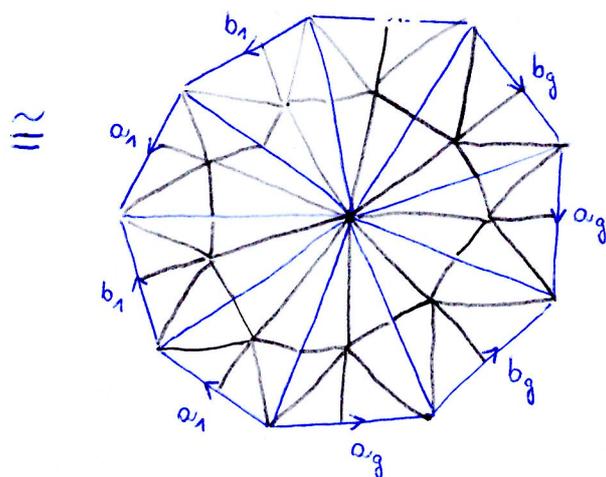
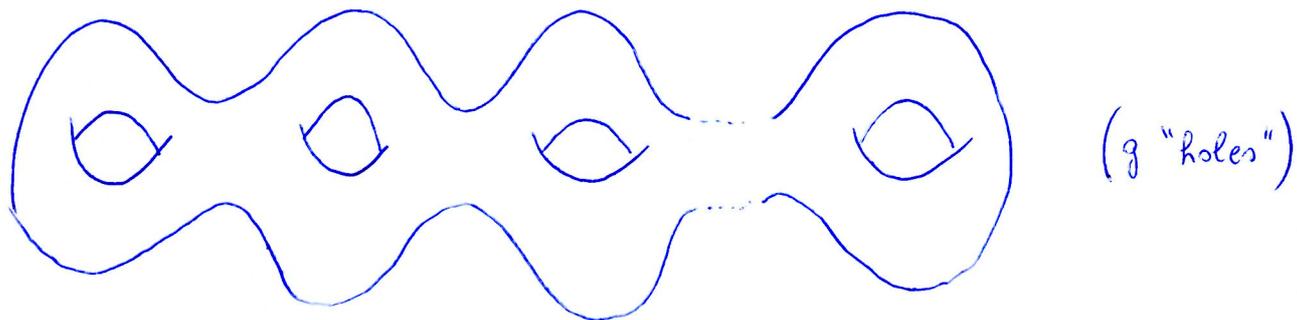


\cup



$$\chi(S^2) = \underset{\text{vertices}}{5} - \underset{\text{edges}}{9} + \underset{\text{faces}}{6} = 2$$

- $\Sigma_g =$ closed surface of genus g



$$\chi(\Sigma_g) = \underbrace{2 + 4g + \frac{5 \cdot 4g}{2}}_{\text{vertices}} - \underbrace{4g \cdot 6 - \frac{4g \cdot 6}{2}}_{\text{- edges}} + \underbrace{4g \cdot 6}_{\text{faces}}$$

$$= 2 - 2g$$

exercise Compute the Euler characteristic of compact surfaces with boundary

Theorem (Gauss-Bonnet)

If (M, g) is a closed oriented surface, then $\int_M K_g \text{vol}_g = 2\pi \chi(M)$

Proof

Fix a triangulation of M with $f > 0$ triangles $\Delta_1, \dots, \Delta_f$

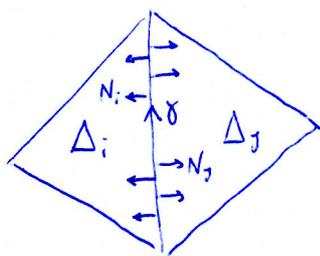
Every edge is shared by two triangles; therefore

$$e = \# \text{ edges} = \frac{3}{2} f$$

Δ_i has inner angles $\vartheta_{i,1}, \vartheta_{i,2}, \vartheta_{i,3}$; therefore

$$\sum_{i=1}^f (\vartheta_{i,1} + \vartheta_{i,2} + \vartheta_{i,3}) = 2\pi v \quad v = \# \text{ vertices}$$

$$\int_M K_g \text{vol}_g = \sum_{i=1}^f \int_{\Delta_i} K_g \text{vol}_g \stackrel{\substack{\uparrow \\ \text{Gauss-Bonnet} \\ \text{formula}}}{=} \sum_{i=1}^f \left(\underbrace{- \int_{\partial \Delta_i} K_{N_i}(t) dt}_{\text{this is an abuse of notation}} - (\pi - \vartheta_{i,1}) - (\pi - \vartheta_{i,2}) - (\pi - \vartheta_{i,3}) + 2\pi \right)$$



$$\int_{\gamma} K_{N_i}(t) dt = - \int_{\gamma} K_{N_j}(t) dt$$

This shows that $\sum_{i=1}^f \int_{\partial \Delta_i} K_{N_i}(t) dt = 0$

$$\Rightarrow \int_M K_g \text{vol}_g = 2\pi v - \underbrace{3\pi f}_{2\pi e} + 2\pi f = 2\pi \chi(M). \quad \square$$

Rmk The Gaussian curvature puts a constraint on the topology of M ,
as a consequence of Gauss-Bonnet theorem:

if $K_g \equiv 0$ then $\chi(M) = 0$, thus $M = \mathbb{T}^2$

if $K_g > 0$ then $2\pi\chi(M) > 0$, thus $\chi(M) = 2$, $M = S^2$

if $K_g < 0$ then $\frac{2\pi\chi(M)}{2-2g} < 0$, $M = \Sigma_g$ with $g \geq 2$

exercise

Extend Gauss-Bonnet theorem to the case of M compact surface
with piecewise smooth boundary.