

## THE HESSIAN OF THE LENGTH FUNCTIONAL

$(M, g)$  Riemannian manifold

We know that the geodesic segments  $\gamma: [a, b] \rightarrow M$  are critical points of the length functional  $L$  on the space of piecewise smooth paths joining  $\gamma(a)$  and  $\gamma(b)$ .

When is  $\gamma$  a local minimum of  $L$ ? In order to address this question, let us compute the Hessian of  $L$  at  $\gamma$ :

Without loss of generality, we assume that  $\|\dot{\gamma}(t)\|_g = 1 \quad \forall t \in [a, b]$ .

Let  $\Gamma: (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$  be a variation of  $\gamma$ :

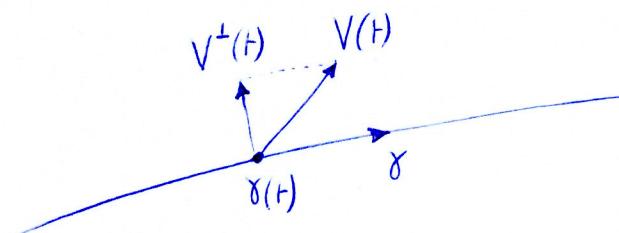
(we write  $\Gamma_n(t) = \Gamma(n, t)$ )

$\Gamma_0 = \gamma, \quad \Gamma_0(a) = \gamma(a), \quad \Gamma_0(b) = \gamma(b) \quad \forall n \in (-\varepsilon, \varepsilon)$

$a = t_0 < t_1 < \dots < t_K = b, \quad \Gamma|_{(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]} \text{ is } C^\infty \quad \forall i = 0, \dots, K-1$

$V(t) = \partial_n \Gamma(n, t)|_{n=0}$  piecewise smooth vector field along  $\gamma$ ;  $V(a) = 0$   
 $V(b) = 0$

$V^\perp(t) = V(t) - g(V(t), \dot{\gamma}(t)) \dot{\gamma}(t) \quad (\text{orthogonal to } \dot{\gamma})$



$$\underline{\text{Proposition}} \quad \left. \frac{d^2}{ds^2} \right|_{s=0} L(\Gamma_s) = \int_a^b \left( |\nabla_t V^\perp|^2 - R(V^\perp, \dot{V}, \dot{V}, V^\perp) \right) dt$$

Proof

- $T(s, t) = \partial_t \Gamma(s, t)$ ,  $S(s, t) = \partial_s \Gamma(s, t)$  so that  $V(t) = S(0, t)$
- we already know that  $\nabla_s T = \nabla_t S$  and

$$\frac{d}{ds} L(\Gamma_s) = \int_a^b \frac{d}{ds} \|T(s, t)\|_g dt = \int_a^b \frac{1}{2\|T\|_g} 2g(\underbrace{\nabla_s T}_\nabla, T) dt$$

$$= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \frac{g(\nabla_t S, T)}{\|T\|_g} dt$$

- $\nabla_s \nabla_t \partial_{x^i} - \nabla_t \nabla_s \partial_{x^i} = R(S, T)$

This can be proved in local coordinates:  $x^1, \dots, x^m$ :

$$\nabla_s \nabla_t \partial_{x^i} = \nabla_s \left( \frac{\partial \Gamma^j}{\partial t} \nabla_{\partial_{x^j}} \partial_{x^i} \right) = \frac{\partial^2 \Gamma^j}{\partial s \partial t} \nabla_{\partial_{x^j}} \partial_{x^i} + \frac{\partial \Gamma^k}{\partial s} \frac{\partial \Gamma^j}{\partial t} \nabla_{\partial_{x^k}} \nabla_{\partial_{x^j}} \partial_{x^i}$$

analogous computation for  $\nabla_t \nabla_s \partial_{x^i}$

$$\begin{aligned} \nabla_s \nabla_t \partial_{x^i} - \nabla_t \nabla_s \partial_{x^i} &= \frac{\partial \Gamma^k}{\partial s} \frac{\partial \Gamma^j}{\partial t} \underbrace{\left( \nabla_{\partial_{x^k}} \nabla_{\partial_{x^j}} \partial_{x^i} - \nabla_{\partial_{x^j}} \nabla_{\partial_{x^k}} \partial_{x^i} \right)}_{R(\partial_{x^k}, \partial_{x^j}) \partial_{x^i}} \\ &= R(S, T) \partial_{x^i} \end{aligned}$$

$$\frac{d^2}{ds^2} L(\Gamma_s) \Big|_{s=0} = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left( \frac{\nabla_t \nabla_s S + R(S, T)S}{\|T\|_g} \right)$$

$$+ \left( \frac{d}{ds} \frac{1}{\|T\|_g} \right) g(\nabla_t S, T) \Big|_{s=0} dt$$

$$= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left( \frac{\frac{d}{dt} g(\nabla_s S, T) - g(\nabla_s S, \nabla_t T)}{\|T\|_g} + R(S, T, S, T) + \|\nabla_t S\|_g^2 \right)$$

$$- \frac{1}{2\|T\|_g^3} 2 g(\nabla_s T, T) g(\nabla_t S, T) \Big|_{s=0} dt$$

$$\nearrow = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left( \underbrace{-R(V, \dot{\gamma}, \dot{\gamma}, V)}_{R(V^\perp, \dot{\gamma}, \dot{\gamma}, V^\perp)} + \underbrace{\|\nabla_t V\|_g^2}_{\|\nabla_t V\|^2} - g(\nabla_t V, \dot{\gamma})^2 \right) dt$$

$$\|T(0, t)\|_g = 1$$

$$\nabla_t T \Big|_{s=0} = \nabla_t \dot{\gamma} = 0 \quad (\text{since } R(w, w, \cdot, \cdot) = 0)$$

□

$$I_a^b(V, V) := \frac{d^2}{ds^2} \Big|_{s=0} L(\Gamma_s) \quad (*)$$

$I_a^b$  is a symmetric bilinear form on the space of piecewise smooth vector fields  $V$  along  $\gamma$  such that  $V(a) = 0$ ,  $V(b) = 0$ ,  $g(V(t), \dot{\gamma}(t)) = 0 \forall t \in [a, b]$ , and it can be found by applying the polarization identity to the quadratic form  $(*)$

$$I_a^b(V, W) = \frac{1}{2} (I_a^b(V+W, V+W) - I_a^b(V, V) - I_a^b(W, W))$$

$$\text{where } V = V^\perp, W = W^\perp$$

$$\Rightarrow I_a^b(V, W) = \int_a^b \left( g(\nabla_t V, \nabla_t W) - R(V, \dot{V}, \dot{W}) \right) dt$$

(We recall the notion of Kernel of a bilinear form:

$$\text{Ker } I_a^b = \{ V \mid I_a^b(V, W) = 0 \ \forall W \}$$

Definition  $\gamma(a)$  and  $\gamma(b)$  are conjugate points along  $\gamma|_{[a,b]}$  when  $\text{Ker } I_a^b \neq \{0\}$

Proposition  $V \in \text{Ker } I_a^b$  if and only if it is a smooth solution of

$$\begin{cases} \nabla_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0 & (*) \\ g(V, \dot{\gamma}) = 0 & (**) \\ V(a) = 0, \quad V(b) = 0 & (***) \end{cases}$$

(The solutions of the linear ODE (\*) are called JACOBI FIELDS)

Proof. By assumption, we only consider vector fields  $V$  that are piecewise smooth. If  $V$  is smooth on every sub-interval  $[t_i, t_{i+1}]$ , where  $a = t_0 < t_1 < \dots < t_{k-1} = b$ , an integration by parts gives

$$\begin{aligned} I_a^b(V, W) &= \sum_{i=0}^{k-1} \left( g(\nabla_t V|_{t=t_i^-} - \nabla_t V|_{t=t_i^+}, W(t_i)) \right. \\ &\quad \left. - \int_{t_i}^{t_{i+1}} g(\nabla_t^2 V + R(V, \dot{\gamma})\dot{\gamma}, W) dt \right) \end{aligned}$$

If  $V$  is a solution of (\*), (\*\*), (\*\*\*) , this expression readily implies that  $V \in \text{Ker } I_a^b$

Assume now that  $V \in \text{Ker } I_a^b$ . Notice that

$$g(\nabla_t^2 V + R(V, \dot{\gamma})\dot{\gamma}, \dot{\gamma}) = g(\nabla_t^2 V, \dot{\gamma}) = \underbrace{\frac{d}{dt} g(V, \dot{\gamma})}_{\equiv 0} \equiv 0 \quad (\text{A})$$

For every  $W$  such that  $W(t_i) = 0 \quad \forall i = 0, \dots, K$ , we have

$$0 = I_a^b(V, W) = - \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} g(\nabla_t^2 V + R(V, \dot{\gamma})\dot{\gamma}, W) dt \quad (\text{B})$$

(A) and (B) imply that

$$\nabla_t^2 V + R(V(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0 \quad \forall t \in (t_i, t_{i+1}), i = 0, \dots, K-1$$

We are left to show that  $V$  is everywhere  $C^1$ .

For every  $Z$  such that  $Z(t_j) = 0 \quad \forall j \neq i$ , we have

$$0 = I_a^b(V, Z) = g(\nabla_t V|_{t=t_i^-} - \nabla_t V|_{t=t_i^+}, Z(t_i))$$

Since this must hold for every value of  $Z(t_i)$ , we have

$$\nabla_t V|_{t=t_i^-} = \nabla_t V|_{t=t_i^+}$$

Therefore  $V$  is  $C^1$ , and since it is the solution of the ODE

$$\nabla_t^2 V = -R(V, \dot{\gamma})\dot{\gamma}$$

by bootstrapping, we conclude that  $V$  is  $C^\infty$

if  $V$  is  $C^1$ , then  $\nabla_t^2 V$  is  $C^0$  as well,

and therefore  $V$  is  $C^{1+2}$



Corollary If  $(M, g)$  has negative sectional curvature

(i.e.  $K(\Pi) < 0 \quad \forall x \in M, \Pi \subset T_x M$  2-plane)

then every geodesic is "locally length minimizing":

[if  $V$  vector field along  $\gamma$  such that  $V(a) = 0, V(b) = 0$   
and  $g(V, \dot{\gamma}) \equiv 0$ , then  
 $\gamma_s(t) := \exp_{\gamma(t)}(sV(t))$   
 $\frac{d^2}{ds^2} \Big|_{s=0} L(\gamma_s) \geq 0, \quad = 0 \text{ if and only if } V \equiv 0$

Proof (Assume, as before,  $\|\dot{\gamma}\|_g \equiv 1$ )

$$\frac{d^2}{ds^2} \Big|_{s=0} L(\gamma_s) = I_a^b(V, V) = \int_a^b \left( \|\nabla_t V\|_g^2 - R(V, \dot{\gamma}, \dot{\gamma}, V) \right) dt$$

$$= \int_a^b \left( \|\nabla_t V\|_g^2 - K(\text{span}\{V, \dot{\gamma}\}) \cdot (\|V\|_g^2 - g(V, \dot{\gamma})^2) \right) dt$$

$$\geq \int_a^b \|\nabla_t V\|_g^2 dt \geq 0, \quad \text{and} = 0 \text{ if and only if } \nabla_t^2 V \equiv 0, \\ \text{thus if and only if } V \equiv 0.$$

□

If you are curious, there is a lot more to learn concerning

Jacobi fields and conjugate points. Check out, for instance,

John Milnor, Morse Theory, Princeton University Press

This is all for us. Please take care and see you soon!

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