

SURFACES EMBEDDED IN THE EUCLIDEAN 3-SPACE

$$(M, g) \xrightarrow{i} (\tilde{M}, \tilde{g}) \quad \text{with} \quad \dim M = 2$$

$$g = i^* \tilde{g}$$

$$\tilde{M} = \mathbb{R}^3$$

$\tilde{g} = \langle \cdot, \cdot \rangle$ Euclidean metric

Recall: $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is flat, i.e. $\tilde{R} = 0$

$$\tilde{\nabla}_Y (X^j \frac{\partial}{\partial x^j}) = Y(X^j) \frac{\partial}{\partial x^j}$$

Let us assume that there exists a unit normal vector field N on M , i.e.

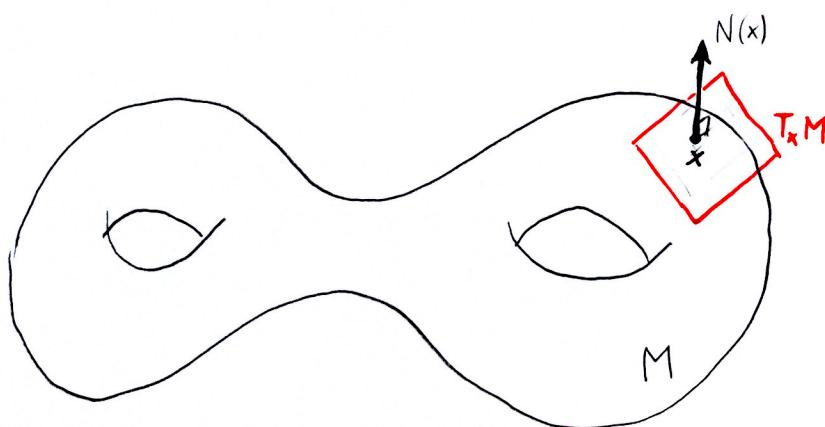
$$N \in \Gamma(NM)$$

$$\|N\| = \sqrt{\langle N, N \rangle} = 1$$

$$\langle N, X \rangle = 0 \quad \forall X \text{ vector field on } M$$

(we can always find such a N locally around any given point of M)

e.g.



Definition GAUSS MAP $G_x : T_x M \xrightarrow{\text{linear}} T_x M$

$$V \longmapsto \tilde{\nabla}_V N = dN^T(V) \frac{\partial}{\partial x^j}$$

in the Euclidean
coordinates of
 $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$

Rmk $\tilde{\nabla}_V N \in T_x M \quad \forall x \in M, V \in T_x M$, indeed

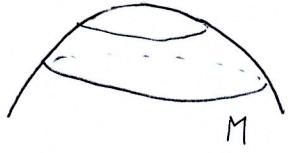
$$\langle \tilde{\nabla}_V N, N \rangle = \sqrt{\underbrace{\|N\|^2}_{=1}} - \langle N, \tilde{\nabla}_V N \rangle = -\langle N, \tilde{\nabla}_V N \rangle,$$

$$\text{thus } \langle \tilde{\nabla}_V N, N \rangle = 0$$

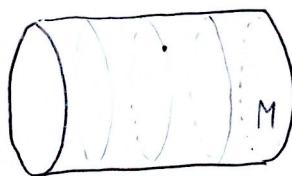
GAUSSIAN CURVATURE of (M, g) : $K_g : M \rightarrow \mathbb{R}$

$$K_g(x) := \det(G_x)$$

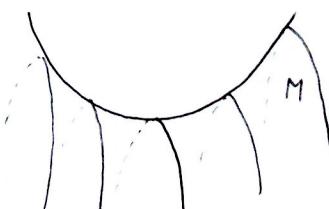
K_g describes how the surface M is curved inside \mathbb{R}^3 :



$$K_g > 0$$



$$K_g = 0$$



$$K_g < 0$$

The definition of the Gaussian curvature K_g depends on the embedding $M \hookrightarrow \mathbb{R}^3$ (since the Gauss maps G_x depend on the unit normal vector field N). Nevertheless, K_g turns out to only depend on the Riemannian metric $g = i^* \tilde{g}$!!!

This is the content of a celebrated theorem due to Gauss. The statement was so striking that Gauss called it "Theorema Egregium" (Latin for "Remarkable theorem")

Theorem of Egnegium

Let $(M, g) \hookrightarrow (\mathbb{R}^3, \langle \cdot, \cdot \rangle = \tilde{g})$ be a Riemannian surface embedded in the Euclidean 3-space (with $g = i^* \tilde{g}$). Then

$$K_g(x) = \underbrace{K(T_x M)}_{\text{sectional curvature of } (M, g)} \quad \forall x \in M$$

In particular, K_g only depends of g : if $\jmath: M \hookrightarrow \mathbb{R}^3$ is another embedding such that $\jmath^* \tilde{g} = i^* \tilde{g} = g$, then we can employ the Gauss map associated to \jmath in order to compute K_g .

Proof:

Let E_1, E_2 be an orthonormal frame defined on a small neighborhood U of any given point $x \in M$, i.e.

$$\|E_1\|_g = \|E_2\|_g \equiv 1, \quad g(E_1, E_2) \equiv 0 \quad \text{on } U$$

We write the Gauss map in this basis:

$$G_x: T_x M \rightarrow T_x M$$

$$G_x = \begin{pmatrix} \langle \tilde{\nabla}_{E_1} N, E_1 \rangle & \langle \tilde{\nabla}_{E_1} N, E_2 \rangle \\ -\langle \tilde{\nabla}_{E_1} N, E_1 \rangle & \langle \tilde{\nabla}_{E_2} N, E_2 \rangle \end{pmatrix} = \begin{pmatrix} -\langle N, \tilde{\nabla}_{E_1} E_1 \rangle & -\langle N, \tilde{\nabla}_{E_1} E_2 \rangle \\ -\langle N, \tilde{\nabla}_{E_2} E_1 \rangle & -\langle N, \tilde{\nabla}_{E_2} E_2 \rangle \end{pmatrix}$$

$$\begin{aligned} K_g(x) &= \det(G_x) = \langle N, \tilde{\nabla}_{E_1} E_1 \rangle \langle N, \tilde{\nabla}_{E_2} E_2 \rangle - \langle N, \tilde{\nabla}_{E_2} E_1 \rangle \langle N, \tilde{\nabla}_{E_1} E_2 \rangle |_x \\ &= (\langle \mathcal{II}(E_1, E_1), \mathcal{II}(E_2, E_2) \rangle - \langle \mathcal{II}(E_2, E_1), \mathcal{II}(E_1, E_2) \rangle) |_x \\ &= R(E_1, E_2, E_2, E_1) |_x = K(T_x M). \end{aligned}$$

formula (*)
on page 3

□

Remark

On any Riemannian surface (M, g) , the Gaussian curvature determines every other curvature tensor:

$$\text{Riemann tensor} \quad R(X, Y, Z, W) = K_g \cdot (g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \quad (\text{exercise})$$

$$\text{Ricci tensor} \quad \text{Ric}(X, Y) = \sum_{i=1,2} R(X, E_i, E_i, Y) =$$

$$= K_g \cdot \left(\underbrace{\sum_{i=1,2} g(E_i, E_i) g(X, Y)}_2 - \underbrace{\sum_{i=1,2} g(X, E_i) g(Y, E_i)}_{g(X, Y)} \right)$$

$$= K_g \cdot g(X, Y) \quad \Rightarrow \text{Every Riemannian surface is an Einstein manifold}$$

$$\text{Scalar curvature} \quad S = \sum_{i=1,2} \text{Ric}(E_i, E_i) = 2 K_g$$

The Gaussian curvature K_g is NOT necessarily a constant function. This is in contrast with the higher-dimensional situation:

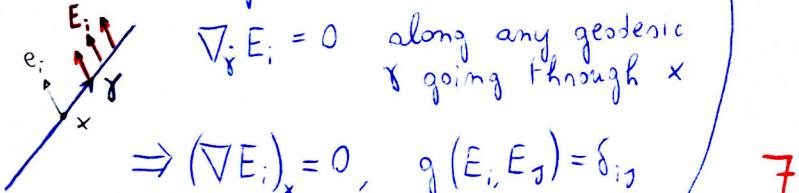
Proposition Any Einstein Riemannian manifold (M, g) of dimension ≥ 3 has constant scalar curvature S . (this is an application of the second Bianchi identity)

Proof $\text{Ric} = \frac{S}{m} g$, where $m = \dim M$

$$\nabla_X \text{Ric} = \nabla_X \left(\frac{S}{m} g \right) = \frac{1}{m} (XS) g + \frac{S}{m} \underbrace{\nabla_X g}_0 = \frac{1}{m} (XS) g \quad (*)$$

Fix $x \in M$, and let E_1, \dots, E_m be an orthonormal frame defined near x and such that $(\nabla E_i)_x = 0 \quad \forall i = 1, \dots, m$

we build it as follows: let e_1, \dots, e_m be an orthonormal basis of $T_x M$; we extend every e_i to a vector field E_i by parallel transport along radial geodesics:



$$\sum_i (\nabla_{E_i} \text{Ric}) (E_i, E_j) \Big|_x = \sum_{i, k} \nabla_{E_i} (R(E_k, \cdot, \cdot, E_k)) (E_i, E_j) \Big|_x$$

$$= \sum_{i, k} (\nabla_{E_i} R) (E_k, E_i, E_j, E_k) \Big|_x$$

since the E_i 's
are parallel at x

$$= \sum_{i, k} \left[-(\nabla_{E_k} R) (E_k, E_i, E_k, E_i) - (\nabla_{E_k} R) (E_k, E_i, E_i, E_j) \right] \Big|_x$$

Bianchi

$$= \sum_{i, k} \left[-E_j (R(E_k, E_i, E_k, E_i)) - E_k (R(E_k, E_i, E_i, E_j)) \right] \Big|_x$$

since the E_i 's
are parallel at x

$$= E_j S - \sum_k E_k (Ric(E_k, E_j)) \Big|_x = E_j S - \sum_k (\nabla_{E_k} Ric)(E_k, E_j) \Big|_x$$

$$\Rightarrow \sum_i (\nabla_{E_i} Ric)(E_i, E_j) = \frac{1}{2} E_j S$$

However, equation (*) on page 7 implies $\sum_i (\nabla_{E_i} Ric)(E_i, E_j) = \frac{1}{m} E_j S$

Therefore, $\frac{1}{2} E_j S = \frac{1}{m} E_j S \quad \forall j = 1, \dots, m$. If $m > 2$, then

$E_j S \equiv 0 \quad \forall j = 1, \dots, m$, that is, $dS \equiv 0$ (S is constant).

□