TD06: GEODESICS AND COMPLETE RIEMANNIAN MANIFOLDS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Exercise 1 (Geodesics of the Model spaces)**.** Determine the geodesics of the following model Riemannian manifolds (M, g). Compute the exponential map and its injectivity radius at any point $p \in M$.

(1) $(M, g) = (\mathbf{R}^n, g_0)$;
(2) $(M, g) = (\mathbf{S}^n, g)$ where g is the induced metric by the Euclidean metric on $\mathbf{R}^{n+1}$;
(3) $(M, g) = (\mathbf{T}^n_L, g_L)$, where $L = \omega_1\mathbf{Z} + \cdots + \omega_n\mathbf{Z}$ with $\omega_i > 0$ and $\mathbf{T}^n_L := \mathbf{R}^n/L$ is the n-torus induced by L endowed with the quotient metric g_L;
(4) Poincaré's Disc $(M, g) = (\mathbf{D}^2, g_{\mathbf{D}})$;
(5) Hyperbolic half-plane $(M, g) = (\mathbf{H}^2, g_{\mathbf{H}^2})$

Exercise 2. We say that two maximal geodesics are *parallel* if they are either disjoint or equal up to reparametrization. Let γ be a geodesic of (M, g) and $p \in M\backslash\mathrm{Im}(\gamma)$ where (M, g) is chosen as model Riemannian manifolds in the previous exercise. How many geodesics passing through p and parallel to γ are there?

Exercise 3. Show that the meridians of a surface of revolution $M \subset \mathbf{R}^3$ are geodesics.

Exercise 4. Let $(g_t)_{t\in\mathbf{R}}$ be a family of Riemannian metrics on $\mathbf{R}^n$. Consider $\mathbf{R}^{n+1}$ endowed with the metric $g = g_t + dt^2$. Show that the curves $\gamma_x : t \mapsto (x, t)$ are geodesics. Are they minimizers of the length functional L_g?

Exercise 5 (Common perpendicular of two submanifolds)**.** Let (M, g) be a Riemannian manifold. Let N_1 and N_2 be two submanifolds of M. Show that if there exists a path γ joining N_1 to N_2 such that γ is length-minimizing (among any path joining N_1 to N_2), then γ is a geodesic and γ is orthogonal to N_1 and N_2 at its extremities.