SOLUTIONS OF THE EXAM – March 27, 2020

Exercise 1. Indicate whether the following statements are true or false (a justification is not required).

(a) If M is a non-orientable manifold, its tangent bundle $\mathrm{T}M$ is non-orientable as well.

(b) The 1-form $\lambda = \sum_{i=1}^{n}(x_i \mathrm{d}y_i - y_i \mathrm{d}x_i)$ on $\mathbb{R}^{2n}$ restricts to a contact form on the convex sphere $Y = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{n}(|x_i - 1|^2 + |y_i|^2) = 1\}$.

(c) If z_0 is a point in a symplectic manifold (M, ω), there exists a Hamiltonian $H : M \to \mathbb{R}$ and a neighborhood $U \subseteq M$ of z_0 with the following property: for each neighborhood $V \subseteq M$ of z_0 there exists $\tau > 0$ such that $\phi_H^t(z) \in V$ for all $z \in U$ and $t \geq \tau$.

Solution.

(a) False. The tangent bundle $\mathrm{T}M$ always admits a symplectic form ω, and therefore ω^n is a volume form for $n = \dim(M) = \dim(\mathrm{T}M)/2$.

(b) False. Indeed, λ vanishes at $0 \in Y$, whereas a contact form is nowhere vanishing.

(c) False. If such H and U existed, we could choose the open neighborhood $V \subset M$ of z_0 to be relatively compact and such that $\overline{V} \subset U$. Since ϕ_H^τ is symplectic, it preserves the symplectic volume form ω^n, where $2n = \dim(M)$. Therefore, if $\phi_H^\tau(U) \subset V$, we would have

$$\mathrm{Vol}(U, \omega) = \int_U \omega^n = \int_{\phi_H^\tau(U)} \omega^n \leq \int_V \omega^n = \mathrm{Vol}(V, \omega).$$

However, since $\overline{V}$ is relatively compact and contained in U, the volume $\mathrm{Vol}(V, \omega)$ is finite and strictly smaller than $\mathrm{Vol}(U, \omega)$.

Exercise 2. Let (Y, α) be a contact manifold with Reeb flow ϕ_α^t. Assume that there exists a connected open subset $U \subseteq Y$ and a smooth function $\tau : U \to (0, \infty)$ such that

$$\phi_\alpha^{\tau(z)}(z) = z, \qquad \forall z \in U.$$

Is τ necessarily a constant function? Prove it, or provide a counterexample.

Solution. We define

$$\mathcal{U} := \{\gamma_z \in C^\infty(\mathbb{R}/\mathbb{Z}, Y) \mid z \in U\},$$

where $\gamma_z(t) = \phi_\alpha^{\tau(z)t}(z)$. Since U is connected, $\mathcal{U}$ is connected as well. Each $\gamma_z \in \mathcal{U}$ is the reparametrization of a closed Reeb orbit, and therefore it is a critical point of the action functional

$$\mathcal{A} : C^\infty(\mathbb{R}/\mathbb{Z}, Y) \to \mathbb{R}, \qquad \mathcal{A}(\gamma) = \int_\gamma \alpha.$$

In particular, $\mathcal{A}$ is constant on $\mathcal{U}$. Since $\tau(z) = \mathcal{A}(\gamma_z)$, we conclude that τ is constant.

Exercise 3. Let Y be a closed manifold of dimension $2n+1$ equipped with two contact forms α_0 and α_1 defining the same Reeb vector field $R := R_{\alpha_0} = R_{\alpha_1}$. Compute the difference of the contact volumes $\mathrm{Vol}(Y, \alpha_1) - \mathrm{Vol}(Y, \alpha_0)$, where

$$\mathrm{Vol}(Y, \alpha_i) = \int_Y \alpha_i \wedge (\mathrm{d}\alpha_i)^n.$$

Hint. Consider the 1-forms $\beta := \alpha_1 - \alpha_0$ and $\alpha_s := (1-s)\alpha_0 + s\alpha_1$ for $s \in [0,1]$. Compute $\alpha_s(R)$, $\mathrm{d}\alpha_s(R, \cdot)$, $\beta(R)$, $\mathrm{d}\beta(R, \cdot)$, $\beta \wedge (\mathrm{d}\alpha_s)^n$, and finally

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_Y \alpha_s \wedge (\mathrm{d}\alpha_s)^n.$$

Solution. We consider the 1-forms $\alpha_s := (1-s)\alpha_0 + s\alpha_1$ and $\beta := \frac{\mathrm{d}}{\mathrm{d}s}\alpha_s = \alpha_1 - \alpha_0$, and compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \int_Y \alpha_s \wedge (\mathrm{d}\alpha_s)^n &= \int_Y \beta \wedge (\mathrm{d}\alpha_s)^n + n \int_Y \alpha_s \wedge \mathrm{d}\beta \wedge (\mathrm{d}\alpha_s)^{n-1} \\ &= \underbrace{\int_Y \beta \wedge (\mathrm{d}\alpha_s)^n}_{(*)} - n \underbrace{\int_Y \mathrm{d}(\alpha_s \wedge \beta \wedge (\mathrm{d}\alpha_s)^{n-1})}_{(**)} + n \underbrace{\int_Y \mathrm{d}\alpha_s \wedge \beta \wedge (\mathrm{d}\alpha_s)^{n-1}}_{=(***)} \end{aligned}$$

Notice that $\alpha_s(R) = 1$ and $\mathrm{d}\alpha_s(R, \cdot) = 0$, and therefore $\beta(R) = 0$ and $\mathrm{d}\beta(R, \cdot) = 0$. This implies that $R \lrcorner (\beta \wedge (\mathrm{d}\alpha_s)^n) = 0$. Since $\beta \wedge (\mathrm{d}\alpha_s)^n = \mathrm{d}\alpha_s \wedge \beta \wedge (\mathrm{d}\alpha_s)^{n-1}$ is a form of top degree $2n+1$, it must vanish, and in particular

$$(*) = (***) = 0.$$

Finally, $(**)$ vanishes by Stokes Theorem, since Y is closed. In particular, we proved that $\mathrm{Vol}(Y, \alpha_0) = \mathrm{Vol}(Y, \alpha_1)$.

Exercise 4. Let $(W, \mathrm{d}\lambda)$ be a symplectic manifold, and $H : W \to \mathbb{R}$ a smooth Hamiltonian whose associated Hamiltonian vector field X_H on W is defined as usual by $\mathrm{d}\lambda(X_H, \cdot) = \mathrm{d}H$. Consider the functional

$$\mathcal{A} : C^\infty(\mathbb{R}/\mathbb{Z}, W) \times (0, \infty) \to \mathbb{R}, \qquad \mathcal{A}(\gamma, \tau) = \int_\gamma \lambda + \tau \int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))\, \mathrm{d}t.$$

Compute the differential $\mathrm{d}\mathcal{A}(\gamma, \tau)$, and characterize the critical points of $\mathcal{A}$ in terms of the flow of the Hamiltonian vector field X_H.

Hint. Any pair $(\gamma, \tau) \in C^\infty(\mathbb{R}/\mathbb{Z}, W) \times (0, \infty)$ can be identified with the τ-periodic curve $\Gamma : \mathbb{R}/\tau\mathbb{Z} \to W$, $\Gamma(t) = \gamma(t/\tau)$.

Solution. Consider a 1-periodic vector field ζ along γ, and $\sigma \in \mathbb{R}$. For each $s \in \mathbb{R}$ close to 0, we define

$$\gamma_s \in C^\infty(\mathbb{R}/\mathbb{Z}, W), \qquad \gamma_s(t) = \exp_{\gamma(t)}(s\zeta(t)),$$

and compute

$$\begin{aligned}
\mathrm{d}\mathcal{A}(\gamma, \tau)(\zeta, \sigma) &= \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \mathcal{A}(\gamma_s, \tau + s\sigma) \\
&= \int_{\mathbb{R}/\mathbb{Z}} \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \gamma_s^* \lambda + \tau \int_{\mathbb{R}/\mathbb{Z}} \mathrm{d}H(\gamma(t))\zeta(t)\, \mathrm{d}t + \sigma \int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))\, \mathrm{d}t \\
&= \int_{\mathbb{R}/\mathbb{Z}} \gamma^* \mathcal{L}_\zeta \lambda + \tau \int_{\mathbb{R}/\mathbb{Z}} \mathrm{d}H(\gamma(t))\zeta(t)\, \mathrm{d}t + \sigma \int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))\, \mathrm{d}t \\
&= \int_{\mathbb{R}/\mathbb{Z}} \Big(\mathrm{d}\lambda(\zeta(t), \dot\gamma(t)) + \tau\, \mathrm{d}\lambda(X_H(\gamma(t)), \zeta(t)) \Big)\, \mathrm{d}t + \sigma \int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))\, \mathrm{d}t \\
&= \int_{\mathbb{R}/\mathbb{Z}} \Big(\mathrm{d}\lambda(\underbrace{-\dot\gamma(t) + \tau X_H(\gamma(t))}_{(*)}, \zeta(t)) \Big)\, \mathrm{d}t + \sigma \underbrace{\int_{\mathbb{R}/\mathbb{Z}} H(\gamma(t))\, \mathrm{d}t}_{(**)}.
\end{aligned}$$

This readily implies that (γ, τ) is a critical point of $\mathcal{A}$ if and only if $(*) \equiv 0$ and $(**) = 0$. The equation $(*) \equiv 0$ can be rewritten for the τ-periodic curve $\Gamma(t) = \gamma(t/\tau)$ as

$$\dot\Gamma(t) = X_H(\Gamma(t)).$$

Namely, $(*) \equiv 0$ if and only if Γ is a τ-periodic orbit; in this case, the function $t \mapsto H(\Gamma(t))$ is constant, and therefore the equation $(**) = 0$ can be rewritten as

$$H \circ \Gamma \equiv 0.$$

Summing up, we showed that (γ, τ) is a critical point of $\mathcal{A}$ if and only if $\Gamma(t) = \gamma(t/\tau)$ is a τ-periodic orbit of X_H on the energy hypersurface $H^{-1}(0)$.

Exercise 5. Let $S^3 \subset \mathbb{C}^2$ be the unit 3-sphere, and $L(p,q) = S^3/\sim$ the (p,q)-lens space, which is defined as the quotient of S^3 under the equivalence relation

$$(z_1, z_2) \sim (e^{i2\pi/p} z_1, e^{i2\pi q/p} z_2), \qquad \forall z_1, z_2 \in \mathbb{C}.$$

Can you find some integers $p, q > 0$ and a Riemannian metric on $L(p,q)$ that is Besse but not Zoll?

Solution. Yes: if we choose $p, q > 0$ to be relatively prime integers, the Euclidean metric on S^3 induces a metric g to the quotient $L(p,q)$ that is Besse but not Zoll. Indeed, the unit speed geodesics of $(L(p,q), g)$ have the form

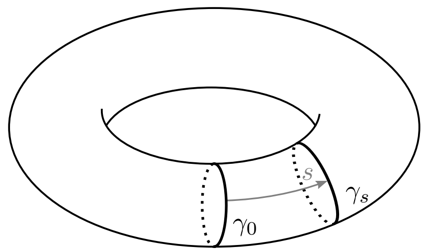
$$\gamma(t) = [\cos(t)\boldsymbol{z} + \sin(t)\boldsymbol{w}],$$

where $\boldsymbol{z} = (z_1, z_2)$ and $\boldsymbol{w} = (w_1, w_2)$ are orthogonal points in S^3. In particular, every unit-speed geodesic has period 2π. If $\boldsymbol{w} = \boldsymbol{i}\boldsymbol{z}$, the geodesic γ can be expressed as $\gamma(t) = [e^{it}\boldsymbol{z}]$; if $\boldsymbol{z} = (z_1, z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$, then γ has minimal period 2π, whereas if $z_2 = 0$ then γ has minimal period $2\pi/p$.

Exercise 6. Consider the 2-torus $\mathbb{T}^2 = S^1 \times S^1$, its space of unparametrized embedded circles $\Pi = \mathrm{Emb}(S^1, \mathbb{T}^2)/\mathrm{Diff}(S^1)$, and the connected component $\mathcal{C} \subset \Pi$ containing the meridian $[\gamma_0]$, where $\gamma_0(t) = (t, 0)$. Find a lower bound for the number of simple closed geodesics of $(\mathbb{T}^2, g)$ contained in $\mathcal{C}$. Is the lower bound that you found sharp?

Hint. You can assume, without providing a proof, the following two facts:

- The fundamental group $\pi_1(\mathcal{C})$ is isomorphic to $\mathbb{Z}$, with generator $\Gamma : S^1 \to \mathcal{C}$, $\Gamma(s) = [\gamma_s]$, where $\gamma_s(t) = (t, s)$.



- Consider the open subset $\mathcal{U}(\ell, \epsilon) = \big\{\zeta \in \mathcal{C} \bigm| L(\zeta) \in [\ell - \epsilon^2, \ell + \epsilon^2],\ \max|\kappa_\zeta| < \epsilon\big\}$, where L is the g-length functional
$$L(\zeta) = \int_{S^1} \|\dot\zeta(t)\|_g \, \mathrm{d}t,$$
and κ_ζ is the g-curvature of ζ. If there is a unique closed geodesic $\gamma \in \mathcal{C}$ of length $L(\gamma) = \ell$, then there exists $\epsilon > 0$ and a contractible open subset $\mathcal{W} \subset \mathcal{C}$ such that $\mathcal{U}(\ell, \epsilon) \subset \mathcal{W}$.

Solution. For every homology class $h \in H_*(\mathcal{C}; \mathbb{Z}_2)$, the min-max
$$c(h) = \inf_{[\sigma]=h} \max_{\sigma} L$$
is the length of a simple closed geodesic $[\zeta] \in \mathcal{C}$ of $(\mathbb{T}^2, g)$. In particular, the shortest $[\zeta] \in \mathcal{C}$ is a simple closed geodesic of $(\mathbb{T}^2, g)$. Indeed, if 1 is the generator of $H_0(\mathcal{C}; \mathbb{Z}_2)$, we have $\ell := \inf L|_{\mathcal{C}} = c(1)$. Since $\pi_1(\mathcal{C}) \cong \mathbb{Z}$, we have $H_1(\mathcal{C}; \mathbb{Z}_2) \cong H^1(\mathcal{C}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and the generators $h \in H_1(\mathcal{C}; \mathbb{Z}_2)$ and $w \in H^1(\mathcal{C}; \mathbb{Z}_2)$ satisfy $h \frown w = w(h) = 1$. By Lusternik-Schnirelmann's Theorem, we have $c(1) = c(h)$ if and only if $w|_{\mathcal{U}(\ell,\epsilon)} \neq 0$ in $H^1(\mathcal{U}(\ell, \epsilon); \mathbb{Z}_2)$ for all $\epsilon > 0$.

We claim that there are always at least two simple closed geodesics in $\mathcal{C}$. Indeed, assume by contradiction that there is only one, which must have length $c(1)$. Then $c(1) = c(h)$, and if we choose $\epsilon > 0$ small enough, $\mathcal{U}(\ell, \epsilon)$ is contained in a contractible open subset $\mathcal{W} \subset \mathcal{U}$. In particular $w|_{\mathcal{W}} \neq 0$ in $H^1(\mathcal{W})$, and therefore $w|_{\mathcal{U}(\ell,\epsilon)} = (w|_{\mathcal{W}})|_{\mathcal{U}(\ell,\epsilon)} = 0$ as well, contradicting Lusternik-Schnirelmann's Theorem.

The lower bound of two simple closed geodesics in $\mathcal{C}$ is sharp: by Clairaut's relation, the following torus of revolution (in which the boundary curves α are glued together) has only two simple closed geodesics in $\mathcal{C}$, that is, α and β.