

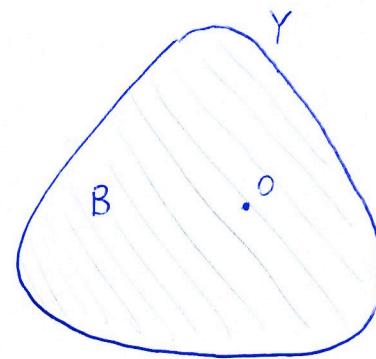
REEB DYNAMICS ON CONVEX SPHERES

Let us recall the setting:

$B \subset \mathbb{R}^{2m}$ convex with smooth $\partial B = Y \cong S^{2m-1}$

α

$$\alpha = \sum_{i=1}^m \frac{1}{2} (x_i dy_i - y_i dx_i) \Big|_Y \quad \text{contact form on } Y$$



$\alpha \in (1, 2)$ fixed

$H: \mathbb{R}^{2m} \rightarrow [0, \infty)$ defined by $H|_Y \equiv 1$, $H(\lambda z) = \lambda^\alpha H(z) \quad \forall \lambda \geq 0, z \in \mathbb{R}^{2m}$

We study two equivalent problems:

closed Reeb orbits on Y

$$(*) \begin{cases} \dot{\gamma}(t) = R_\alpha(\gamma(t)) \text{ on } Y \\ \gamma(0) = \gamma(T_\gamma) \text{ for some } T_\gamma > 0 \end{cases}$$

1-periodic Hamiltonian orbits on \mathbb{R}^{2m}

$$(**) \begin{cases} \dot{\gamma}(t) = J \nabla H(\gamma(t)) \text{ on } \mathbb{R}^{2m} \\ \gamma(0) = \gamma(1) \neq 0 \end{cases}$$

We have that γ is a solution of $(*)$ iff $\gamma_K(t) = \left(\frac{2KT_\gamma}{\alpha}\right)^{\frac{1}{\alpha-2}} \gamma(KT_\gamma t)$ is a solution of $(**)$ $\forall K=1, 2, 3, \dots$

These problems can be studied by means of a variational principle:

Clarke action functional

$$\Psi: \underbrace{L^b_c(S^1, \mathbb{R}^{2m})}_{\{u \in L^b(S^1, \mathbb{R}^{2m}) \mid \int_{S^1} u(t) dt = 0\}} \rightarrow \mathbb{R}, \quad \Psi(\dot{\xi}) = \int_{S^1} \left(-\frac{1}{2} \langle J\xi, \dot{\xi} \rangle + H^*(-J\dot{\xi}) \right) dt$$

where $S^1 = \mathbb{R}/\mathbb{Z}$

$$b = \frac{\alpha}{\alpha-1} \in (2, \infty)$$

$$H^*: \mathbb{R}^{2m} \rightarrow [0, \infty), \quad H^*(w) = \max_{z \in \mathbb{R}^{2m}} (\langle w, z \rangle - H(z))$$

H^* is positively b -homogeneous

Ψ is C^1 , and invariant under the S^1 -action: on $L^b_c(S^1, \mathbb{R}^{2n})$:

$$n \cdot u = u(n \cdot \cdot) \quad \forall n \in S^1, u \in L^b_c(S^1, \mathbb{R}^{2n})$$

γ is a solution of $(*)$ iff $S^1 \cdot \gamma_K$ is a critical circle of Ψ (different from 0)
 $\forall K = 1, 2, 3, \dots$

$$\Psi(\gamma_K) = -\left(1 - \frac{\alpha}{2}\right)\left(\frac{2K}{\alpha} T_\gamma\right)^{-\frac{\alpha}{2-\alpha}} < 0$$

$\Rightarrow \Psi$ has no positive critical values

The origin in $L^b_c(S^1, \mathbb{R}^{2n})$ is the only critical point with critical value 0

We are interested in the negative critical values

Let us show that Ψ has good variational properties

Proposition Ψ is bounded from below

Proof $\Psi(\dot{\xi}) = \int_{S^1} \left(-\frac{1}{2} \underbrace{\langle J\xi, \dot{\xi} \rangle}_{\langle \xi, -J\dot{\xi} \rangle} + \underbrace{H^*(-J\dot{\xi})}_{\geq c|\dot{\xi}|^b} \right) dt$ for some $c > 0$ depending only on Y and α

$$\geq -\frac{1}{2} \underbrace{\|\xi\|_{L^\infty} \|\dot{\xi}\|_{L^1}}_{\leq \|\dot{\xi}\|_{L^1}} + c \|\dot{\xi}\|_{L^b}^b$$

we choose the primitive ξ such that $\xi(0) = 0$

$$\geq -\frac{1}{2} \underbrace{\|\dot{\xi}\|_{L^2}^2}_{\leq \|\dot{\xi}\|_{L^b}^2} + c \|\dot{\xi}\|_{L^b}^b \geq \|\dot{\xi}\|_{L^b}^2 \left(c \|\dot{\xi}\|_{L^b}^{b-2} - \frac{1}{2} \right)$$

Holder $b > 2$

$$\geq -\frac{1}{4c} \quad \square$$

Proposition

Ψ satisfies the so-called PALAIS-SMALE condition:

$$\forall \{\dot{\xi}_m | m \in \mathbb{N}\} \subset L^b_0(S^1, \mathbb{R}^{2n})$$

such that $\begin{cases} \Psi(\dot{\xi}_m) \xrightarrow[m \rightarrow \infty]{} c \in \mathbb{R} \\ \|d\Psi(\dot{\xi}_m)\|_{(L^b_0)^*} \xrightarrow[m \rightarrow \infty]{} 0 \end{cases}$

(we say that
 $\{\dot{\xi}_m\}$ is a
PALAIS-SMALE
sequence)

$$\exists \text{ a converging subsequence } \dot{\xi}_{m_k} \xrightarrow[k \rightarrow \infty]{} \dot{\xi}$$

$$(Rmk \quad \dot{\xi} \in \text{crit}(\Psi), \quad \Psi(\dot{\xi}) = c)$$

Proof.

In the proof of the previous proposition, we showed

$$\Psi(\dot{\xi}) \geq \|\dot{\xi}\|_{L^b}^2 \left(\text{const} \cdot \|\dot{\xi}\|_{L^b}^{b-2} - \frac{1}{2} \right)$$

By this equation, if $\{\dot{\xi}_m\}$ is a Palais-Smale sequence, then

$$\|\dot{\xi}_m\|_{L^b} \leq \text{const}$$

Therefore, up to extracting a subsequence, $\{\dot{\xi}_m\}$ converges

weakly in $L^b_0(S^1, \mathbb{R}^{2n})$:

$$\dot{\xi}_m \xrightarrow[\text{weakly}]{} \dot{\xi} \quad \text{as } m \rightarrow \infty$$

Since $\dot{\xi}_m \in L^b_0(S^1, \mathbb{R}^{2n})$, we have $\nabla H^*(-J\dot{\xi}_m) \in L^\infty(S^1, \mathbb{R}^{2n})$ (exercise)

We set

$$u_m := -J\dot{\xi}_m + J\nabla H^*(-J\dot{\xi}_m) \in L^\infty(S^1, \mathbb{R}^{2n})$$

$$\lambda_m := \int_{S^1} u_m(t) dt \in \mathbb{R}^{2n}$$

We already computed

$$d\Psi(\dot{\xi}_m)w = \int_{S^1} \langle u_m(t), w(t) \rangle dt \quad \forall w \in L^b_0(S^1, \mathbb{R}^{2n})$$

Since $\|\dot{\psi}(\dot{\xi}_m)\|_{(L^b_0)^*} \xrightarrow[m \rightarrow \infty]{} 0$, we have

$$\begin{aligned} w \in L^b_0 \\ \|w\|_{L^b} \leq 1 \end{aligned} \quad \int_{S^1} \langle \mu_m(t), w(t) \rangle dt \xrightarrow[m \rightarrow \infty]{} 0$$

Therefore

$$0 \leftarrow \sup_{\substack{z \in L^b \\ \|z\| \leq 1}} \int_{S^1} \langle \mu_m(t), z(t) - \int_{S^1} z(s) ds \rangle dt$$

$$\int_{S^1} \langle \mu_m(t) - \lambda_m, z(t) \rangle dt$$

which implies

$$\frac{\mu_m - \lambda_m}{\|\cdot\|} \xrightarrow[m \rightarrow \infty]{L^\omega} 0 \quad (*)$$

$$-J\dot{\xi}_m + J\nabla H^*(-J\dot{\xi}_m)$$

Since $\|\dot{\xi}_m\|_{L^b_0} \leq \text{const}$ and $\dot{\xi}_m \xrightarrow[L^b_0]{\text{weakly}} \dot{\xi}$, we have that ξ_m converges in L^ω (up to extracting a subsequence), and $\|\nabla H^*(-J\dot{\xi}_m)\|_{L^\omega} \leq \text{const}$.

Therefore, equation $(*)$ implies that $|\lambda_m| \leq \text{const}$, and thus

$$\lambda_m \rightarrow \lambda \in \mathbb{R}^{2n} \quad (\text{up to extracting a subseq.})$$

which implies

$$\begin{aligned} \mu_m &\xrightarrow{L^\omega} \mu \\ J\nabla H^*(-J\dot{\xi}_m) &\xrightarrow{L^\omega} w \\ \dot{\xi}_m = J\nabla H(\nabla H^*(-J\dot{\xi}_m)) &\xrightarrow{L^b} -Jw \quad (\text{and } -Jw = \dot{\xi}) \end{aligned}$$

□

As an immediate corollary, we prove the Weinstein conjecture for convex contact spheres:

Corollary $\exists \dot{\xi} \in L^b_0(S^1, \mathbb{R}^{2n})$ such that $\Psi(\dot{\xi}) = \inf \Psi$.

In particular $\dot{\xi} \in \text{crit}(\Psi)$ with $\Psi(\dot{\xi}) < 0$, and therefore (Y, α) has a closed Reeb orbit.

Proof

- $\inf \Psi < 0$, indeed consider

$$\dot{\gamma}(t) = e^{J2\pi t} v \quad \text{for some } v \in \mathbb{R}^{2n} \quad \left(\begin{array}{l} \text{thus} \\ \gamma(t) = -\frac{1}{2\pi} J e^{J2\pi t} v \end{array} \right)$$

we have

$$\begin{aligned} \Psi(\dot{\gamma}) &= \int_{S^1} \left(-\frac{1}{2} \left\langle \frac{1}{2\pi} e^{J2\pi t} v, e^{J2\pi t} v \right\rangle + H^*(-J e^{J2\pi t} v) \right) dt \\ &\leq -\frac{1}{4\pi} \|v\|^2 + \text{const} \|v\|^b < 0 \quad \text{if } \|v\| \neq 0, \|v\| \text{ small enough} \\ &\quad (\text{since } b > 2) \end{aligned}$$

- $c := \inf \Psi < 0$

We claim that $\lim_{m \rightarrow \infty} \inf_{\substack{u \in L^b_0 \\ \Psi(u) \leq c + \frac{1}{m}}} \|\delta \Psi(u)\|_{(L^b_0)^*} = 0$

Otherwise $\exists \delta > 0$ and $m \in \mathbb{N}$ such that

$$\|\partial\Psi(u)\|_{(L^b_0)^*} > \delta \quad \forall u \in L^b_0 \text{ with } \Psi(u) \in [c + \frac{1}{m}]$$

we fix $u \in L^b_0(S^1, \mathbb{R}^{2m})$ with $\Psi(u) < c + \frac{1}{m}$, and consider the maximal solution $\Gamma : [0, T) \rightarrow L^b_0(S^1, \mathbb{R}^{2m})$ of the O.D.E.

$$\begin{cases} \dot{\Gamma}(t) = -\nabla\Psi(\Gamma(t)), & t \in [0, T) \\ \Gamma(0) = u \end{cases}$$

If $T < \infty$, then $\|\Gamma(t)\| \xrightarrow[t \rightarrow T]{} \infty$, and therefore $\|\Gamma(0) - \Gamma(t)\| > \frac{1}{\delta m}$ for $t \in [0, T)$ large enough. However

$$\begin{aligned} \Psi(\Gamma(t)) &= \Psi(\Gamma(0)) - \int_0^t \partial\Psi(\Gamma(s)) \dot{\Gamma}(s) ds \\ &\leq c + \frac{1}{m} - \delta \|\Gamma(0) - \Gamma(t)\| < c = \inf \Psi \end{aligned}$$

which is impossible.

We must have $T = \infty$, but again

$$\begin{aligned} \Psi(\Gamma(t)) &= \Psi(\Gamma(0)) - \int_0^t \partial\Psi(\Gamma(s)) \dot{\Gamma}(s) ds \\ &= \Psi(\Gamma(0)) - \int_0^t \|\nabla\Psi(\Gamma(s))\|^2 ds \\ &\geq \Psi(\Gamma(0)) - t \delta^2 \xrightarrow[t \rightarrow \infty]{} -\infty \end{aligned}$$

which is impossible since $\inf \Psi > -\infty$.

- Now, let $\dot{\xi}_m \in L_0^b(S^1; \mathbb{R}^{2n})$ be such that $\Psi(\dot{\xi}_m) \leq c + \frac{1}{m}$. We must have $\Psi(\dot{\xi}_m) \rightarrow c$, $\|d\Psi(\dot{\xi}_m)\|_{(L_0^b)^*} \rightarrow 0$. The Palais-Smale condition implies that a subsequence $\dot{\xi}_{m_k}$ converges

$$\dot{\xi}_{m_k} \xrightarrow[k \rightarrow \infty]{L_0^b} \dot{\xi}$$

$$\Psi(\dot{\xi}) = c = \inf \Psi < 0.$$

□