HOMOMORPHISMES EN HOMOLOGIE

L'homologie est évidemment un invariant topologique
On va montrer qu'elle est aussi un invariant homotopique

$f : X \to Y$ continue

f induit un homomorphisme $f_n : C_n(X) \to C_n(Y)$

$$\sigma \longmapsto f \circ \sigma$$

Propriétés

- $f : X \to Y$, $g : Y \to Z$ $\Rightarrow$ $(g \circ f)_n = g_n \circ f_n$

- $\partial_n \circ f_* = f_* \circ \partial_n$

donc f_* passe au quotient, en homologie:

$$f_* : H_n(X) \longrightarrow H_n(Y)$$
$$[\sigma] \longmapsto [f_* \sigma]$$

et évidemment

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_* = g_* \circ f_*"'] & H_n(Y) \arrow[r, "g_*"] & H_n(Z) \end{tikzcd}$$

commute

f injective $\not\Longrightarrow$ f_* injectif

Thm Si $f : X \to Y$ et $g : X \to Y$ sont homotopes, alors

$$f_* = g_* \quad H_n(X) \to H_n(Y) \quad \forall n \in \mathbb{N}$$

Preuve

- On va construire une homotopie de chaînes

$$P_n \quad C_n(X) \longrightarrow C_{n+1}(Y) \quad \text{homomorphisme}$$

$$\text{tq} \qquad P_{n-1} \circ \partial_n + \partial_{n-1} \circ P_n = g_* - f_*$$

- Si on a une telle P_n, alors $\forall [\sigma] \in H_n(X)$

$$g_*[\sigma] = [g_*\sigma] = [f_*\sigma + P_{n-1} \circ \underbrace{\partial_n \sigma}_{0} + \partial_{n-1} \circ P_n \sigma]$$

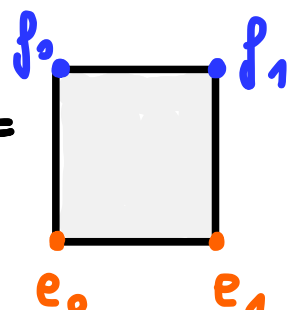
$$= [f_*\sigma] + \underbrace{[\partial_{n-1} \circ P_n \sigma]}_{0} = f_*[\sigma]$$

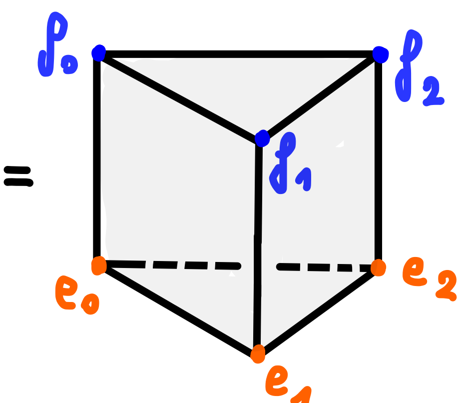
- **Construction de P**

$h_t \ X \to Y$ homotopie, $h_0 = f$, $h_1 = g$

On considère $\Delta^m \times [0,1]$, où

$\Delta^m \times \{0\} = [e_0, .., e_m]$, $\Delta^m \times \{1\} = [f_0, .., f_m]$

ex $\Delta^1 \times [0,1] =$ 

$\Delta^2 \times [0,1] =$ 

$\forall\ \sigma : \Delta^n \to X$ simplexe singulier

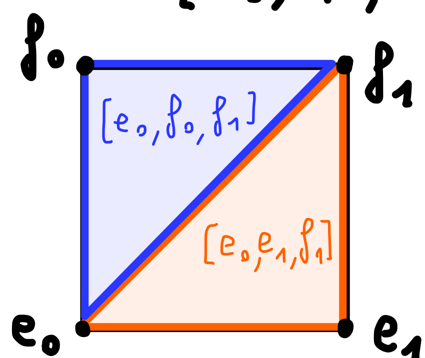
on définie

$\tilde{\sigma} : \Delta^n \times [0,1] \to X, \qquad \tilde{\sigma}(z,t) = h_t(\sigma(z))$

$P_n : C_n(X) \to C_{n+1}(Y)$

$$P_n(\sigma) := \sum_{i=0}^{n} (-1)^i \, \tilde{\sigma}\big|_{[e_0, \dots, e_i, f_i, \dots, f_n]}$$

ex $\quad n = 1$



Vérifions que P_n est l'homotopie de chaînes cherchée.

$$\partial_{n+1} P_n \sigma - P_{n-1} \partial_n \sigma$$

$$= \sum_{j \leq i} (-1)^{i+j} \tilde{\sigma}|_{[e_0 \ldots \cancel{e_j} \ldots e_i f_i \ldots f_n]}$$

$$+ \sum_{j \geq i} (-1)^{i+j+1} \tilde{\sigma}|_{[e_0 \ldots e_i f_i \ldots \cancel{f_j} \ldots f_n]}$$

$$- \sum_{i < j} (-1)^{i+j} \tilde{\sigma}|_{[e_0 \ldots e_i f_i \ldots \cancel{f_j} \ldots f_n]}$$

$$- \sum_{i > j} (-1)^{i+j-1} \tilde{\sigma}|_{[e_0 \ldots \cancel{e_j} \ldots e_i f_i \ldots f_n]}$$

$$= \tilde{\sigma}\big|_{[f_0 \dots f_n]} - \tilde{\sigma}\big|_{[e_0 \dots e_n]}$$

$$= h_1 \circ \sigma - h_0 \circ \sigma = g \circ \sigma - f \circ \sigma \qquad \square$$

Corollaire

Si $f : X \to Y$ est une equivalence d'homotopie alors $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ est un isomorphisme $\forall n \in \mathbb{N}$

Preuve

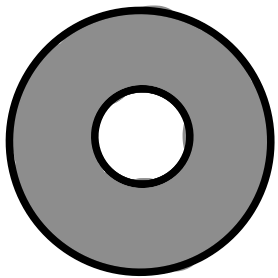
$\exists\, g : Y \to X$ inverse homotopique de f

$$g \circ f = id \qquad f \circ g = id$$

$$\Longrightarrow \quad g_* \circ f_* = (g \circ f)_* = id, \quad f_* \circ g_* = (f \circ g)_* = id_*$$

$\square$

$$H_n(B^d) \cong H_n(\text{point}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

$A =$  $= S^1 \times [0,1]$

$$H_n(A) \cong H_n(S^1)$$

⚠ On ne sait pas encore calculer $H_n(S^d)$!!!

UN PEU D'ALGÈBRE HOMOLOGIQUE

Oublions pour un instant les espaces topologiques !!!

complexe de chaînes (algébrique)

$$\xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}}$$

où C_n groupe abélien

∂_n homomorphisme

$\partial_{n+1} \circ \partial_n = 0$

On écrit (C_*, ∂_*)

homologie

$$H_n(C_*) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_n}$$

homomorphisme de complexes de chaînes

$$(C_*, \partial_*) \xrightarrow{f_*} (B_*, \partial_*)$$

i.e. $f_n \; C_n \to B_n$ homomorph

$$f_n \circ \partial_{n+1} = \partial_{n+1} \circ f_{n+1}$$

$$\begin{tikzcd}
{} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r, "\partial_{n-2}"] \arrow[d, "f_{n-2}"] & {} \\
{} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & C_{n-2} \arrow[r, "\partial_{n-2}"] & {}
\end{tikzcd}$$

(le diagramme commute)

f_* induit un homomorphisme en homologie

$$f_* \quad H_n(C_*) \longrightarrow H_n(B_*)$$
$$[\sigma] \longmapsto [f_n(\sigma)]$$

(bien défini, car

$$\begin{aligned}[f_n(\sigma + \partial_{n+1}\mu)] &= [f_n(\sigma)] + [f_n(\partial_{n+1}\mu)] \\ &= [f_n(\sigma)] + [\partial_{n+1} f_n(\mu)] \\ &= [f_n(\sigma)] \end{aligned}$$
)

suite exacte

$$A \xrightarrow{f} B \xrightarrow{g} C$$

t.q. A, B, C groupes abéliens

f, g homomorphismes

$$\mathrm{Im}(f) = \mathrm{Ker}(g)$$

(si la suite est plus longue (voir même infini), on dit qu'elle est **exacte** si chaque triple est exacte

exacte

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow \cdots$$

exacte

exacte)

Suite exacte "courte"

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$\operatorname{Im} f = \ker g$$

$\Rightarrow$ f injectif, g surjectif

exemple

$$0 \to A \hookrightarrow A \oplus B \twoheadrightarrow B \to 0$$

$$a \mapsto (a, 0)$$

$$(a, b) \mapsto b$$

suite exacte courte de complexes de chaînes

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

$$\begin{tikzcd}
 & {} \arrow[d, "\partial_{n+1}"] & {} \arrow[d, "\partial_{n+1}"] & {} \arrow[d, "\partial_{n+1}"] & \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\partial_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "f_{n-2}"] \arrow[d, "\partial_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] \arrow[d, "\partial_{n-2}"] & C_{n-2} \arrow[r] \arrow[d, "\partial_{n-2}"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

chaque ligne est exacte

Thm Une suite exacte courte

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0$$

induit une suite exact "longue" en homologie

$$\cdot \xrightarrow{\partial_*} H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \rightarrow \cdot$$

Preuve

- Définition de ∂_* + "diagram chasing"

$$
\begin{tikzcd}
 & {} \arrow[d, "\partial_{n+1}"] & {} \arrow[d, "\partial_{n+1}"] & {} \arrow[d, "\partial_{n+1}"] & \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\partial_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "f_{n-2}"] \arrow[d, "\partial_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] \arrow[d, "\partial_{n-2}"] & C_{n-2} \arrow[r] \arrow[d, "\partial_{n-2}"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

$$[\sigma] \in H_n(C_*) \implies \sigma = g_n(\tilde{\sigma}), \quad \partial_n \tilde{\sigma} = f_{n-1}(\tilde{\tilde{\sigma}})$$

$$\partial_*[\sigma] = [\tilde{\tilde{\sigma}}]$$

$$
\begin{tikzcd}
 & {} \arrow[d, "\partial_{m+1}"] & {} \arrow[d, "\partial_{m+1}"] & {} \arrow[d, "\partial_{m+1}"] & \\
0 \arrow[r] & A_m \arrow[r, "f_m"] \arrow[d, "\partial_m"] & B_m \arrow[r, "g_m"] \arrow[d, "\partial_m"] & C_m \arrow[r] \arrow[d, "\partial_m"] & 0 \\
0 \arrow[r] & A_{m-1} \arrow[r, "f_{m-1}"] \arrow[d, "\partial_{m-1}"] & B_{m-1} \arrow[r, "g_{m-1}"] \arrow[d, "\partial_{m-1}"] & C_{m-1} \arrow[r] \arrow[d, "\partial_{m-1}"] & 0 \\
0 \arrow[r] & A_{m-2} \arrow[r, "f_{m-2}"] \arrow[d, "\partial_{m-2}"] & B_{m-2} \arrow[r, "g_{m-2}"] \arrow[d, "\partial_{m-2}"] & C_{m-2} \arrow[r] \arrow[d, "\partial_{m-2}"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

$\operatorname{Im} f_* \subset \operatorname{Ker} g_*$ (facile)

$$g_*[\sigma] = 0 \Rightarrow g_{m-1}(\sigma) = \partial_m(\tilde{\sigma}) = \partial_m(g_m(\tilde{\tilde{\sigma}}))$$

$$\Rightarrow \sigma - \partial_m \tilde{\tilde{\sigma}} = f_{m-1}(\tilde{\tilde{\tilde{\sigma}}}) \Rightarrow [\sigma] = f_*[\tilde{\tilde{\tilde{\sigma}}}]$$

Compléter la preuve par exercice. $\square$

Rmq Si $A_* \subset B_*$

i.e $\partial_n A_n \subset A_{n-1}$ $\forall n$

on a une suite exacte courte

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow B_*/A_* \longrightarrow 0$$

(complexe quot)

et une suite exacte longue

$$\cdots \xrightarrow{\partial_*} H_n(A_*) \to H_n(B_*) \to H_n(B_*/A_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \to$$

Rmq (Naturalité)

Un morphisme de suites exactes

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "a_n"] & B_n \arrow[r, "g_n"] \arrow[d, "b_n"] & C_n \arrow[r] \arrow[d, "c_n"] & 0 \\ 0 \arrow[r] & A'_n \arrow[r, "f'_n"] & B'_n \arrow[r, "g'_n"] & C'_n \arrow[r] & 0 \end{tikzcd}$$

(toute flèche est un homomorph de chaînes)

induit un morphisme de suites exactes longues

$$\begin{tikzcd} {} \arrow[r, "\partial_*"] & H_n(A_*) \arrow[r, "f_*"] \arrow[d, "a_*"] & H_n(B_*) \arrow[r, "g_*"] \arrow[d, "b_*"] & H_n(C_*) \arrow[r, "\partial_*"] \arrow[d, "c_*"] & H_{n-1}(A_*) \arrow[r, "f_*"] \arrow[d, "a_*"] & {} \\ {} \arrow[r, "\partial_*"] & H_n(A'_*) \arrow[r, "f'_*"] & H_n(B'_*) \arrow[r, "g'_*"] & H_n(C'_*) \arrow[r, "\partial_*"] & H_{n-1}(A'_*) \arrow[r, "f_*"] & {} \end{tikzcd}$$

HOMOLOGIE SINGULIÈRE RELATIVE

$Y \subset X$ espaces topologiques

complexe de chaînes singulières relatives

$$C_n(X,Y) = \frac{C_n(X)}{C_n(Y)}$$

$$\xrightarrow{\partial_{n+1}} C_n(X,Y) \xrightarrow{\partial_n} C_{n-1}(X,Y) \xrightarrow{\partial_{n-1}} C_{n-2}(X,Y) \xrightarrow{\partial_{n-2}}$$

homologie singulière relative

$$H_n(X,Y) = \frac{\mathrm{Ker}\left(\partial_n : C_n(X,Y) \to C_{n-1}(X,Y)\right)}{\mathrm{Im}\left(\partial_{n+1} : C_{n+1}(X,Y) \to C_n(X,Y)\right)}$$

$$\left(\frac{\text{chaînes relatives}}{\text{bords relatifs}}\right)$$

Liée à $H_n(X)$ et $H_n(Y)$ par la suite longue exacte

$$\cdots \xrightarrow{\partial_*} H_n(Y) \to H_n(X) \to H_n(X,Y) \xrightarrow{\partial_*} H_{n-1}(Y) \longrightarrow \cdots$$

<u>Rmq</u> $H_n(X) = H_n(X, \emptyset)$

Plus généralement.

$Z \subseteq Y \subseteq X$ espaces topologiques

La suite exacte courte

$$0 \longrightarrow C_n(Y,Z) \longrightarrow C_n(X,Z) \longrightarrow C_n(X,Y) \longrightarrow 0$$

induit la suite exacte longue

$$\cdot \xrightarrow{\partial_*} H_n(Y,Z) \longrightarrow H_n(X,Z) \longrightarrow H_n(X,Y) \xrightarrow{\partial_*} H_{n-1}(Y,Z) \longrightarrow \cdot$$

Prop (Invariance par homotopie)

Si $f, g : (A,B) \to (X,Y)$ sont homotopes

(i.e. $\exists$ homotopie $h_t : (A,B) \to (X,Y)$ t.q. $h_0 = f$ et $h_1 = g$)

alors $f_* = g_* \quad H_n(A,B) \to H_n(X,Y) \quad \forall n$

Preuve

Analogue au cas de l'homologie absolute

$\square$

Quelque calcul élémentaire

- Si $Y \subseteq X$, $Y \neq \emptyset$, et X est connexe par arcs, alors $H_0(X,Y) = 0$

suite exacte longue

$$\begin{tikzcd} {} \arrow[r, "\partial_*"] & H_0(Y) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\neq$}"] & H_0(X) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & H_0(X,Y) \arrow[r, "\partial_*"] & 0 \\ & 0 & \mathbb{Z} & & \end{tikzcd}$$

surjectif

- $$H_n([0,1],\{0,1\}) = \begin{cases} \mathbb{Z}, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

$\forall n \geq 2$

$$H_n(B^k, \partial B^k) = \tilde{H}_{n-1}(\partial B^k)$$

homologie réduite :

$\tilde{H}_n(X) = H_n(X) \quad \forall n \neq 0$

$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

si $X \neq \emptyset$

suite exacte longue

$$\begin{tikzcd}
H_n(B^k) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(B^k, \partial B^k) \arrow[r, "\partial_*"] & H_{n-1}(\partial B^k) \arrow[r] & H_{n-1}(B^k) \arrow[d, phantom, "\shortparallel"] \\
0 & & & 0
\end{tikzcd}$$

si $n \neq 0$

isomorphisme si $n \neq 0, 1$

si $n \neq 1$

$$
\begin{tikzcd}
H_1(B^k) \arrow[r] \arrow[d, equal] & H_1(B^k, \partial B^k) \arrow[r, "\partial_*"] & H_0(\partial B^k) \arrow[r] & H_0(B^k) \arrow[d, equal] \\
0 & & & \mathbb{Z}
\end{tikzcd}
$$

$\mathbb{Z}$ si $k > 1$

$\mathbb{Z} \oplus \mathbb{Z}$ si $k = 1$

isomorphisme si $k > 1$

surjectif de noyau $\mathbb{Z} \oplus 0$ si $k = 1$