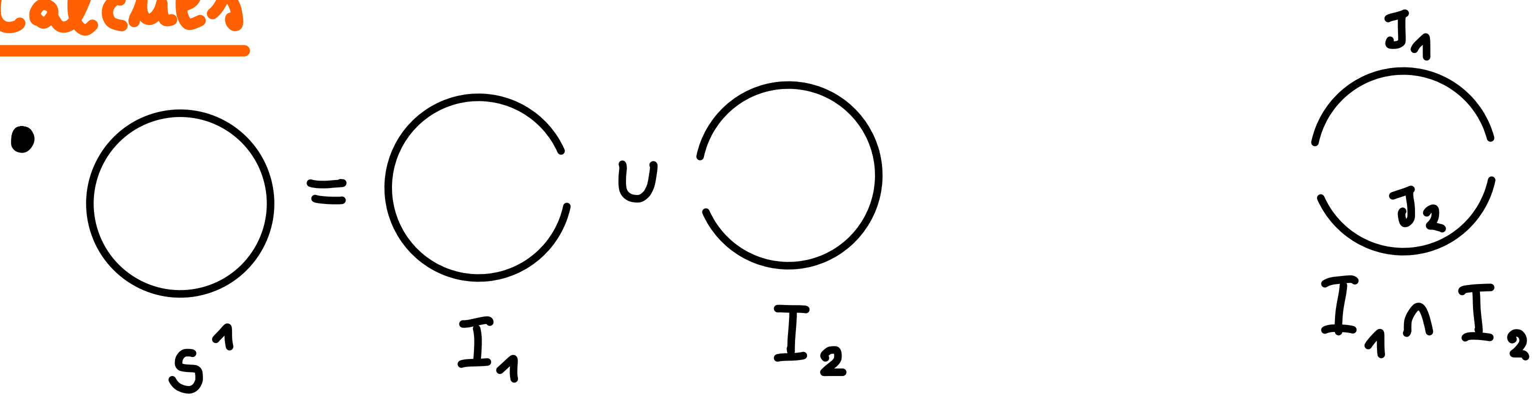
Calculs



$$H_n(I_1) \cong H_n(I_2) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$H_n(I_1 \cap I_2) = H_n(J_1) \oplus H_n(J_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

si $n \geq 2$

$$\to \underbrace{H_n(I_1) \oplus H_n(I_2)}_{=0} \to H_n(S^1) \xrightarrow{\partial_*} \underbrace{H_{n-1}(I_1 \cap I_2)}_{=0} \to$$

$$\Rightarrow \boxed{H_n(S^1) = 0 \quad \forall n \geq 2}$$

si $n = 1$

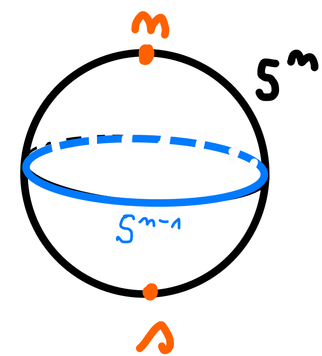
$$\to H_1(I_1) \oplus H_1(I_2) \to H_1(S^1) \to H_0(I_1 \cap I_2) \xrightarrow{i_*} H_0(I_1) \oplus H_0(I_2)$$

$$H_1(I_1) \oplus H_1(I_2) = 0, \quad H_0(I_1 \cap I_2) = \mathbb{Z} \oplus \mathbb{Z}, \quad H_0(I_1) \oplus H_0(I_2) = \mathbb{Z} \oplus \mathbb{Z}$$

$$i_*(\sigma_1, \sigma_2) = (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2)$$

$$H_1(S^1) \cong \mathrm{Ker}(i_*) \cong \mathbb{Z}$$

• $S^n = U \cup V$, $\quad n \geq 2$



$U = S^n \setminus \{n\}$, $\quad V = S^n \setminus \{s\}$

$U \cong B^n$, $\quad V \cong B^n$

$U \cap V = S^n \setminus \{n, s\} \simeq S^{n-1}$

$p > 1$

$$\to \underbrace{H_p(U) \oplus H_p(V)}_{=0} \to H_p(S^n) \xrightarrow{\cong} \underbrace{H_{p-1}(U \cap V)}_{\cong H_{p-1}(S^{n-1})} \to \underbrace{H_{p-1}(U) \oplus H_{p-1}(V)}_{=0} \to 0$$

$$H_p(S^n) \cong H_{p-1}(S^{n-1}) \quad \forall\, n > 1,\ p > 1$$

on sait déjà $\quad H_1(S^n) = 0 \quad \forall\, n > 1$

$$H_0(S^n) \cong \mathbb{Z} \quad \forall\, n \geq 1$$

$\forall n > 1$

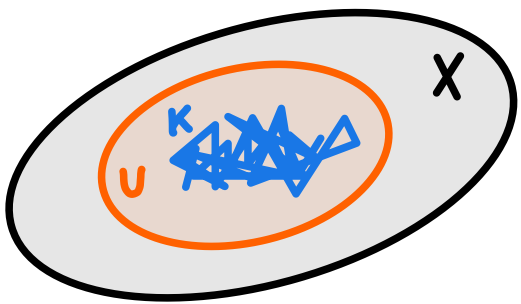
$$H_p(S^n) \cong \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & p \neq 0, n \end{cases}$$

APPLICATIONS

- Homologie locale de $K \subset X$ $\qquad H_n(X, X \setminus K)$

Par excision $\quad H_n(X, X \setminus K) \cong H_n(U, U \setminus K)$

$\forall\, U \subset X$ voisinage de K



Thm (Invariance du domaine)

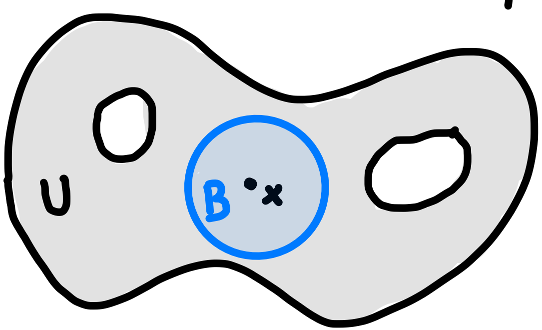
$\nexists$ homeomorphisme $\Phi\ U \xrightarrow{\cong} V$, où $U \subset \mathbb{R}^n$ ouvert, $V \subset \mathbb{R}^m$ ouvert, $n \neq m$

Preuve

Un tel homeo induirait un isomorphisme

$$\Phi_*: H_p(U, U \setminus x) \longrightarrow H_p(V, V \setminus \Phi(x))$$

$\forall x \in U$



$B \subset U$ voisinage de x homeomorphe à la boule compacte B^n

$$H_p(U, U \setminus \{x\}) \cong H_p(B, \underbrace{B \setminus \{x\}}_{\substack{\simeq\, \partial B \\ \text{(homotopiquement} \\ \text{equivalent)}}})$$

$$\cong H_p(B, \partial B) = \begin{cases} \mathbb{Z}, & p = \dim(U) \\ 0, & p \neq \dim(U) \end{cases}$$

et par le même argument.

$$H_p(V, V \setminus \{\phi(x)\}) \cong \begin{cases} \mathbb{Z}, & p = \dim(V) \\ 0, & p \neq \dim(V) \end{cases}$$

$\square$

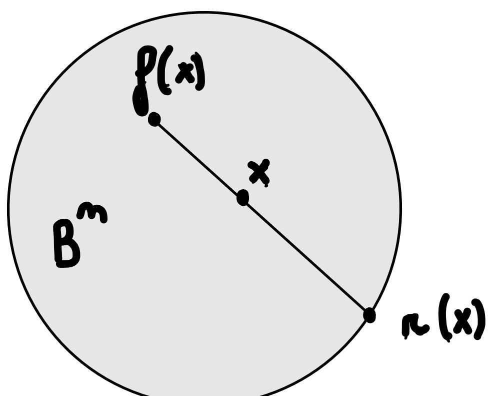
Thm du point fixe de Brouwer

($B^n \subset \mathbb{R}^n$ n-boule fermée)

Toute $f : B^n \longrightarrow B^n$ continue a un point fixe

i e $x \in B^n$ t.q $f(x) = x$

Preuve



Si $\nexists\, x \in B^n$ t.q. $f(x) = x$

on a une retraction

$$r : B^n \longrightarrow \partial B^n$$

$$\begin{tikzcd} \partial B^n \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}"'] & B^n \arrow[r, "r"] & \partial B^n \end{tikzcd}$$

$$\begin{tikzcd} H_{n-1}(\partial B^n) \arrow[r, "i_*"] \arrow[rr, bend right, "\cong", "\mathrm{id}_*"'] & H_{n-1}(B^n) \arrow[r, "r_*"] & H_{n-1}(\partial B^n) \end{tikzcd}$$

$\Rightarrow r_*$ surjectif

mais $H_{n-1}(B^n) = \begin{cases} 0, & n > 1 \\ \mathbb{Z}, & n = 1 \end{cases}$ $\qquad H_{n-1}(\partial B^n) = \begin{cases} \mathbb{Z}, & n > 1 \\ \mathbb{Z} \oplus \mathbb{Z}, & n = 1 \end{cases}$

$\square$

DEGRÉ TOPOLOGIQUE (sur S^n)

$f : S^n \to S^n$ continue, $n \geq 0$

$$f_* \quad \underset{\displaystyle \mathbb{Z}}{\underset{\rotatebox{90}{$\cong$}}{\tilde{H}_n(S^n)}} \longrightarrow \underset{\displaystyle \mathbb{Z}}{\underset{\rotatebox{90}{$\cong$}}{\tilde{H}_n(S^n)}} \qquad f_*(\sigma) = \underbrace{\deg(f)}_{\in \mathbb{Z}}\, \sigma$$

Rmqs

- $\deg(\mathrm{id}) = 1$
- $\deg(f) = \deg(g) \quad \forall f \simeq g$ (homotopes)

- $\deg(f \circ g) = \deg(f) \cdot \deg(g) \quad \forall f, g : S^n \to S^n$

  (car $(f \circ g)_* = f_* \circ g_*$)

- $\deg(f) = 0 \quad \forall f : S^n \to S^n$ non surjective

Preuve

$x \notin f(S^n) \Rightarrow f : S^n \to S^n \setminus \{x\}$

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & \tilde{H}_n(S^n) \\
& \tilde{H}_n(S^n \setminus \{x\}) \arrow[ur, "incl_*"'] \arrow[d, phantom, "\cong" rotate=90] & \\
& \tilde{H}_n(B^n) = 0 &
\end{tikzcd}$$

$\Rightarrow f_* = 0$

$\square$

- $S^n \subset \mathbb{R}^{n+1}$ sphère unité

$f_i : S^n \longrightarrow S^n$, $\quad i \in \{1, .., n+1\}$

$f_i(x_1, .., x_{n+1}) = (x_1, \ldots, -x_i, .., x_{n+1})$

Prop $\deg(f_i) = -1$

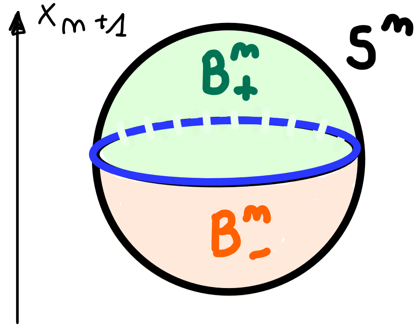
Preuve (on peut supposer $i = 1$, $f = f_1$)

- $\underline{n = 0}$ $\quad S^0 = \{-1, 1\}$ $\quad C_0(S^0) = \{\underset{\underset{-1}{|||}}{\sigma_-}, \underset{\underset{1}{|||}}{\sigma_+}\}$

$\tilde{H}_0(S^0) = \langle[\sigma_- - \sigma_+]\rangle$ $\quad f_*[\sigma_- - \sigma_+] = [\sigma_+ - \sigma_-]$

- Supposons vrai pour $S^0, \dots, S^{n-1}$

- $\underline{n}$.



$$B^n_\pm = \{ x \in S^n \mid \pm x_{n+1} \geq 0 \}$$

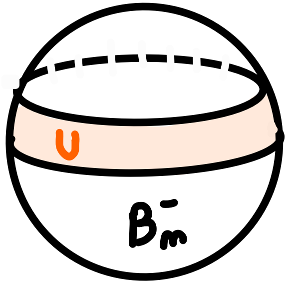
$x = (x_1, \dots, x_{n+1})$

$$B^n_- \cap B^n_+ = S^{n-1}$$

$$H_n(B^n_-, S^{n-1}) \xrightarrow[\cong]{\mathrm{incl}_*} H_n(S^n, B^n_+)$$

Essentiellement à cause de l'excision

(Plus précisement : $B_n^- \xrightarrow{\text{incl}} B_n^- \cup U$, $S^{n-1} \xrightarrow{\text{incl}} U$ sont équiv. d'homotopie



$$H_n(B_n^-, S^{n-1}) \xrightarrow{\cong} H_n(B_n^- \cup U, U) \underset{\text{exc.}}{\xrightarrow{\cong}} H_n(S^n, B_n^+)$$
)

suites exactes.

$$\underset{\underset{0}{\parallel}}{\tilde{H}_n(B_+^n)} \longrightarrow \tilde{H}_n(S^n) \xrightarrow{\cong} \tilde{H}_n(S^n, B_+^n) \xrightarrow{\partial_*} \underset{\underset{0}{\parallel}}{\tilde{H}_{n-1}(B_+^n)}$$

$$\underset{\underset{0}{\parallel}}{\tilde{H}_n(B_-^n)} \longrightarrow \tilde{H}_n(B_-^n, S^{n-1}) \underset{\cong}{\xrightarrow{\partial_*}} \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \underset{\underset{0}{\parallel}}{\tilde{H}_{n-1}(B_-^n)}$$

Donc.

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "f_*"] & \tilde{H}_n(S^n, B^n_+) \arrow[d, "f_*"] & \tilde{H}_n(B^n_-, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong", "\partial_*"'] \arrow[d, "f_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "f_* = -1"] \\
\tilde{H}_n(S^n) \arrow[r, "\cong"] & \tilde{H}_n(S^n, B^n_+) & \tilde{H}_n(B^n_-, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong", "\partial_*"'] & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

$\square$

Cor $-id \quad S^n \longrightarrow S^n$

$x \longmapsto -x$

$$\deg(-id) = (-1)^{n+1}$$

Thm de la sphère chevelue

$\exists$ champ de vecteurs V sur S^n
qui ne s'annule nulle part

ssi

n est impair

(from Wikipedia)

Preuve

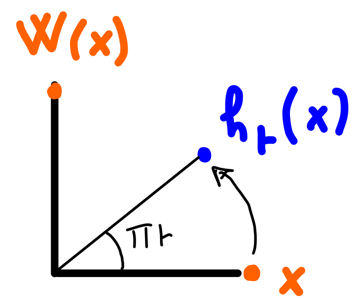
- Soit V un tel champ de vect. sur S^n

$$V(x) \neq 0 \quad \forall x \in S^n$$

$$W(x) = \frac{V(x)}{|V(x)|} \implies \begin{array}{l} |W(x)| = 1 \\ \langle x, W(x) \rangle = 0 \end{array} \quad \forall x \in S^n$$

$$h_t : S^n \longrightarrow S^n, \quad h_t(x) = \cos(\pi t)\, x + \sin(\pi t)\, W(x)$$

$t \in [0,1]$



$h_0 = \mathrm{Id}, \quad h_1 = -\mathrm{Id}$

$$\underbrace{\deg(h_0)}_{=\,1} = \underbrace{\deg(h_1)}_{=\,(-1)^{n+1}} \Longrightarrow n \text{ impair}$$

- Sur $S^{2n-1} \subset \mathbb{C}^n$ (sphère unité)

on a le champ de vecteurs

$$V(z_1, .., z_n) = (iz_1, .., iz_n) \quad \forall (z_1 . z_n) \in S^{2n-1}$$

$$\left(\langle z, V(z) \rangle = 0 \;\; \forall z \in S^{2n-1}, \quad \text{donc } V(z) \in T_z S^{2n-1} \right)$$

$\square$

CW-COMPLEXES

(C = closure-finite
W = weak topology)

Un CW complexe est un espace topologique de la forme

$$X = \bigcup_{n \geq 0} X^n$$

où

1) le 0-squelette X^0 est discret

2) le n-squelette X^n est un espace topologique de la forme

$$X^n = \frac{X^{n-1} \amalg \coprod_\alpha B^n_\alpha}{\sim}$$

où $B_\alpha^n \subset \mathbb{R}^n$ est la boule unité compacte

$$\phi_\alpha \quad \partial B_\alpha^n \longrightarrow X^{n-1} \quad (\text{continue})$$

application d'attachement

$$x \sim \phi_\alpha(x) \quad \forall x \in \partial B_\alpha^n$$

n-cellule $e_\alpha^n \subset X^n$ image de B_α^n dans X^n

⚠ $(e_\alpha^n)^\circ \overset{\text{homeom}}{\cong} (B^n)^\circ$, mais ~~$e_\alpha^n \cong B^n$~~

3) $X = \bigcup_{n \geq 0} X^n$ est muni de la **topologie faible**

i e $A \subset X$ est ouvert

ssi

$A \cap X^n$ ouvert de X^n

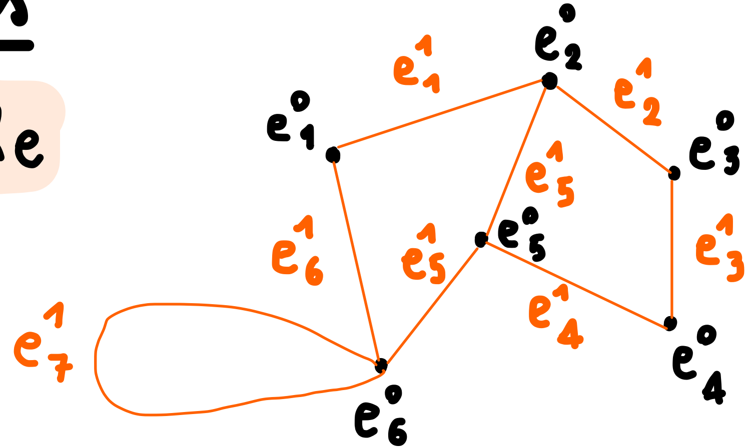
(On travaillera surtout avec CW complexes finis.

$X = X^0 \cup X^1 \cup X^2 \cup \dots \cup X^d$

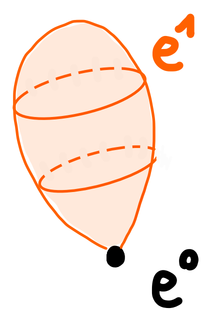
$X^n = X^{n-1} \cup e^n_{\alpha_1} \cup \dots \cup e^n_{\alpha_{K_n}} \quad \forall n \geq 0$)

exemples

- graphe



- sphère $S^n = e^0 \cup e^1$

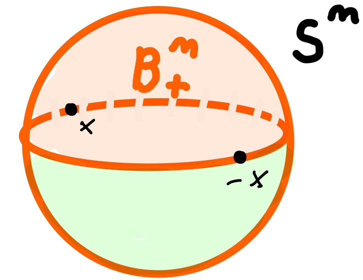


- espace projectif réel

$$\mathbb{R}P^n = S^n/\sim \; = B^n_+/\sim$$

où $x \sim -x$

$$= \underbrace{\frac{\partial B^n_+}{\sim}}_{\mathbb{R}P^{n-1}} \cup \underbrace{(B^n_+)^\circ}_{(e^n)^\circ}$$



$$\mathbb{R}P^0 = e^0$$

$$\mathbb{R}P^n = \underbrace{e^0 \cup .. \cup e^{n-1}}_{\mathbb{R}P^{n-1}} \cup e^n$$

- espace projectif complexe $\mathbb{C}P^n$

$S^{2n+1} \subset \mathbb{C}^{n+1}$ sphère unité

$$\mathbb{C}P^n = S^{2n+1}/\sim \; = \; B^{2n}/\sim$$

où $z \sim e^{i\theta} z \quad \forall z \in S^{2n+1}, \; e^{i\theta} \in S^1$

$$B^{2n} = \left\{ z = (z_1, \dots, z_{n+1}) \in S^{2n+1} \;\middle|\; z_{n+1} \geq 0 \right\}$$

$$z_{n+1} = \sqrt{1 - |z_1|^2 - \dots - |z_{n-1}|^2}$$

$$B^{2n}/\sim \; = \; \underbrace{\frac{\partial B^{2n}}{\sim}}_{= \mathbb{C}P^{n-1}} \cup \underbrace{(B^{2n})^\circ}_{= (e^{2n})^\circ}$$

$$\mathbb{C}P^1 \cong S^2 = e_0 \cup e_2$$

$$\mathbb{C}P^n = \underbrace{e^0 \cup e^2 \cup \dots \cup e^{2n-2}}_{\mathbb{C}P^{n-1}} \cup e^{2n}$$