HOMOLOGIE CELLULAIRE

Thm Soit X un CW-complexe. Il existe un complexe de chaînes $(C_*^{cw}(X), \delta_*)$ tq

$$C_n^{cw}(X) = \text{groupe abelien libre engendré par les } n\text{-cellules}$$

$$H_n(C_*^{cw}(X)) \cong H_n(X)$$

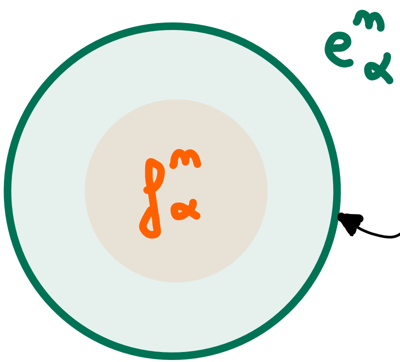
$$\left(C_n^{cw}(X) \cong H_n(X^n, X^{n-1}) \right)$$

Preuve

$$X = X_0 \cup X_1 \cup X_2 \cup \dots \qquad \text{CW-complexe}$$

$$\boxed{X^n = \frac{X^{n-1} \cup \bigcup_\alpha B^n_\alpha}{\sim} \qquad x \sim \phi_\alpha(x), \quad \phi_\alpha : \partial B^n_\alpha \to X^{n-1}}$$

- $e^n_\alpha \subset X^n$ n-cellule

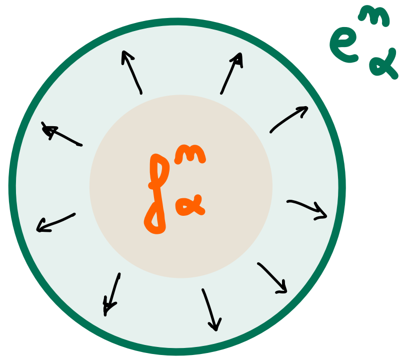


le borde ∂e^n_α n'est pas forcement homeomorphe à S^{n-1}

f^n_α n-boule ouverte t.q. $\overline{f^n_\alpha} \subset \underbrace{(e^n_\alpha)^\circ}_{\cong B^n \text{ boule ouverte}}$

$\exists$ retraction par déformation $r_t : e_\alpha^n \to e_\alpha^n$

t q $r_0 = id$, $r_t|_{\partial e_\alpha^n} = id$, $r_1(e_\alpha^n \setminus f_\alpha^n) = \partial e_\alpha^n$



On peut recoller ces retractions par deformation.

- $Y^n := X^n \setminus \bigcup_\alpha f_\alpha^n$

  on a une retraction par déformation

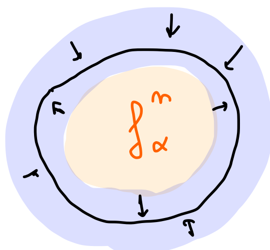
  $r_t : X^n \longrightarrow X^n$ t.q $r_0 = \mathrm{id}$, $r_t|_{X^{n-1}} = \mathrm{id}$ $\forall t$

  $r_1(Y^n) = X^{n-1}$

- $H_k(X^n, X^{n-1}) \underset{\mathrm{incl}_*}{\xrightarrow{\cong}} H_k(X^n, Y^n)$ (grace à r_t)

- $H_k(X^n, Y^n) \xleftarrow{\cong} H_k\left(\bigcup_\alpha (e_\alpha^n)^\circ, \bigcup_\alpha (e_\alpha^n)^\circ \setminus f_\alpha^n\right)$ (par excision)

• $$H_k\left(\bigcup_\alpha (e^m_\alpha)^\circ, \bigcup_\alpha (e^m_\alpha)^\circ \setminus f^m_\alpha\right) = \bigoplus_\alpha \underbrace{H_k\left((e^m_\alpha)^\circ, (e^m_\alpha)^\circ \setminus f^m_\alpha\right)}_{\cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m \end{cases}}$$

$(e^m_\alpha)^\circ =$ 

$$H_k(X^m, X^{m-1}) \cong \begin{cases} \bigoplus_\alpha \mathbb{Z} & \text{si } k=m \\ 0 & \text{si } k \neq m \end{cases}$$

$$C^{cw}_m(X) = H_m(X^m, X^{m-1})$$

- suite exacte longue de $X^{n-1} \subset X^n$

$$\underset{\substack{\parallel \\ 0 \text{ si } k+1 \neq n}}{H_{k+1}(X^n, X^{n-1})} \xrightarrow{\partial_*} H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow \underset{\substack{\parallel \\ 0 \text{ si } k \neq n}}{H_k(X^n, X^{n-1})}$$

$$\Rightarrow H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n) \quad \forall k \neq n-1, n$$

- $H_k(X^n) \cong H_k(X^0) = 0 \quad \forall k > n$

- $H_k(X^n) \xrightarrow[\cong]{incl_*} H_k(X^m) \quad \forall k < n \leq m$

- $\boxed{H_k(X^n) \xrightarrow[\cong]{incl_*} H_k(X) \quad \forall k < n}$

$[\sigma] \in H_k(X)$ $\qquad \sigma = \sigma_1 + . + \sigma_\ell$, où σ_i sont simplexes sing

$\sigma_i(\Delta^k)$ compact $\Rightarrow \sigma_i(\Delta^k) \subset X^{m_i}$

$m := \max\{n, m_1, ., m_\ell\}$

$\sigma \in C_k(X^m)$

$$\Rightarrow \begin{array}{ccccc} H_k(X^n) & \xrightarrow{\cong} & H_k(X^m) & \longrightarrow & H_k(X) \\ \Psi & & \Psi & & \Psi \\ h & \longmapsto & [\sigma] & \longmapsto & [\sigma] \end{array}$$

$$\begin{array}{ccc} H_k(X^n) & \xrightarrow{\ incl_*\ } & H_k(X) \\ [\sigma] & \longmapsto & 0 \end{array}$$

donc $\exists\, \mu \in C_{k+1}(X)$ t.q $\sigma = \partial \mu$

par le même argument de la page précédente

$\mu \in C_{k+1}(X^m)$ pour un certain $m \geq n$

donc
$$\begin{array}{ccc} H_k(X^n) & \xrightarrow{\ incl_*\ } & H_k(X^m) \\ [\sigma] & \longmapsto & 0 \end{array}$$

$\Rightarrow [\sigma] = 0$, car $H_k(X^n) \xrightarrow{incl_*} H_k(X^m)$ isomorphisme $\forall\, k < n \leq m$

•

$$
\begin{tikzcd}
 & & & & & 0 & & \\
 & & 0 = H_m(X^{m-1}) \arrow[dr] & & H_m(X^{m+1}) \arrow[ur] & \cong H_m(X) & & \\
 & & & H_m(X^m) \arrow[ur] \arrow[dr] & & & & \\
 {} \arrow[rr, "\delta_{m+2}"] & & H_{m+1}(X^{m+1}, X^m) \arrow[ur, "\partial_*"] \arrow[rr, "\delta_{m+1}"] & & H_m(X^m, X^{m-1}) \arrow[dr, "\partial_*"'] \arrow[rr, "\delta_m"] & & H_{m-1}(X^{m-1}, X^{m-2}) \arrow[r, "\delta_{m-1}"] & {} . \\
 & & & & & H_{m-1}(X^{m-1}) \arrow[ur] & & \\
 & & & & 0 = H_{m-1}(X^{m-2}) \arrow[ur] & & &
\end{tikzcd}
$$

exact

exact

exact

$$\boxed{\delta_m \circ \delta_{m+1} = 0}$$

•

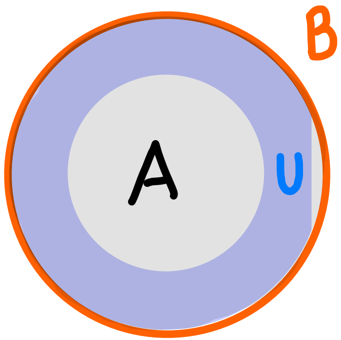
$$\begin{tikzcd}
 & & & H_m(X^{m+1}) \arrow[r, phantom, "\cong"] & H_m(X) & & \\
 & & H_m(X^m) \arrow[ur] \arrow[dr, hook, "\iota_*"] & & & & \\
 {} \arrow[r, "\delta_{m+2}"] & H_{m+1}(X^{m+1}, X^m) \arrow[ur, "\partial_{m+1}"] \arrow[rr, "\delta_{m+1}"] & & H_m(X^m, X^{m-1}) \arrow[dr, "\partial_m"'] \arrow[rr, "\delta_m"] & & H_{m-1}(X^{m-1}, X^{m-2}) \arrow[r, "\delta_{m-1}"] & . \\
 & & & & H_{m-1}(X^{m-1}) \arrow[ur, hook, "j_*"'] & &
\end{tikzcd}$$

$$\begin{aligned}
H_m(X) &\cong H_m(X^m)/\operatorname{Im}(\partial_{m+1}) \cong \operatorname{Im}(\iota_*)/\operatorname{Im}(\delta_{m+1}) \\
&\cong \operatorname{Ker}(\partial_m)/\operatorname{Im}(\delta_{m+1}) \cong \operatorname{Ker}(\delta_m)/\operatorname{Im}(\delta_{m+1})
\end{aligned}$$

$$\operatorname{Ker}(\delta_m)/\operatorname{Im}(\delta_{m+1}) \cong H_m(C_*^{cw}(X))$$

$\square$

Rmq (homologie relative vs homologie quotient)



soit $B \subset A$ fermé tq

$\exists\, U \subset A$ voisinage de B, $r_t : U \to U$

avec $r_0 = id$, $r_t|_B = id$ $\forall t$, $r_1 : U \to B$

L'application quotient q $A \longrightarrow A/B$
induit un isomorphisme

$$q_* \quad H_n(A,B) \longrightarrow H_n(A/B, B/B) \cong \widetilde{H}_n(A/B)$$

Preuve

- $$\begin{tikzcd}
H_n(B) \arrow[r] \arrow[d, "\cong"] & H_n(A) \arrow[r] \arrow[d, "="] & H_n(A,B) \arrow[r, "\partial_*"] \arrow[d] & H_{n-1}(B) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(B) \arrow[d, "="] \\
H_n(U) \arrow[r] & H_n(A) \arrow[r] & H_n(A,U) \arrow[r, "\partial_*"] & H_{n-1}(U) \arrow[r] & H_{n-1}(A)
\end{tikzcd}$$

isomorphisme (par lemme des 5)

- $$\begin{tikzcd}
H_n(A,B) \arrow[r, "\cong"] \arrow[d, "q_*"] & H_n(A,U) \arrow[d, "q_*"] & H_n(A \setminus B, U \setminus B) \arrow[l, "\cong", "\text{(excision)}"'] \arrow[d, "q_*", "\cong"'] \\
H_n(A/B, B/B) \arrow[r, "\cong"] & H_n(A/B, U/B) & H_n((A/B) \setminus (B/B), (U/B) \setminus (B/B)) \arrow[l, "\cong"]
\end{tikzcd}$$

isom. car $B/B \hookrightarrow U/B$ equiv d'homot. (pour cela, il faut B fermée)

$\square$

⚠ L'hypothèse sur B est nécessaire

$$H_n(B^2, B^2 \setminus \{0\}) \cong H_n(B^2, \partial B^2) \cong \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

$$\tilde{H}_n\Big(\underbrace{B^2/(B^2\setminus\{0\})}_{\{x,y\} \text{ avec topologie. } \emptyset,\ \{x\},\ \{x,y\}}\Big) = 0$$

it is contractible: $h_t : \{x,y\} \longrightarrow \{x,y\}$

$h_0 = id$

$h_t \equiv x \quad \forall t > 0$

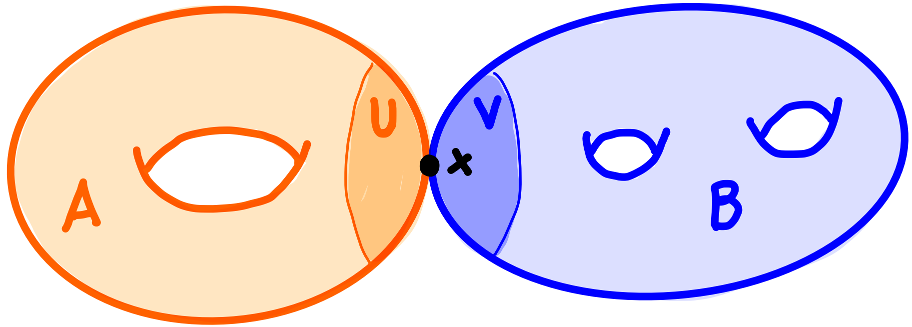
Rmq (homologie d'un bouquet d'espaces)

A, B espaces topologiques pointés
(i e avec deux points fixés $a \in A$, $b \in B$)

bouquet $A \vee B = (A \amalg B)/\sim$, où $a \sim b$

Si a et b sont retractions par déformation de certains leurs voisinages, alors
$$\tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow[\cong]{i_* + j_*} \tilde{H}_n(A \vee B) \quad \text{isom.}$$
où $i : A \to A \vee B$, $j : B \to A \vee B$ sont inclusions

Preuve



$x = a \equiv b$

$$
\begin{tikzcd}
\tilde{H}_n(A) \oplus \tilde{H}_n(B) \arrow[r, "i_* + j_*"] \arrow[d, "\cong"] & \tilde{H}_n(A \vee B) \arrow[d, "\cong"] \\
H_n(A, a) \oplus H_n(B, b) \arrow[r, "i_* + j_*"] \arrow[d, "\cong"] & H_n(A \vee B, x) \arrow[d, "\cong"] \\
H_n(A, U) \oplus H_n(B, V) \arrow[r, "i_* + j_*"] & H_n(A \vee B, U \vee V) \\
H_n(A \setminus a, U \setminus a) \oplus H_n(B \setminus b, V \setminus b) \arrow[u, "\text{exc}", "\cong"'] \arrow[r, phantom, "="] & H_n(A \vee B \setminus x, U \vee V \setminus x) \arrow[u, "\text{exc.}"', "\cong"]
\end{tikzcd}
$$

$\square$

NOTATION.

$\phi_\alpha \quad \partial B_\alpha^n \longrightarrow X^{n-1}$ application d'attach

$$\begin{array}{l} \tilde{\phi}_\alpha \quad B_\alpha^n \longrightarrow e_\alpha^n \\ \qquad\qquad\qquad\quad \cap \\ \qquad\qquad\qquad\quad X^n \end{array}$$

$\tilde{\phi}_\alpha |_{\partial B_\alpha^n} = \phi_\alpha$

$\phi_\alpha |_{\mathring{B}_\alpha^n} = \text{inclusion}$

X CW complexe, $n > 1$

$$
\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[rrrr, "\delta_n"] \arrow[drr, "\partial_*"] \arrow[dd, "q_*"', "\cong"] & & & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[dd, "\cong"', "q_*"] \\
& & H_{n-1}(X^{n-1}) \arrow[urr] \arrow[drr, "q_*"'] & & \\
H_n(X^n/X^{n-1}) & & & & H_{n-1}(X^{n-1}/X^{n-2}) \\
\bigoplus_\alpha H_n(B^n_\alpha/\partial B^n_\alpha) \arrow[u, "\sum_\alpha \tilde{\phi}_{\alpha *}", "\cong"'] & & \bigoplus_\alpha H_{n-1}(\partial B^n_\alpha) \arrow[uu, "\sum_\alpha \phi_{\alpha *}"] \arrow[urr, "\sum_\alpha \phi_{\alpha *}"] \arrow[rr, dashed] & & \bigoplus_\beta H_{n-1}(B^{n-1}_\beta/\partial B^{n-1}_\beta) \arrow[u, "\sum_\beta \tilde{\phi}_{\beta *}", "\cong"'] \\
\bigoplus_\alpha H_n(B^n_\alpha, \partial B^n_\alpha) \arrow[u, "q_*", "\cong"'] \arrow[urr, "\partial_*", "\cong"'] \arrow[rrrr, dashed] & & & & \bigoplus_\beta H_{n-1}(B^{n-1}_\beta, \partial B^{n-1}_\beta) \arrow[u, "q_*", "\cong"']
\end{tikzcd}
$$

par rmq précédente

homol. non réduite

<u>Récapitulatif</u>.

X CW-complexe, $\tilde{\Phi}_\alpha : (B^n_\alpha, \partial B^n_\alpha) \to (e^n_\alpha, \partial e^n_\alpha) \cap (X^n, X^{n-1})$

$$C^{CW}_n(X) = \bigoplus_\alpha \underbrace{H_n(B^n_\alpha, \partial B^n_\alpha)}_{\langle h_\alpha \rangle}$$

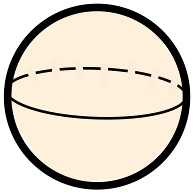
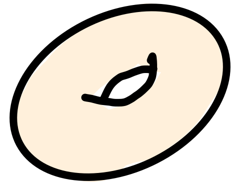
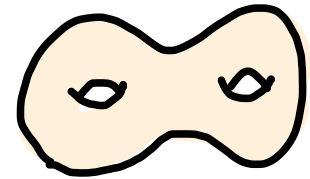
$$\delta_n \quad C^{CW}_n(X) \longrightarrow C^{CW}_{n-1}(X) \qquad \delta_n(h_\alpha) = \sum_\beta \deg(\varphi_{\alpha\beta})\, h_\beta$$

où $\quad S^{n-1} \equiv \partial B^n_\alpha \qquad B^{n-1}_\beta / \partial B^{n-1}_\beta \equiv \begin{cases} S^{n-1} & n > 1 \\ \{p\} & n = 1 \end{cases}$

$$
\begin{tikzcd}
S^{n-1} \equiv \partial B^n_\alpha \arrow[r, "\phi_\alpha"] \arrow[dr, "\psi_{\alpha\beta}"'] & X^{n-1} / \left(X^{n-1} \setminus \overset{\circ}{e}{}^{n-1}_\beta \right) \\
& B^{n-1}_\beta / \partial B^{n-1}_\beta \arrow[u, "\tilde{\phi}_\beta", "\cong"'] \equiv \begin{cases} S^{n-1} & n > 1 \\ \{p\} & n = 1 \end{cases}
\end{tikzcd}
$$

exemples

- Σ_g = surface orientable de genre g

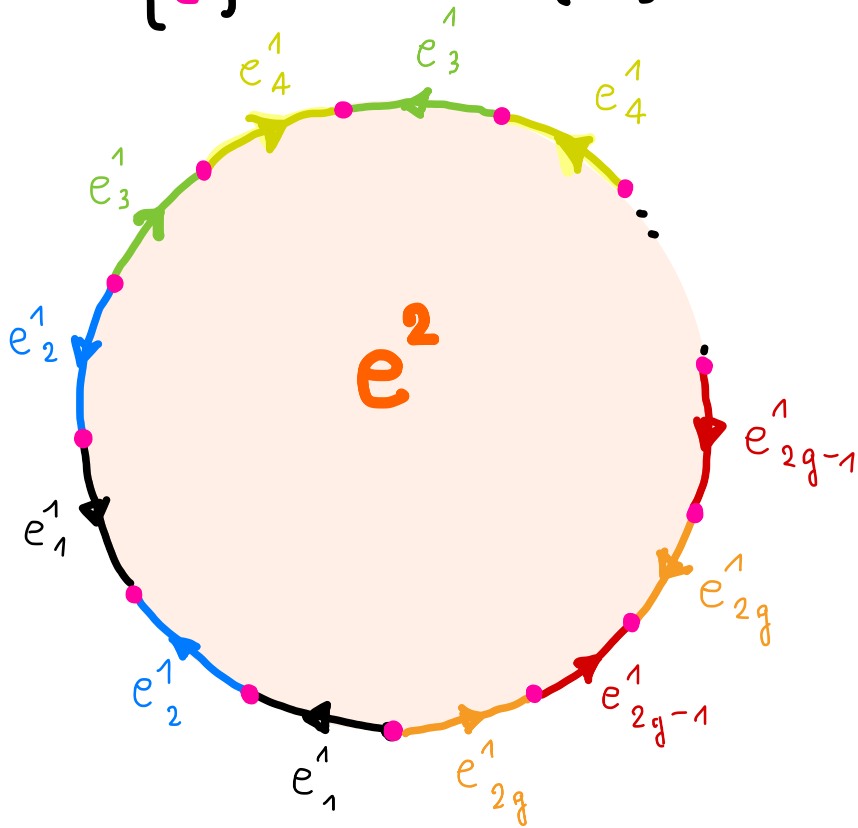
$g=0$  $g=1$  $g=2$ 

Σ_g est bien un CW-complexe

de dimension 2

avec une 0-cellule, $2g$ 1-cellule, une 2-cell.

$$\Sigma_g = X = \underset{\{e^0\}}{\underset{\shortparallel}{X^0}} \cup X^1 \cup \underset{\{e^2\}}{\underset{\shortparallel}{X^2}}$$



$$C_0^{cw}(\Sigma_g) = \langle h^0 \rangle \cong \mathbb{Z}$$

$$C_1^{cw}(\Sigma_g) = \langle h_1^1, h_2^1, \dots, h_{2g-1}^1, h_{2g}^1 \rangle \cong \mathbb{Z}^{2g}$$

$$C_2^{cw}(\Sigma_g) = \langle h^2 \rangle \cong \mathbb{Z}$$

$$\delta_1(h_\alpha^1) = h^0 - h^0 = 0$$

$$\delta_2(h^2) = h_1^1 + h_2^1 - h_1^1 - h_2^1 + . + h_{2g-1}^1 + h_{2g}^1$$
$$- h_{2g-1}^1 - h_{2g}^1 = 0$$

$$0 \xrightarrow{\delta_3} C_2^{cw}(\Sigma_g) \xrightarrow{\delta_2 = 0} C_1^{cw}(\Sigma_g) \xrightarrow{\delta_1 = 0} C_0^{cw}(\Sigma_g)$$

$$\begin{array}{ccc} \cong & \cong & \cong \\ \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \end{array}$$

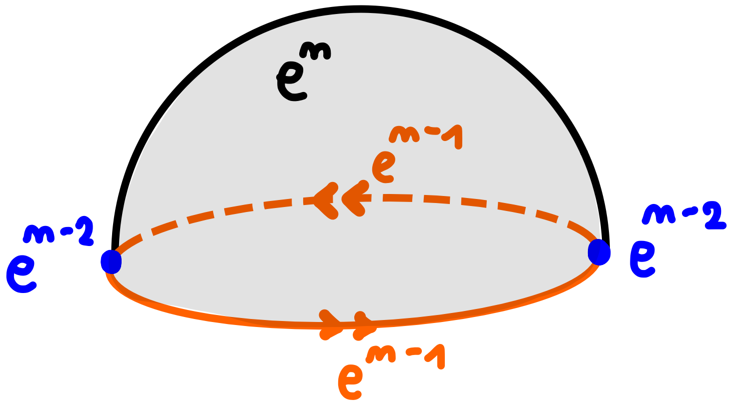
$$H_m(\Sigma_g) \cong H_m(C_*^{cw}(\Sigma_g)) = \begin{cases} \mathbb{Z} & m = 0, 2 \\ \mathbb{Z}^{2g} & m = 1 \\ 0 & m \neq 0, 1, 2 \end{cases}$$

- $\mathbb{C}P^d = e^0 \cup e^2 \cup \dots \cup e^{2d}$

$$H_n(\mathbb{C}P^d) \cong \begin{cases} \mathbb{Z} & n = 0, 2, \dots, 2d \\ 0 & n \text{ autrement} \end{cases}$$

• $\mathbb{RP}^d = \underbrace{e^0 \cup e^1 \cup \dots \cup e^{d-1}}_{\mathbb{RP}^{d-1}} \cup e^d$

$$C_m^{CW}(\mathbb{RP}^d) = \begin{cases} \langle h_m \rangle \cong \mathbb{Z} & \text{si } m = 0, \dots, d \\ 0 & \text{si } m \notin \{0, \dots, d\} \end{cases}$$



$$\delta_m(h_m) = \deg(\psi_m)\, h_{m-1}$$

où $\tilde{\Phi}_m : B^m \longrightarrow e^m$ appl. d'attach

$$\Phi_m = \tilde{\Phi}_m\big|_{\partial B^m} \quad \underset{\displaystyle S^{m-1}}{\underset{\cong}{\partial B^m}} \longrightarrow \underset{\displaystyle S^{m-1}}{\underset{\cong}{e^{m-1}/e^{m-2}}}$$

$$
\begin{tikzcd}
\partial B^n \arrow[rr, "\Phi_n"] \arrow[d, phantom, "\cong" rotate=90] & & e^{n-1}/e^{n-2} \arrow[d, phantom, "\cong" rotate=90] \\
S^{n-1} \arrow[r, "q"] \arrow[dr, "g"'] & S^{n-1}/S^{n-2} \arrow[d, phantom, "\cong" rotate=90] & S^{n-1} \\
& S_a^{n-1} \vee S_b^{n-1} \arrow[ur, "f"'] &
\end{tikzcd}
$$

$$f|_{S_a^{n-1}} = id$$

$$g(y) = (y, y)$$

$$f|_{S_b^{n-1}} \simeq -id$$

$$
\begin{aligned}
H_{n-1}(S^{n-1}) &\xrightarrow{g_*} H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1}) \xrightarrow{f_*} H_{n-1}(S^{n-1}) \\
h &\longmapsto (h, h) \\
& \qquad (k, l) \longmapsto k + (-1)^n l \\
h &\longmapsto (1 + (-1)^n) h
\end{aligned}
$$

$$\left(C_*^{cw}(\mathbb{RP}^d), \delta_*\right).$$

- si $d = 2p$ pair

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow 0 \\
& \shortparallel & & & & & & \shortparallel & & \shortparallel & \\
& C_d^{cw} & & & & & & C_1^{cw} & & C_0^{cw} &
\end{array}$$

$$H_m(\mathbb{R}^{2p}) \cong \begin{cases} \mathbb{Z} & m = 0 \\ \mathbb{Z}/2\mathbb{Z} & m = 1, 3, 5, \dots, 2p-1 \\ 0 & \text{autrement} \end{cases}$$

• si $d = 2p+1$ impair

$$0 \longrightarrow \underset{\substack{\parallel \\ C_d^{cw}}}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \underset{\substack{\parallel \\ C_1^{cw}}}{\mathbb{Z}} \xrightarrow{0} \underset{\substack{\parallel \\ C_0^{cw}}}{\mathbb{Z}} \longrightarrow 0$$

$$H_n(\mathbb{R}^{2p+1}) \simeq \begin{cases} \mathbb{Z} & n = 0,\ 2p+1 \\ \mathbb{Z}/2\mathbb{Z} & n = 1, 3, 5, \dots, 2p-1 \quad (p \geq 1) \\ 0 & \text{autrement} \end{cases}$$