CHARACTÉRISTIQUE D'EULER

$$\chi(W) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rang}\big(H_n(W)\big)$$

$$H_n(W) = \mathbb{Z}^r \oplus T$$

$\chi(W)$ est un invariant topologique
(pas toujours défini)

Thm Si W est un CW complexe $\overbrace{\text{fini}}^{\#\text{ fini de cellules}}$,

$$c_n(W) := \operatorname{rang}\big(H_n(W^n, W^{n-1})\big) = \#\ n\text{-cellules}$$

Alors $$\chi(W) = \sum_{n\geq 0} (-1)^n c_n(W)$$

Preuve

Conséquence de l'homologie cellulaire et du thm suivant.

$$0 \to C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdot \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$ complexe des chaînes

Si chaque C_n est de type fini, alors

$$\sum_{n \geq 0} (-1)^n \operatorname{rang}(C_n) = \sum_{n \geq 0} (-1)^n \operatorname{rang}(H_n(C_*))$$

$$H_n(C_*) = \frac{Z_n}{B_n} \qquad \text{où} \quad Z_n = \operatorname{Ker} \partial_n, \; B_n = \operatorname{im} \partial_{n+1}$$

$$0 \longrightarrow Z_n \xrightarrow{\text{incl}} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0 \quad \text{exacte}$$

$$0 \longrightarrow B_n \xrightarrow{\text{incl}} Z_n \xrightarrow{\text{quot}} H_n(C_*) \longrightarrow 0 \quad \text{exacte}$$

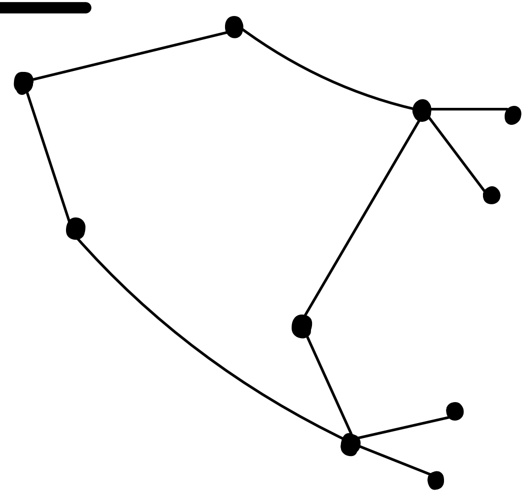
$$\operatorname{rang} C_n = \operatorname{rang} Z_n + \operatorname{rang} B_{n-1}$$

$$\operatorname{rang} Z_n = \operatorname{rang} B_n + \operatorname{rang} H_n(C_*)$$

$$\Rightarrow \quad \operatorname{rang} C_n = \operatorname{rang} H_n(C_*) + \operatorname{rang} B_n + \operatorname{rang} B_{n-1}$$

$\square$

exemple

$X =$ 

$\chi(X) = 10 - 10 = 0$

en fait

$X \simeq S^1$ (homot equiv)

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & n \neq 0,1 \end{cases}$$

$\chi(S^1) = 0$

HOMOLOGIE SINGULIÈRE avec coefficients

X espace topologique

G groupe abelien

$$C_n(X;G) := C_n(X) \otimes G$$

produit tensoriel sur $\mathbb{Z}$

$\sigma \in C_n(X;G)$

Rmq

$C_n(X;\mathbb{Z}) = C_n(X)$

$\sigma = g_1\sigma_1 + \dots + g_k\sigma_k$, où $g_i \in G$, $\sigma_i : \Delta^n \to X$ simp sing

$$\partial_n \cdot C_n(X) \longrightarrow C_{n-1}(X)$$

$\Big\downarrow$ extension

$$\partial_n : C_n(X, G) \longrightarrow C_{n-1}(X; G)$$

$$g\sigma \longmapsto g\,\partial_n\sigma$$

$g \in G$

σ simp sing

$$\xrightarrow{\partial_{n+1}} C_n(X, G) \xrightarrow{\partial_n} C_{n-1}(X, G) \xrightarrow{\partial_{n-1}} C_{n-2}(X; G) \xrightarrow{\partial_{n-2}}$$

$$\partial_{n-1} \circ \partial_n = 0$$

$H_n(\ ,G)$ a propriété similaires à $H_n(\ )$.

- $H_0(X;G) \cong G$ si X connexe par arcs
- $H_n(\{p\};G) = 0 \quad \forall n > 0$
- $H_n(S^d, G) \cong \begin{cases} G & n = 0, d \\ 0 & n \notin \{0, d\} \end{cases} \cong H_n(S^d) \otimes G$

  $d > 0$
- degré, Mayer-Vietoris, excision, homologie reduite, homologie relative, homologie cellulaire,

On utilise surtout $G = \mathbb{Q}, \mathbb{R}, \mathbb{Z}_p$

$$\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}, \ p \text{ premier}$$

⚠ $\cancel{H_n(X,G) \cong H_n(X) \otimes G}$

exemple

$$H_n(\mathbb{R}P^d) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n < d \quad \text{impair} \\ \mathbb{Z} & n = d \quad \text{impair} \\ 0 & \text{autrement} \end{cases}$$

Avec $G = \mathbb{Z}_2$, homologie cellulaire.

$$C_n^{cw}(\mathbb{R}P^d; \mathbb{Z}_2) = \begin{cases} \langle h_n \rangle \cong \mathbb{Z}_2 & \text{si } n = 0, \dots, d \\ 0 & \text{si } n \notin \{0, \dots, d\} \end{cases}$$

$$\cdots \xrightarrow{\delta_4 = 2} C_3^{cw} \xrightarrow{\delta_3 = 0} C_2^{cw} \xrightarrow{\delta_2 = 2} C_1^{cw} \xrightarrow{\delta_1 = 0} C_0^{cw}$$

mais $2 = 0$ dans $\mathbb{Z}_2$

$$\Rightarrow \boxed{H_n(\mathbb{R}P^d; \mathbb{Z}_2)} \cong C_n^{cw}(\mathbb{R}P^d; \mathbb{Z}_2) \cong \boxed{\begin{cases} \mathbb{Z}_2, & n = 0, \dots, d \\ 0, & \text{autrem} \end{cases}}$$

COHOMOLOGIE SINGULIÈRE

X espace topologique, G groupe abelien

$$\boxed{C^n(X,G)} := \mathrm{Hom}(C_n(X), G)$$

groupe des n-cochaînes singulières

$$\boxed{d_n} : C^n(X,G) \longrightarrow C^{n+1}(X;G)$$
$$\mu \longmapsto \mu \circ \partial_{n+1}$$

$$d_n = \partial^*_{n+1}$$

$$d_{n+1} \circ d_n = \partial^*_{n+2} \circ \partial^*_{n+1} = (\underbrace{\partial_{n+1} \circ \partial_{n+2}}_{=0})^*$$

$$0 \longrightarrow C^0(X;G) \xrightarrow{d_0} C^1(X;G) \xrightarrow{d_1} C^2(X;G) \xrightarrow{d_2} C^3(X,G) \xrightarrow{d_3} \cdots$$

complexe des cochaînes singulières

cohomologie singulière

$$\boxed{H^n(X, G)} = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n-1}}$$

(on écrit simplement $H^n(X) := H^n(X; \mathbb{Z})$)

⚠ $$\cancel{H^n(X; G) \cong \operatorname{Hom}(H_n(X), G)}$$

Une parenthèse d'algèbre homologique

G groupe abelien (coefficients)

$$A^* := \mathrm{Hom}(A, G)$$

groupe abelien duale

groupe abelien

- Si $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ est exacte, alors

$0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$ est exacte

$\alpha^* g = g \circ \alpha$

$\beta^* h = h \circ \beta$

(exercice)

•

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \quad \text{exacte}$$

$$\Downarrow$$

$$0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \to 0 \quad \text{exacte}$$

<u>**exemple**</u> $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \underset{\mathbb{Z}_n}{\overset{\mathbb{Z}/n\mathbb{Z}}{\shortparallel}} \to 0$ exacte

mais, si $G = \mathbb{Z}$, la suite duale

$$0 \to \underset{0}{\underset{\shortparallel}{\mathbb{Z}_n^*}} \to \underset{\mathbb{Z}}{\underset{\shortparallel}{\mathbb{Z}^*}} \xrightarrow{n} \underset{\mathbb{Z}}{\underset{\shortparallel}{\mathbb{Z}^*}} \to 0 \quad \text{pas exacte}$$

• Si $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ suite exacte courte **scindée**

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[dr, "i"'] & B \arrow[r, "\beta"] \arrow[d, "\psi", "\cong"'] & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, "p"'] & & \end{tikzcd}$$

$i(a) = (a, 0)$, $p(a,b) = b$

alors $0 \longrightarrow C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \longrightarrow 0$ exacte

$$\left(\begin{array}{l} \qquad\qquad\qquad\qquad \psi \\ \qquad\qquad\qquad\qquad \phi \\ \phi = \alpha^* \tilde{\phi} = \tilde{\phi} \circ \alpha, \\ \text{où} \quad \tilde{\phi} = (\phi, 0) \circ \psi \end{array} \right)$$

Rmq Toute $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exacte avec C libre est scindée

S base de C,
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] \arrow[l, bend left, dashed, "\phi"] & 0 \end{tikzcd}$$

On définie ϕ sur S t.q. $\phi(s) \in \beta^{-1}(s)$ $\forall s \in S$

et on étende à C

$\beta \circ \phi = id$ $\quad \psi : A \oplus C \xrightarrow{\cong} B$

$$(a, c) \longmapsto \alpha(a) + \phi(b)$$

$$\psi^{-1}(b) = \left(\alpha^{-1}(b - \phi \circ \beta(b)), \beta(b) \right)$$

Propriétés de la cohomologie singulière

- $$\begin{array}{ccc} & \partial_{n+1}^* & \\ & \parallel & \\ C^n(X,A;G) & \xrightarrow{d_n} & C^{n+1}(X,A;G) \\ \parallel & & \\ \mathrm{Hom}(C_n(X,A),G) & & \end{array}$$ $A \subset X$ esp top

cohomologie singulière relative

$$\boxed{H^n(X,A,G)} = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n-1}}$$

- $f : (X,A) \to (Y,B)$ continue

$$\left(\begin{array}{l} f : X \longrightarrow Y \quad \text{continue} \\ \quad\;\, \cup \\ \quad\;\, A \longrightarrow f(A) \subset B \end{array} \right)$$

$\Big\downarrow$ induit

$$f_* : C_n(X,A) \longrightarrow C_n(Y,B)$$

$$(\partial_{n+1} \circ f_*)^* = (f_* \circ \partial_{n+2})^*$$

$$f^* : C^n(Y,B;G) \longrightarrow C^n(X,A;G), \qquad f^* \circ d_n = d_{n+1} \circ f^*$$

$$\phi \longmapsto \phi \circ f_*$$

$$\boxed{f^* : H^n(Y,B,G) \longrightarrow H^n(X,A;G)}$$

cohomologie est contravariant

$$(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$$

$$\Rightarrow \quad (g \circ f)^* = f^* \circ g^*$$

- $$0 \longrightarrow C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{j_*} \underbrace{C_n(X,A)}_{\text{libre}} \longrightarrow 0$$

$$\Big\downarrow \text{ duale}$$

$$0 \to C^n(X,A,G) \xrightarrow{j^*} C^n(X) \xrightarrow{i^*} C^n(A) \longrightarrow 0 \quad \text{exacte}$$

on obtient une **suite exacte longue**

- $$\xrightarrow{\partial} H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\partial} H^{n+1}(X,A;G) \xrightarrow{j^*} \cdots$$

∂ homomorphisme de connexion

- cohomologie singulière réduite $\tilde{H}^n(X;G)$

$$\cdots \xrightarrow{\partial_4} C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} \boxed{C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}}$$

duale

$$\boxed{\mathrm{Hom}(\mathbb{Z};G) \xrightarrow{\varepsilon^*} C^0(X;G)} \xrightarrow{d_0} C^1(X,G) \xrightarrow{d_1} C^2(X;G) \xrightarrow{d_2}$$

$$\tilde{H}^0(X;G) := \frac{\mathrm{Ker}\ d_0}{\mathrm{Im}\ \varepsilon^*}$$

$$\tilde{H}^0(X;G) := \frac{\ker d_0}{\operatorname{Im} \varepsilon^*}$$

$$\ker d_0 = \left\{ \phi : X \to G \;\middle|\; \begin{array}{c} \phi|_Y \text{ const} \\ \forall \text{ comp conn par arc} \\ Y \subseteq X \end{array} \right\} \cong H^0(X,G)$$

$$\operatorname{Hom}(\mathbb{Z},G) \xrightarrow{\cong} G$$
$$\psi \longmapsto \psi(1)$$

$$(\varepsilon^*\psi)(\sigma) = \psi(1) \quad \forall \sigma \text{ 0-simp sing}$$

$$\varepsilon^*\psi = \psi \circ \varepsilon$$

$$\operatorname{Im} \varepsilon^* = \{ \phi\ X \to G \text{ constant} \} \cong G$$

- **Invariance par homotopie**

$\forall\ f, g : (X,A) \longrightarrow (Y,B)$ homotopes

$\Big(h_t : (X,A) \longrightarrow (Y,B),\quad h_0 = f,\ h_1 = g$
$t \in [0,1] \Big)$

on a $f^* = g^* \quad H^n(Y,B;G) \longrightarrow H^n(X,A,G)$

$\Big(f_n - g_n = P_{n-1} \circ \partial_n + \partial_{n+1} P_n, \quad P_n : C_n(X,A) \to C_{n+1}(X,A)$
homotopie des chaînes

$\Downarrow$

$f_n^* - g_n^* = d_{n-1} \circ P_{n-1}^* + P_n^* \circ d_n \Big)$

- Excision.

$$H^n(X,A;G) \xrightarrow[\text{incl}^*]{\cong} H^n(X\setminus Z, A\setminus Z; G)$$

$$\forall\, Z \subset A \subset X, \quad \text{avec } \overline{Z} \subset A^\circ$$

- Suite exacte de Mayer-Vietoris. $X = A^\circ \cup B^\circ$

$$\cdots \xrightarrow{d} H^n(X,G) \xrightarrow{\psi^*} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi^*} H^n(A\cap B;G) \xrightarrow{d} H^{n+1}(X,G) \xrightarrow{\psi^*} \cdots$$

$$\left.\begin{aligned} &\psi^*(\mu) = (\iota_A^*\mu, \iota_B^*\mu), \quad \iota_A : A \to X, \quad \iota_B : B \to X \\ &\phi^*(\alpha,\beta) = j_A^*\alpha - j_B^*\beta, \quad j_A : A\cap B \to A, \quad j_B : A\cap B \to B \end{aligned}\right\} \text{incl.}$$