Homotopy, fundamental group

Exercise 1. Generalities.
Let X and Y be topological spaces, $f, g : X \to Y$ be continuous and $n \geqslant 1$ be an integer.

1. Suppose that X is path connected and that f and g are homotopic. Show that $f(X)$ and $g(X)$ are contained in the same path connected component of Y.
2. Suppose that X is contractible. Show that f and g are homotopic.
3. Suppose that $Y = \mathbb{R}^n \setminus \{0\}$ and that for any $x \in X$, $||f(x) - g(x)|| < ||f(x)||$. Show that f and g are homotopic.
4. Suppose that $Y = \mathbb{S}^n$ and that f is non-surjective. Show that f is homotopic to a constant map.
5. Suppose that $Y = \mathbb{S}^n$ and that for any $x \in X$, $||f(x) - g(x)|| < 2$. Show that f and g are homotopic. Deduce that any continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ with no fixed point is homotopic to the map $x \mapsto -x$.

Exercise 2. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Exercise 3. Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. If $i_0 \in I$, let $p_{i_0} : \prod_{i \in I} X_i \to X_{i_0}$ be the canonical projection. Show that the map

$$\prod_{i \in I} (p_i)_* : \pi_1\left(\prod_{i \in I} X_i, (x_i)_{i \in I}\right) \to \prod_{i \in I} \pi_1(X_i, x_i)$$

is a group isomorphism.

Exercise 4. Homotopy equivalences.

1. The *Möbius strip* is the quotient space M of $[0,1]^2$ under the equivalence relation $(x, 0) \sim (1-x, 1)$. Draw M. Show that it is homotopy equivalent to $\mathbb{S}^1$.
2. Let X, X' (resp. Y, Y') be homotopy equivalent topological spaces. Show that $X \times Y$ and $X' \times Y'$ are homotopy equivalent.
3. Let X and C be topological spaces. Assume thatd C is contractible. Show that $X \times C$ and X are homotopy equivalent.

Exercise 5. Homotopy type of a complement. Let $n \geqslant 1$ be an integer.

1. Let E be a linear subspace of dimension $k < n$ of $\mathbb{R}^n$. Show that $\mathbb{R}^n \setminus E$ and $\mathbb{S}^{n-k-1}$ have the same homotopy type.
2. Let C be a bounded convex subset of $\mathbb{R}^n$. Show that $\mathbb{R}^n \setminus C$ has the same homotopy type as $\S^{n-1}$.
3. Let X be a topological space and A, B be subspaces of X. Assume that A and B have the same homotopy type. Do $X \setminus A$ and $X \setminus B$ still have the same homotopy type ?
4. Show that the once-punctured torus and the wedge sum of two circles have the same homotopy type.

Exercise 6. Fundamental group of $\mathbb{S}^1$. The aim of this exercise is to compute the fundamental group of $\mathbb{S}^1$. As a model of $\mathbb{S}^1$, we chose the unit circle inside the complex plane.

1. Let $\gamma : I \to \mathbb{S}^1$ be a loop with base point 1. Show that there is an integer $n > 0$ such that for all $0 \leqslant i < n$, we have either $\gamma\left(\left[\frac{i}{n}, \frac{i+1}{n}\right]\right) \subseteq \mathbb{S}^1 \setminus \{i\}$ or $\gamma\left(\left[\frac{i}{n}, \frac{i+1}{n}\right]\right) \subseteq \mathbb{S}^1 \setminus \{-i\}$.

2. Let $e : \mathbb{R} \to \mathbb{S}^1$ be the map $x \mapsto e^{2i\pi x}$. Let $m \in \mathbb{Z}$. Show that there is a unique map $\widetilde{\gamma} : \left[0, \frac{1}{n}\right] \to \mathbb{R}$ such that $\widetilde{\gamma}(0) = m$ and such that the diagram

$$\begin{array}{ccc} \left[0, \frac{1}{n}\right] & \xrightarrow{\widetilde{\gamma}} & \mathbb{R} \\ & \searrow^{\gamma} & \downarrow e \\ & & \mathbb{S}^1 \end{array}$$

commutes. We call such a map a lifting of γ.

3. Show that there is a unique map $\widetilde{\gamma} : I \to \mathbb{R}$ such that $\widetilde{\gamma}(0) = m$ and such that the diagram

$$\begin{array}{ccc} I & \xrightarrow{\widetilde{\gamma}} & \mathbb{R} \\ & \searrow^{\gamma} & \downarrow e \\ & & \mathbb{S}^1 \end{array}$$

commutes.

4. Show that the integer $\widetilde{\gamma}(1) - \widetilde{\gamma}(0)$ does not depend on m. We call that integer the *degree* of γ and denote it by $\deg(\gamma)$.

5. Let $H : I^2 \to \mathbb{S}^1$ be an homotopy between two loops γ and γ'. Show that H lifts to a map $\widetilde{H} : I^2 \to \mathbb{R}$. Show that $\widetilde{H}$ is an homotopy between $\widetilde{\gamma}$ and $\widetilde{\gamma'}$. Deduce that the degree defines a map

$$\deg : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}.$$

6. Show that the map deg is a group isomorphism.

Exercise 7. Sphere-filling loops Let $n \geqslant 1$ be an integer.

1. Let γ be a loop on $\mathbb{S}^n$. Assume that the image of γ is not $\mathbb{S}^n$ itself. Show that γ is homotopic to the constant loop.

2. There are loops on the sphere $\mathbb{S}^n$ whose image is $\mathbb{S}^n$ itself (we do not ask to prove this). Show that there is a sphere-filling loop which is homotopic to the constant loop.

3. Assume $n \geqslant 2$. Let $\gamma : I \to \mathbb{S}^n$ be a path. Show that there is an integer $m > 0$ such that for all $0 \leqslant i < m$, the path $\gamma|_{\left[\frac{i}{m}, \frac{i+1}{m}\right]}$ is homotopic relative to the subset $\{\frac{i}{m}, \frac{i+1}{m}\}$ to a path which is nowhere dense in $\mathbb{S}^n$.

4. Deduce that the fundamental group of $\mathbb{S}^n$ is trivial if $n \geqslant 2$.

Exercise 8. Fundamental group of a topological group

1. **Eckmann-Hilton principle.** Let X be a set. Assume that X is endowed with two compatible products *i.e.* with two maps $* : X \times X \to X$ and $\cdot : X \times X \to X$ such that:

   - Each binary operation $*$ and $\cdot$ has a unit (denoted by 1_* and $1_\cdot$).
   - For all $x, x', y, y' \in X$, we have:

   $$0(x \cdot x') * (y \cdot y') = (x * y) \cdot (x' * y').$$

   Show that those binary operations are equal and define a commutative monoid structure on X.

2. Show that the fundamental group of a topological group is commutative.