ENS de Lyon **Master 1 – Algebraic topology**
TD10 **Spring 2024**

Künneth Formula-Orientability

1. Let n, m, g and h be positive integers and let R be a commutative ring. Compute the cohomology rings (with multiplication given by the cup product), with coefficients in R, of the following topological spaces.

(i) $S^n \times S^m$.

(ii) The n-dimensional torus $\mathbb{T}^n$.

(iii) Σ_g, the compact orientable surface of genus g.

2. Show that if M is a differentiable manifold, both notions of orientability defined in the lecture coincide.

3. Show that the surface Σ'_g is not orientable. Show that if n is an even integer, the space $\mathbb{RP}^n$ is not orientable.

4. Let n and g be positive integers. Show that the following manifolds are orientable.

(i) Σ_g.

(ii) S^n.

(iii) $\mathbb{CP}^n$.

(iv) $\mathbb{RP}^{2n+1}$.

(v) Every Lie group.

(vi) Every complex manifold, i.e. admitting an atlas with holomorphic transition maps.

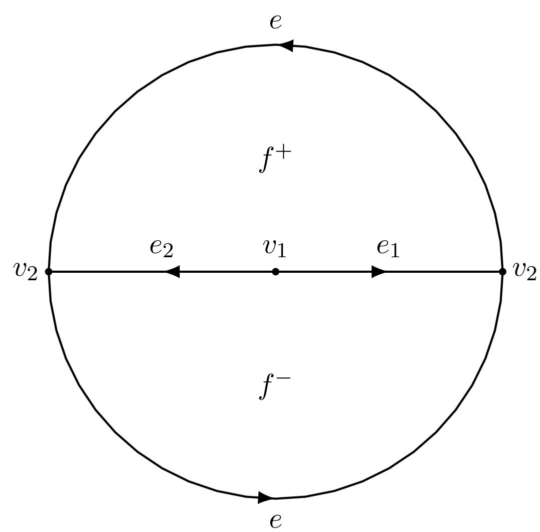
(vii) TM for any manifold. *Hint.* It is easier to prove it for T^*M.

5. If X is any space, we denote $H^*(X)_2 = H^*(X, \mathbb{Z}/2\mathbb{Z})$. Let n be a positive integer. We denote $\mathbb{P}^n = \mathbb{RP}^n$, the real projective space of dimension n. We know that $H^i(\mathbb{P}^n)_2 = \mathbb{Z}/2\mathbb{Z}$ for all $i = 0, \cdots, n$, and 0 for all other values of i. Thus we have a group isomorphism to a truncated polynomial algebra:

$$H^*(\mathbb{RP}^n)_2 \simeq (\mathbb{Z}/2\mathbb{Z})[t]/(t^{n+1}).$$

We want to prove that this is in fact an isomorphism of rings (or of $\mathbb{Z}/2\mathbb{Z}$-algebras). In other words, denoting by $\alpha_{n,i}$ the generator of $H^i(\mathbb{P}^n)_2$ for $0 \leq i \leq n$, we want to prove that $\alpha_{n,1}^n$ generates $H^n(\mathbb{P}^n)_2$. [Equivalently, $\alpha_{n,1}^n \neq 0$; but this would not extend to the case of $\mathbb{CP}^n$].

(i) We put on $\mathbb{P}^2$ a degenerate simplicial structure, more precisely a structure of Δ-*complex* in the sense of Hatcher, as follows: one glues, along corresponding oriented sides e_1, e_2, e, two triangles f^+, f^- each of which has already two vertices v_2 identified:



Let $C_*(\mathbb{P}^2)$ be the associated simplicial chain complex. Show that the simplicial cochain $a = C_2(\mathbb{P}^2) \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$a(e) = a(e_1) = 1, a(e_2) = 0$$

is a cocycle in $C^*(\mathbb{P}^2, \mathbb{Z}/2\mathbb{Z})$, and that the $\mathbb{Z}/2\mathbb{Z}$-chain $f^+ - f^-$ is a cycle in $C_*(\mathbb{P}^2) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Compute $(a \cup a)(f^+ + f^-)$ and deduce the result for $n = 2$.

(ii) Show that the inclusion $\mathbb{P}^{n-1} \to \mathbb{P}^n$ induces an isomorphism on H^i if $i \leq n-1$ and that this isomorphism respects the cup product.

(iii) Deduce that it suffices to show that

$$(*) \qquad 1 \leq i, j \leq n \text{ and } i + j = n \Rightarrow \alpha_{n,i} \cup \alpha_{n,j} \text{ generates } H^n(\mathbb{P}^n)_2.$$

(iv) Recall that the points of $\mathbb{P}^n$ can be given by homogenous coordinates $[x_0 : \cdots : x_n]$. We define two copies of $\mathbb{P}^i$ and $\mathbb{P}^j$ in $\mathbb{P}^n$, which meet at a unique point p:

$$\begin{aligned} \Pi^i &= \{[x_0 : \cdots : x_i : 0 : \cdots : 0] \mid (x_0, \cdots, x_i) \in \mathbb{R}^{i+1} \setminus \{(0,0)\} \\ \Pi'^j &= \{[0 : \cdots : x_i : \cdots : x_n] \mid (x_i, \cdots, x_n) \in \mathbb{R}^{j+1} \setminus \{(0, \cdots, 0)\} \\ p_i &= [0 : \cdots : 1_i : 0 : \cdots : 0]. \end{aligned}$$

(v) Show that the following diagram is well defined (in particular, precise the meaning of v_1, v_2 and j) and commutes.

$$\begin{array}{ccc} H^i(\mathbb{P}^n)_2 \times H^j(\mathbb{P}^n)_2 & \xrightarrow[c]{\smile} & H^n(\mathbb{P}^n)_2 \\ \uparrow{\scriptstyle u_1^* \times u_2^*} & & \uparrow{\scriptstyle u^*} \\ H^i(\mathbb{P}^n, \mathbb{P}^n \setminus \Pi'^j)_2 \times H^j(\mathbb{P}^n, \mathbb{P}^n \setminus \Pi^i)_2 & \xrightarrow[c_1]{\smile} & H^n(\mathbb{P}^n, \mathbb{P}^n \setminus \{p_i\})_2 \\ \downarrow{\scriptstyle v_1^* \times v_2^*} & & \downarrow{\scriptstyle v^*} \\ H^i(\mathbb{R}^n, \mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^j)_2 \times H^j(\mathbb{R}^n, \mathbb{R}^n \setminus (\mathbb{R}^i \times \{0\})_2 & \xrightarrow[c_2]{\smile} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})_2 \end{array}$$

Show that the bottom horizontal arrow is an isomorphism. Deduce that to prove $(*)$ it suffices to show that all vertical arrows are isomorphisms.

(vi) Show that v_1^*, v_2^* and v^* are isomorphisms.

(vii) Show that $\mathbb{P}^n \setminus \Pi'^j$ and $\Pi^i \setminus \{p_i\}$ deformation retract onto Π^{i-1}. Considering the following diagram, deduce that u_1^* is an isomorphism:

$$\begin{array}{ccc} H^i(\mathbb{P}^n, \mathbb{P}^n \setminus \Pi'^j)_2 & \xrightarrow{u_1^*} & H^i(\mathbb{P}^n)_2 \\ \downarrow & & \downarrow \\ H^i(\Pi^i, \Pi^i \setminus \{p_i\})_2 & \longrightarrow & H^i(\Pi^i)_2. \end{array}$$

(viii) Show that u_2^* is an isomorphism and conclude.

6. Sketch the proof that $H^*(\mathbb{CP}^n; \mathbb{Z}/2\mathbb{Z})$ is isomorphic as a ring to $\mathbb{Z}[t]/(t^{n+1})$.