ENS de Lyon **Master 1 – Algebraic topology**
TD11 **Spring 2024**

Poincaré Duality

1. Show that $\mathbb{CP}^2$ does not admit a homeomorphism reversing the orientation.

2. Let M be a connected topological manifold of dimension $n > 0$. We define the *orientation covering* $\pi : \widehat{M} \to M$ by

$$\widehat{M} = \{(x, \mu) \mid x \in M, \mu \text{ generates } H_n(M, M \setminus \{x\})\} \ , \ \pi(x, \mu) = x.$$

(i) Define a topology on $\widehat{M}$ such that π is a local homeomorphism, and show that π is a covering of degree two. Show that if M is differentiable, $\widehat{M}$ has a unique differentiable structure such that π is smooth.

(ii) Show that the manifold $\widehat{M}$ is orientable.

(iii) Show that M is orientable if and only if $\widehat{M}$ is not connected and that in this case, the manifold $\widehat{M}$ is homeomorphic to a disjoint union of two copies of M.

(iv) Deduce that simply connected manifolds (or more generally, manifolds whose fundamental group has no subgroup of index 2) are orientable.

(v) If $\gamma \in C(S^1, M)$, define $w(\gamma) \in \mathbb{Z}/2\mathbb{Z}$ as 0 if γ lifts to $\widehat{M}$ and 1 otherwise. Show that it defines an element

$$w_1(M) \in H^1(M, \mathbb{Z}/2\mathbb{Z}) \approx \mathrm{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}),$$

called the *first Stiefel-Whitney class* of M. Show that M is orientable if and only if $w_1(M) = 0$.

3. Let M be a compact connected n-manifold for some $n > 0$.

(i) Considering the short exact sequence

$$0 \longrightarrow C_*(M) \xrightarrow{\times 2} C_*(M) \longrightarrow C_*(M, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

define the *Bockstein homomorphism*

$$\beta_i : H_i(M, \mathbb{Z}/2\mathbb{Z}) \to H_{i-1}(M) \ , \ 1 \leq i \leq n.$$

Show that $\mathrm{im}(\beta_i) \subset \mathrm{Tors}_2(H_{i-1}(M))$, where $\mathrm{Tors}_2(E) = \{e \in E \mid 2e = 0\}$ for a $\mathbb{Z}$-module E.

(ii) Show that M is non-orientable if and only if $\beta_n \neq 0$. *One needs to use: $H_n(M) = 0$ if M is non-orientable. This follows from the method of proof of the course, which shows that the map $H_n(M) \to H_n(M, M \setminus \{x\})$ is always injective.*

(iii) Define $\overline{\beta}_n = \beta_n \bmod 2 : H_n(M, \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(M, \mathbb{Z}/2\mathbb{Z})$. Using the fact that Poincaré duality holds with $\mathbb{Z}/2\mathbb{Z}$ coefficients, show that $\overline{\beta}_n([M]_{\mathbb{Z}/2\mathbb{Z}})$ is Poincaré dual to $w_1(M)$.

(iv) *We admit that there exists a smooth map $\psi : M \to \mathbb{RP}^n$ which is transversal to $\mathbb{RP}^{n-1}$ and such that $\psi^*(t) = w_1(M)$ where t is the generator of $H^1(\mathbb{RP}^{2n+1}, \mathbb{Z}/2\mathbb{Z})$ (in fact this is true for all classes in $H^1(M, \mathbb{Z}/2\mathbb{Z})$; one constructs f skeleton by skeleton, using obstruction theory, then perturbs it to obtain the transversality (Sard)).

Show that $N = \psi^{-1}(\mathbb{RP}^{2n})$ is a compact oriented hypersurface in M, maybe non-connected, whose homology class $[N] = \sum [N_i] \in H_{n-1}(M, \mathbb{Z})$ is equal to $\beta_n([M]_{\mathbb{Z}/2\mathbb{Z}})$. Show that one of the components of N is one sided.

(v) Conversely, if M admits a compact oriented hypersurface which is one-sided, show that M is non-orientable.

4. Let M be a compact connected n-manifold.

(i) If M is orientable, show that $H_{n-1}(M)$ is torsion-free.

(ii) Assume that M is a non-orientable. Show that

$$H_n(M; \mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}/2\mathbb{Z}$.

5. Let M be a compact 3-manifold. Write $H_1(M) = \mathbb{Z}^r \oplus F$ with F a finite Abelian group.

(i) Assume that M is simply connected. Compute $H_1(M)$ and $H_2(M)$. Using Hurewicz-Whitehead, show that M is homotopy equivalent to S^3 (by Perelman, M is homeomorphic (or diffeomorphic) to S^3).

(ii) Assume that M is orientable. Compute $H_2(M)$ in terms of r, F.

Now we assume that M is non-orientable.

(iii) Compute $H_1(M, \mathbb{Z}/2\mathbb{Z})$ and $H_2(M, \mathbb{Z}/2\mathbb{Z})$ in terms of r, F.

(iv) Deduce that $H_2(M) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ (in particular $r > 0$).

(v) Deduce that the fundamental group of M is infinite.

(vi) *Find an example where $\pi_1(M) = \mathbb{Z}$.

6. Let X be a compact oriented n-manifold, with fundamental class $[X] \in H_n(X)$, and let k be an integer between 1 and $n-1$.

(i) Show that the cup product $H^k(X) \times H^{n-k}(X) \to H^n(X)$ induces a bilinear form

$$(a, b) \in H_k(X) \otimes H_{n-k}(X) \mapsto a.b \in \mathbb{Z},$$

called the *intersection product.*

(ii) Let M be a compact oriented submanifold of X of dimension k. We still denote by $[M]$ the image of the fundamental class of M in $H_k(X)$. If N is another compact submanifold of X, of dimension $n-k$. We define the *intersection number*

$$M.N := [M].[N].$$

Compute this number

i. if $X = M \times N$ with M identified to $M \times \{y\}$ and N identified to $\{x\} \times N$;

ii. if $X = T^n$ and M, N are subtori. A subtorus $M \subset T^n$ is $A.(T^k \times \{y\})$ with $A \in \mathrm{GL}(n, \mathbb{Z})$ and $y \in T^{n-k}$.