ENS de Lyon — Master 1 – Algebraic topology
TD 3 — 2023-2024

Exercise 1. First computations of homology groups.

Compute the homology groups of the following complexes:

1. $\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}$.
2. $\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}$.
3. $\cdots 0 \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$, where n is an integer.
4. $\cdots 0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z}$, where g is an integer and f is the map $n \in \mathbb{Z} \mapsto (2n, \ldots, 2n) \in \mathbb{Z}^g$.

Exercise 2. Some applications of Hurewicz's theorem.

Compute the first homology group of the following spaces:

1. The sphere $\mathbb{S}^n$ where $n \geqslant 1$ is an integer.
2. The Möbius strip.
3. The Klein bottle K which is the quotient space of $[0,1]^2$ under the relations $(0,y) \sim (1, 1-y)$ and $(x,0) \sim (1-x, 1)$ if $x, y \in [0,1]$. *You can use the fact that the fundamental group of K is isomorphic to $\langle a, b \rangle / aba^{-1}b$.*
4. The torus Σ_g with g holes. *You can use the fact that the fundamental group of Σ_g is isomorphic to $\langle a_1, b_1, \ldots, a_g, b_g \rangle / a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.*

Exercise 3. Examples of chain complexes

1. Let $(M_n)_{n \geqslant 0}$ be a sequence of abelian groups.
   (a) Construct a chain complex C such that for all $n \geqslant 0$, we have $H_n(C) = M_n$.
   (b) Same question but the C_i have to be free abelian groups if $i \geqslant 0$.
2. Give a complex C of abelian groups such that the C_i are not all of finite type but for all n, the abelian group $H_n(C)$ is of finite type.

Exercise 4. Abelianization

If G is a group, we denote G^{ab} its abelianization.

1. Show that if $\phi\colon H \to G$ is a group homomorphism and G is abelian, there is a unique group homomorphism $\overline{\phi} : H^{ab} \to G$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ \downarrow & \nearrow_{\overline{\phi}} & \\ H^{ab} & & \end{array}$$

   commutes.
2. If $\phi\colon H \to G$ is a group homomorphism, deduce that there is a unique group homomorphism $\phi^{ab} : H^{ab} \to G^{ab}$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \\ H^{ab} & \xrightarrow{\phi^{ab}} & G^{ab} \end{array}$$

   commutes.
3. If $f : X \to Y$ is a continuous map of topological spaces, describe the morphism

$$(f_*)^{ab} : H_1(X) \to H_1(Y).$$

Exercise 5. Suspensions and cones of chain complexes

1. Let (C, ∂) be a chain complex. We let $(C[1])_n = C_{n-1}$ and $(\partial[1])_n = -\partial_{n-1}$ if $n \geqslant 1$ and we let $(C[1])_0 = 0$. Show that $(C[1], \partial[1])$ is a chain complex. We call it the *suspension of* C.

2. Let $f : C \to D$ be a chain complex morphism. We let

$$C_0(f) = D_0 \text{ and } C_i(f) = C_i \oplus D_{i-1} \text{ if } i \geqslant 1.$$

   Moreover, if $(x, y) \in C_i \oplus D_{i+1}$, we let $\delta(x, y) = (-\partial x, \partial y + f(x)) \in C_{i-1} \oplus D_i$ for $i \geqslant 1$ and if $i = 0$, we let $\delta(x, y) = \partial y + f(x) \in D_0$.

   Show that $(C(f), \delta)$ is a chain complex. We call it the *cone of* f.

3. Show that we have a short exact sequence of complexes:

$$0 \to D \to C(f) \to C[1] \to 0$$

4. Using the snake lemma, show that we have a long exact sequence:

$$\ldots \to H_i(D) \to H_i(C(f)) \to H_i(C[1]) \xrightarrow{u} H_{i-1}(D) \to \cdots$$

   ans that the map u is $H_{i-1}(f)$.

Exercise 6. A simple Van Kampen's theorem in homology.

Recall Van Kampen's theorem: *Let X be a path connected topological space. Let U_1 and U_2 be path connected open subsets of X such that $U_1 \cap U_2$ is simply connected. Let $x \in U_1 \cap U_2$. Then, $\pi_1(X, x)$ is isomorphic to the free product*

$$\pi_1(U_1, x) * \pi_1(U_2, x).$$

1. Let G and H be groups. Show that the abelianization of $G * H$ is isomorphic to the product of the abelianizations of G and H.

2. Let X be a path connected topological space. Let U_1 and U_2 be path connected open subsets of X such that $U_1 \cap U_2$ is simply connected. Let $x \in U_1 \cap U_2$. Show that $H_1(X)$ is isomorphic to $H_1(U_1) \oplus H_1(U_2)$.

3. Compute the first homology group of a wedge sum of circles.