

### Relative homology groups; Mayer Vietoris exact sequence

#### Exercise 1. Some properties of relative homology groups.

Let  $(X, A)$  be a topological pair.

1. Show that  $H_0(X, A) = 0$  if and only if  $A$  meets each path-component of  $X$ .
2. Show that  $H_1(X, A) = 0$  if and only if the canonical map  $H_1(A) \rightarrow H_1(X)$  is surjective and each path-component of  $X$  contains at most one path-component of  $A$ .
3. Assume that  $A$  is a single point space. Compute the relative homology groups  $H_n(X, A)$  in terms of the homology groups of  $X$ .
4. Assume that  $i : A \rightarrow X$  is a retract, show that the map  $H_n(A) \rightarrow H_n(X)$  is injective.

#### Exercise 2. Suspensions and cones.

Let  $X$  be a topological space. The *cone over  $X$*  is the topological space  $CX := X \times [0, 1]/X \times \{1\}$ . The *suspension of  $X$*  is the topological space  $\Sigma X := CX/X \times \{0\}$

1. Show that  $CX$  is contractible.
2. Compute the homology groups of  $\Sigma X$  in terms of the homology groups of  $X$ .
3. Let  $n \geq 0$  be an integer. What is the suspension of the sphere  $S^n$  ?
4. Compute the homology groups of  $S^n$ .

**Exercise 3.** Show that the relative homology group  $H_1(\mathbb{R}, \mathbb{Q})$  is free and find a basis.

#### Exercise 4. A description of relative homology groups.

Let  $(X, A)$  be a topological pair. Consider the topological space

$$X \cup CA = (X \sqcup CA)/(a \sim (a, 0) \mid a \in A).$$

Let  $p$  be the image of  $(a, 1)$  (which is the same for any  $a \in A$ ) in  $X \cup CA$ .

1. Let  $n \geq 1$  be an integer. Let  $B^n$  be the closed unit ball. What is  $B^n \cup C(\partial B^n)$  ?
2. Show that the map of pairs  $(X \cup CA, p) \rightarrow (X \cup CA, CA)$  induces an isomorphism of homology groups.
3. Show that the map of pairs  $(X, A) \rightarrow (X \cup CA, CA)$  induces an isomorphism of homology groups.
4. Deduce that for all  $n$ ,

$$H_n(X, A) \simeq \tilde{H}_n(X \cup CA).$$

5. Deduce another way to compute the homology groups of the sphere  $S^n$ .

#### Exercise 5. Long exact sequence of the triple.

Let  $(X, A, B)$  be a topological triple.

1. Show that the following sequence is exact:

$$\cdots \rightarrow H_n(A, B) \xrightarrow{(i_*)^n} H_n(X, B) \xrightarrow{(j_*)^n} H_n(X, A) \xrightarrow{d_n} H_{n-1}(A, B) \rightarrow \cdots$$

where  $i : (A, B) \rightarrow (X, B)$  and  $j : (X, B) \rightarrow (X, A)$  are the inclusions and  $d_n$  is the composition

$$H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \xrightarrow{(k_*)^{n-1}} H_{n-1}(A, B)$$

where  $\delta_n$  is the connecting map of the exact sequence of the pair  $(X, A)$  and  $k : (A, \emptyset) \rightarrow (A, B)$  is the inclusion.

2. Deduce the exact sequence of the pair in reduced homology.

**Exercise 6. Homology groups of the parachute.**

Let  $\Delta^2 = [e_0, e_1, e_2]$  be the standard 2-simplex. Compute the homology groups of  $\Delta^2/\{e_0, e_1, e_2\}$ .

**Exercise 7. Wedge sum of spaces.**

Let  $(X, x)$  and  $(Y, y)$  be pointed topological spaces. We define their wedge sum  $X \vee Y = X \sqcup Y / (x \sim y)$ .

Assume that  $x$  and  $y$  have contractible neighborhoods respectively in  $X$  and  $Y$ . Compute the homology groups of  $X \wedge Y$ .

**Exercise 8. A counterexample.**

Show that the torus  $T^2 = S^1 \times S^1$  and the space  $S^1 \vee S^1 \vee S^2$  have the same homology groups but not the same fundamental group.

**Exercise 9. Relative homology of good pairs.**

1. Let  $(X, A)$  be a topological pair. Assume that  $A$  is a *good pair* i.e. that there is a neighborhood  $V$  of  $A$  which deformation retracts onto  $A$ .

Let  $q : (X, A) \rightarrow (X/A, A/A)$  be the quotient map. Let  $n \geq 0$  be an integer. Show that in the commutative diagram:

$$\begin{array}{ccccc}
 H_n(X, A) & \longrightarrow & H_n(X, V) & \longleftarrow & H_n(X - A, V - A) \\
 \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\
 H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & & H_n(X/A - A/A, V/A - A/A)
 \end{array}$$

all the maps are isomorphisms.

Deduce that the map  $q_* : H_n(X, A) \rightarrow \tilde{H}_n(X/A)$  is an isomorphism.

2. Compute once again the homology groups of the sphere  $S^n$ .

**Exercise 10. Homology groups of some pathological topological spaces.**

Compute the homology groups of the line with 2 origins. Same question with the closure in  $\mathbb{R}^2$  of the graph of the function  $(x > 0) \mapsto \sin(1/x)$ .