Recall (?) a theorem of Hurewicz+Whitehead: if X and Y are simply connected CW-complexes and $f \in C(X,Y)$ inducing an isomorphism $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$, then f is a homotopy equivalence.

1. Complements on Moore spaces. Let $n > 1$, G be an Abelian group. A *simply connected Moore space* $M(G,n)$ is a simply connected CW-complex X such that $\widetilde{H}_i(X,\mathbb{Z}) = 0$ for $i \neq n$ and $\widetilde{H}_n(X,\mathbb{Z}) \approx G$.

(i) Show that G admits a free presentation $\mathbb{Z}^{(J)} \xrightarrow{\alpha} \mathbb{Z}^{(I)} \xrightarrow{\pi} G$ with α injective.

(ii) Define M to be a CW-complex of dimension$\leq n+1$ such that $M^{(n)} = \bigvee_{i\in I} S^n_i$ and there are J $n+1$-cells σ_j, attached by maps $a_j : S^n \to M$ such that the map $H_n(\bigvee_{j\in J} S^n_j) \to M^{(n)}$ is α. Show that M is a simply connected Moore space $M(G,n)$.

(iii) If X is a simply connected Moore space $M(G,n)$, construct a map $M \to X$ which is a homotopy equivalence.

2. Complement on Poincaré's hypercubic variety. Recall that this is

$$V = [0,1]^3/_{(0,y,z)\sim(1,-z,y),(x,0,z)\sim(z,1,-x),(x,y,0)\sim(-y,x,1)}.$$

(i) Compute the integer homology of V (if it has not already been done).

(ii) Compute $\pi_1(V)$ and compare it to $H_1(V,\mathbb{Z})$.

(iii) * Show that V is a topological manifold, which admits a structure of differentiable manifold.

3. Let $X = S^1 \vee S^n$ for some $n > 1$.

(i) What is the universal covering $\widetilde{X}$?

(ii) Lifting a CW structure on X, show that $H_n(\widetilde{X},\mathbb{Z}) \approx \mathbb{Z}^{(\mathbb{Z})}$.

Remark. Since $H_n(\widetilde{X},\mathbb{Z}) = \pi_n(X)$, this shows that the higher homotopy groups of a finite complex need not be finitely generated.

(iii) * Show that $H_n(\widetilde{X},\mathbb{Z})$ has a natural structure of module over $R = \mathbb{Z}[t,t^{-1}]$ (Laurent polynomials), and that it is isomorphic to R as an R-module.

4. Let $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0$ be a free and finite complex of $\mathbb{Z}$-modules (or $\mathbb{Z}$-complex), for instance the chain complex of a finite CW-complex. Denote

$$Z_i = \ker \partial_i\ ,\ B_i = \partial_{i+1}C_i\ ,\ H_i = H_i(C) = Z_i/B_i.$$
$$h_i = \mathrm{rank}_{\mathbb{Z}}(H_i(C)) = \dim_{\mathbb{R}}(H_i \otimes \mathbb{R}).$$

(i) Considering the short exact sequences $B_i \to Z_i \to H_i$ and $Z_i \to C_i \to B_{i-1}$, show the *Morse inequalities* (plus Euler characteristic identity)

$$(\forall i = 0,1,\cdots,n) \sum_{j=0}^{i}(-1)^j c_{i-j} \geq \sum_{j=0}^{i}(-1)^j h_{i-j}$$
$$\text{or } c_i - c_{i-1} + \cdots + (-1)^i c_0 \geq h_i - h_{i-1} + \cdots + (-1)^i h_0.$$

What happens when $i = n$?

(ii) Recall that if A is a finitely generated $\mathbb{Z}$-module, it is isomorphic to $\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/e_i\mathbb{Z}$, with $e_i | e_{i+1}$.

Denote by $r(A)$ the minimal number of generators of A. Show that

$$\begin{aligned} k &= r(\mathrm{Tors}(A)) \text{ where } \mathrm{Tors}(A) = \{a \in A \mid (\exists m \in \mathbb{N}^*)\ ma = 0\} \\ r + k &= r(A). \end{aligned}$$

(iii) If $A \to B \to C$ is a short exact sequence of finitely generated $\mathbb{Z}$-modules, show that $r(B) = r(A) + r(C)$. Deduce that, if we set $t_i = r(\mathrm{Tors}(H_i(C))$, we have

$$(\forall i = 0, \cdots, n)\ c_i \geq h_i + t_i + t_{i-1}.$$

(iv) We want to find a free and finite $\mathbb{Z}$-complex $\widetilde{C}_*$ such that $(\forall i)\ \widetilde{c}_i = h_i + t_i + t_{i+1}$ and

$$C_* \approx \widetilde{C}_* \oplus \bigoplus_{j=1}^{r} T_*^j,$$

where T_*^j is a "trivial" complex, meaning that it is localized in degrees $i, i-1$ for some i and that the map $\partial_{i(j)}^j : T_i^j \to T_{i-1}^j)$ being an isomorphism.

For this, assuming that $C_n \neq 0$, show that there is a $\mathbb{Z}$-module decomposition $C_{n-1} = C'_{n-1} \oplus C''_{n-1}$ where $C'_{n-1} = \mathbb{Q}\partial_n C_n \cap C_{n-1}$. Consider the decomposition of $\mathbb{Z}$-complexes,

$$C_* = (C_n \to C'_{n-1} \to 0 \cdots \to 0) \oplus (C''_{n-1} \to C_{n-2} \to \cdots \to C_0 \to 0)$$

and make an induction on $N(C_*) = n + c_n + \cdots + c_0$.

5. Let X be a simply connected CW-complex with $H_n(X, \mathbb{Z})$ is finitely generated for all n. We want to prove that X is homotopy equivalent to a complex Y with finite skeleta.

For this, we first construct an increasing sequence of finite CW-complexes and maps

$$K_0 \subset K_1 \subset K_2 \subset \cdots\ ,\ \dim K_n \leq n\ ,\ f_n \in C(K_n, X)$$

such that $(\forall n \in \mathbb{N})\ f_n$ is *n-connected*, i.e. it induces an isomorphism on H_i for $i < n$ and a surjection for $i = n$.

(i) Construct K_0, K_1 and f_0, f_1.

(ii) Assume that (K_n, f_n) has been constructed for some $n \geq 2$. Let $a_i : (D^{n+1}, S^n) \to (X^{(n+1}, X^{(n)})$, $i \in I_n$, be the $(n+1)$-cells of X_n, and $[a_i]$ be their images in $H_{n+1}(X_n^{(n+1)}, X_n^{(n)})$.

Show that one can find a finite number of $(n+1)$-cells $a_1, \cdots, a_k$ such that $\partial[a_1], \cdots, \partial[a_k]$ generate $H_n(X_n^{(n)}, X_n^{(n-1)})$.

(iii) Show that one can take K_{n+1} obtained from K_n by attaching the $(n+1)$-cells $a_1, \cdots, a_k$, then by induction the $(n+k+1)$ cells of X_n whose boundary lands in $X_{n+1}^{(n+k)}$ for $k = 1, 2, \cdots$.

(iv) Find the desired complex Y.

Remark. The methods of exercises (i) and (ii) can be combine to prove that if X is a simply connected CW-complex with $H_n(X, \mathbb{Z})$ finitely generated for all n, it is homotopy equivalent to a CW-complex Y with $h_n + t_n + t_{n-1}$ cells of each dimension n. In particular, if $H_*(X, \mathbb{Z})$ is finitely generated, then Y is finite.